TOPOLOGICAL SELF-JOININGS OF CARTAN ACTIONS BY TORAL AUTOMORPHISMS

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Abstract. We show that if $r \geq 3$ and α is a faithful $\mathbb{Z}^r$-Cartan action on a torus $\mathbb{T}^d$ by automorphisms, then any closed subset of $(\mathbb{T}^d)^2$ which is invariant and topologically transitive under the diagonal $\mathbb{Z}^r$-action by α is homogeneous, in the sense that it is either the full torus $(\mathbb{T}^d)^2$, or a finite set of rational points, or a finite disjoint union of parallel translates of some d-dimensional invariant subtorus. A counterexample is constructed for the rank 2 case.

Contents

1. Introduction

1.1. Backgrounds.

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, then its group of automorphisms is $\mathrm{Aut}(\mathbb{T}^d) = \mathrm{SL}(d,\mathbb{Z})$. A toral automorphism $\phi \in \mathrm{SL}(d,\mathbb{Z})$ is *irreducible* if it leaves invariant no non-trivial proper subtorus of $\mathbb{T}^d$, or equivalently its characteristic polynomial is irreducible over $\mathbb{Q}$. ϕ is *totally irreducible* if ϕ^n is irreducible for all non-zero integer n.

A *faithful $\mathbb{Z}^r$-action* on $\mathbb{T}^d$ by automorphisms is a group embedding $\alpha : \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ of $\mathbb{Z}^r$ into $\mathrm{SL}(d,\mathbb{Z})$.

Furstenberg proved the rigidity of multiplicative actions by higher-rank semigroups of $\mathbb{N}$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ in [Fur67]. For example, $\mathbb{T}$ itself is the only infinite closed subset of $\mathbb{T}$ that is invariant under both $\times 2$ and $\times 3$. The higher dimensional generalization was obtained by Berend [Ber83]. In particular, as a special case concerning only group actions, the following holds:

Theorem 1.1. [Ber83] *Let $\alpha : \mathbb{Z}^r \hookrightarrow \mathrm{SL}(d,\mathbb{Z})$ be a faithful $\mathbb{Z}^r$-action on $\mathbb{T}^d$ by toral automorphisms that satisfies:*

1. *$r \geq 2$;*
2. *$\exists \mathbf{n} \in \mathbb{Z}^d$ such that $\alpha^{\mathbf{n}}$ is a totally irreducible toral automorphism;*
3. *If $v \in \mathbb{C}^d$ is a nonzero common eigenvector of the commutative group of matrices $\alpha(\mathbb{Z}^r)$, then $\exists \mathbf{n} \in \mathbb{Z}^r$ such that $|\alpha^{\mathbf{n}}.v| > |v|$.*

Then the only infinite α-invariant closed subset of $\mathbb{T}^d$ is $\mathbb{T}^d$ itself. In particular, $\forall y \in \mathbb{T}^d$, $\{\alpha^{\mathbf{n}}.y : \mathbf{n} \in \mathbb{Z}^r\}$ is dense in $\mathbb{T}^d$ unless y is rational.

Remark 1.2. *Notice if $\alpha : \mathbb{Z}^r \curvearrowright \mathbb{T}^d$ is as in Theorem 1.1 and $H \leq \mathbb{Z}^r$ is a finite-index subgroup, then $H \cong \mathbb{Z}^r$ and $\alpha|_H$ satisfies the same conditions in Theorem 1.1 as well. To see this we only need to check conditions (2) and (3). The key fact is if $\mathbf{n} \in \mathbb{Z}^r$, then there is $k \in \mathbb{N}$ such that $k\mathbf{n} \in H$. Moreover, if $\alpha^{\mathbf{n}}$ is totally irreducible then so is $\alpha^{k\mathbf{n}} = (\alpha^{\mathbf{n}})^k$; and if $\alpha^{\mathbf{n}}$ expands a certain eigenvector then so does $\alpha^{k\mathbf{n}}$.*

In particular, Berend's theorem covers the special case of Cartan actions which contain totally irreducible elements (see Lemma 2.4), where a Cartan action is defined by the following maximality condition:

Definition 1.3. *A faithful $\mathbb{Z}^r$-action α on $\mathbb{T}^d$ by automorphisms is* **Cartan** *if the abelian subgroup $\alpha(\mathbb{Z}^r) < \mathrm{SL}(d,\mathbb{Z})$ contains an irreducible element and is maximal in rank, i.e. there exists no intermediate abelian subgroup G such that $\alpha(\mathbb{Z}^r) < G < \mathrm{SL}(d,\mathbb{Z})$ and $\mathrm{rank}(G) > r$.*

Here and throughout the rest of paper, $\mathrm{rank}(G)$ always refers to the torsion-free rank of a finitely generated group G.

We remark that our definition of Cartan action here is more general than what was adopted by some previous authors, as in our case a Cartan action is not necessarily $\mathbb{R}$-split, i.e. some of the common eigenspaces of $\alpha(\mathbb{Z}^r)$ may be defined only over $\mathbb{C}$ but not over $\mathbb{R}$.

Cartan actions are of particular number-theoretical interest since a Cartan action is, up to finite index, conjugate to the multiplicative action of U_K, the group of units of some number field K of degree d, on an arithmetic compact quotient of $K \otimes_{\mathbb{Q}} \mathbb{R}$ (see Proposition 2.2).

1.2. **Main results.** In this paper, we try to understand what happens if the action is no longer irreducible. More precisely, for a $\mathbb{Z}^r$-Cartan action α on $\mathbb{T}^d$ as in Theorem 1.1, we consider the diagonal $\mathbb{Z}^r$-action $\alpha_\triangle$ on $(\mathbb{T}^d)^2$ given by

$$\alpha_\triangle^{\mathbf{n}}.y = (\alpha^{\mathbf{n}}.y^{(1)}, \alpha^{\mathbf{n}}.y^{(2)}), \forall \mathbf{n} \in \mathbb{Z}^r, \forall y = (y^{(1)}, y^{(2)}) \in (\mathbb{T}^d)^2. \tag{1.1}$$

Recall the notion of topological transitivity:

Definition 1.4. *For group action $\rho : G \curvearrowright \Omega$ on a compact metric space Ω, a ρ-invariant closed subset $A \subset \Omega$ is* **topologically transitive** *if for any two non-empty subsets $U, V \subset A$ that are both relatively open inside A, $\exists g \in G$ such that $(\rho(g).U) \cap V \neq \emptyset$.*

Remark 1.5. *the orbit closure $\overline{\{\rho(g).\omega : g \in G\}}$ of any $\omega \in \Omega$ is invariant and topologically transitive.*

One hopes to classify all closed subsets of $(\mathbb{T}^d)^2$ that are invariant and topologically transitive under the diagonal action $\alpha_\triangle$. A few candidate types are listed below.

(0-dimensional) Clearly if $x \in (\mathbb{T}^d)^2$ is rational then the orbit $\{\alpha_\triangle^{\mathbf{n}}.x : \mathbf{n} \in \mathbb{Z}^r\}$ is a finite set of rational points.

(2d-dimensional) It is not hard to see that $\alpha_\triangle$ is ergodic with respect to the Lebesgue measure on $(\mathbb{T}^d)^2$, hence the orbit closure of almost every point is $(\mathbb{T}^d)^2$ itself.

(d-dimensional) There is an intermediate category of d-dimensional sets. For example, let $y = (y^{(1)}, 0)$ where $y^{(1)} \in \mathbb{T}^d$ is irrational. Then by Theorem 1.1, $\{\alpha^{\mathbf{n}}.y^{(1)} : \mathbf{n} \in \mathbb{Z}^r\}$ is dense in $\mathbb{T}^d$. So the orbit closure of y under $\alpha_\triangle$ is $\mathbb{T}^d \times \{0\}$, and thus $\mathbb{T}^d \times \{0\}$ is a topologically transitive $\alpha_\triangle$-invariant subtorus. More generally, there are many d-dimensional subtori of $(\mathbb{T}^d)^2$ that are invariant under the action $\alpha_\triangle$. In general, if A_0 is such an invariant subtorus and $A_1 \subset (\mathbb{T}^d)^2$ is the translate of A_0 by a rational point, then the stabilizer $H = \{\mathbf{n} : \alpha_\triangle^{\mathbf{n}}.A_1 = A_1\}$ is a finite-index subgroup of $\mathbb{Z}^r$. Choose representatives $\mathbf{n}_1, \cdots, \mathbf{n}_s$ such that $\mathbb{Z}^r/H = \{\mathbf{n}_1 + H, \cdots, \mathbf{n}_s + H\}$, and denote $A_t = \alpha_\triangle^{\mathbf{n}_t}.A_1, \forall t = 1, \cdots, s$.

Then the disjoint union $\bigsqcup_{t=1}^{s} A_t$ is $\alpha_\triangle$-invariant and topologically transitive.

All three types of invariant sets above are said to be *homogeneous*. One may speculate that these exhaust all topologically transitive $\alpha_\triangle$-invariant closed subsets of $(\mathbb{T}^d)^2$. It turns out this is true when $r \geq 3$ but fails in rank 2.

Theorem 1.6. *Let $r \geq 2$ and α be a faithful $\mathbb{Z}^r$-Cartan action on $\mathbb{T}^d$ by automorphisms such that $\alpha^{\mathbf{n}}$ is a totally irreducible toral automorphism for at least one $\mathbf{n} \in \mathbb{Z}^r$. Let $\alpha_\triangle : \mathbb{Z}^r \curvearrowright \mathbb{T}^d$ be the diagonal action in (1.1).*

(1) When $r \geq 3$, if an infinite proper closed subset A of $(\mathbb{T}^d)^2$ is invariant and topologically transitive under $\alpha_\triangle$, then there is an $\alpha_\triangle$-invariant d-dimensional subtorus $A_0 \subset \mathbb{T}^{2d}$ such that A decomposes into a finite disjoint union $\bigsqcup_{t=1}^{s} A_t$, where each A_t is a translate of A_0.

(2) If $r = 2$, then there is a point $\mathbf{y} \in (\mathbb{T}^d)^2$ whose orbit closure is an infinite proper subset of $(\mathbb{T}^d)^2$ but not homogeneous. Actually there are three d-dimensional subtori $T_1, T_2, T_3 \subset (\mathbb{T}^d)^2$ transverse to each other, such that the orbit closure is a disjoint union:

$$\overline{\{\alpha_\triangle^{\mathbf{n}}.\mathbf{y} : \mathbf{n} \in \mathbb{Z}^2\}} = \{\alpha_\triangle^{\mathbf{n}}.\mathbf{y} : \mathbf{n} \in \mathbb{Z}^2\} \sqcup \Big(\bigcup_{i=1}^{3} T_i\Big). \tag{1.2}$$

Theorem 1.6 is the main result of this paper. It actually classifies all topologically transitive self-joinings of the $\mathbb{Z}^r$-action α when $r \geq 3$. Recall that a *topological joining* between two group actions $\rho_k : G \curvearrowright \Omega_k, k = 1, 2$ is a subset $A \subset \Omega_1 \times \Omega_2$ which is invariant under the product action $\rho_1 \times \rho_2$ such that $\pi_k(A) = \Omega_k$ where π_k is the projection to Ω_k. Here $\rho_1 \times \rho_2$ is defined by $(\rho_1 \times \rho_2)(g).(\omega_1, \omega_2) = \big(\rho_1(g).\omega_1, \rho_2(g).\omega_2\big)$ (see [Gla03]).

We note that the analogous question in the measure theoretic category was studied by Kalinin and Katok [KK02]. Examples such as Maucourant's [Mau10] show that the measure theoretic and topological questions behave quite differently in this context.

Besides $\alpha_\triangle$-orbits of generic points, we study those of rational ones as well. Recall a subset of a metric space is said to be *ϵ-dense* if it intersects every open ball of radius ϵ.

Theorem 1.7. *Suppose $r \geq 3$ and α is a faithful $\mathbb{Z}^r$-Cartan action on $\mathbb{T}^d$ by automorphisms such that $\alpha^{\mathbf{n}}$ is a totally irreducible toral automorphism for at least one $\mathbf{n} \in \mathbb{Z}^r$. Then $\forall \epsilon > 0$, there exist a finite number of subsets $A_1, \cdots, A_s \subset \mathbb{T}^d$, each of which is a translate of a (possibly different) d-dimensional $\alpha_\triangle$-invariant subtorus in $(\mathbb{T}^d)^2$, such* *that $\bigcup_{t=1}^{s} A_t$ covers all rational points in $(\mathbb{T}^d)^2$ whose $\alpha_\triangle$-orbit is not ϵ-dense in $(\mathbb{T}^d)^2$.*

1.3. **Organization of the paper.** Section 2 gives some basic facts about Cartan actions by toral automorphisms. In particular, it will be established that the action $\alpha : \mathbb{Z}^r \curvearrowright \mathbb{T}^d$ is equivalent to an algebraic $\mathbb{Z}^r$-action ζ on a twisted torus X which arises from some number field K. And in the rest of the paper, instead of $\alpha_\triangle$, we will study the diagonal action $\zeta_\triangle : \mathbb{Z}^r \curvearrowright X^2$. In Section 3, by analyzing characters of X we describe homogeneous $\zeta_\triangle$-invariant subsets in X^2. Besides the 0-dimensional and $2d$-dimensional ones, which are respectively finite collections of torsion points and X^2 itself, the most interesting homogeneous invariant subsets are d-dimensional and each of them is a finite disjoint union of parallel translates of some d-dimensional invariant subtorus. Moreover such subtori are parametrized by their slopes, which are either algebraic numbers from the aforementioned number field K or ∞.

In Section 4, it is shown that any infinite $\zeta_\triangle$-invariant closed subset contains a d-dimensional homogeneous invariant set; which holds even when $r = 2$.

Section 5 establishes the rigidity of the diagonal action $\zeta_\triangle$ when the rank is at least 3. The strategy is that, once a topologically transitive infinite $\zeta_\triangle$-invariant closed subset A is known to contain a d-dimensional homogeneous invariant set as a proper subset, we can repeatedly add new d-dimensional homogeneous invariant subsets to it. But the union of infinitely many d-dimensional homogeneous invariant sets would become dense in X^2 so A must be X^2. Two main ingredients of this proof are respectively a self-returning argument (Lemma 5.6) that moves a point around while keeping it away from known homogeneous subsets; and an extension of Berend's Theorem to rigidity properties of non-hyperbolic abelian actions by toral automorphisms (Proposition 2.8, which is a special case of a more general fact proved in [Wan10b]). Using the same techniques, we also provide a proof of Theorem 1.7 in Section 5.

In Section 6, we construct a non-homogeneous orbit closure in the $r = 2$ case. It is well known that Theorem 1.1 fails for $r = 1$ in which case, for example, homoclinic points have non-homogeneous orbit closures. In [Mau10], Maucourant gave examples of higher rank algebraic diagonal actions which have orbits wandering back and forth between several homogeneous submanifolds. Our counterexample bears resemblance to these examples. We construct a point $\mathbf{x} = (x^{(1)}, x^{(2)}) \in X^2$

such that $x^{(1)}$, $x^{(2)}$ and $x^{(1)}+x^{(2)}$ lie respectively in one of three foliations through the origin of X along different common eigenspaces of the group action. So as elements of the group action all have determinant 1, when a large element from the action is applied, at least one of these three expressions is attracted towards 0.

2. A number-theoretical model of the group action

For future convenience, we translate the problem into a number-theoretical setting.

2.1. Cartan actions and groups of units in number fields.

Let K be a degree d number field with r_1 real embeddings and r_2 conjugate pairs of complex embeddings, where $r_1+2r_2=d$. Denote the real embeddings by $\sigma_1,\cdots,\sigma_{r-1}$ and the conjugate pairs of complex ones by $(\sigma_{r_1+1},\sigma_{r_1+r_2+1})$, $(\sigma_{r_1+2},\sigma_{r_1+r_2+2})$, $\cdots$, $(\sigma_{r_1+r_2},\sigma_{r_1+2r_2})$. Let U_K be the group of units, which is of rank r_1+r_2-1. Define $\sigma: K\mapsto\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$ by

$$\sigma:\theta\mapsto\big(\sigma_1(\theta),\cdots,\sigma_{r_1}(\theta),\sigma_{r_1+1}(\theta),\cdots,\sigma_{r_1+r_2}(\theta)\big). \tag{2.1}$$

then σ is an embedding between additive groups and induces an isomorphism $K\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$.

Write $\theta.(\mu\otimes x)=\theta\mu\otimes x,\forall\theta,\mu\in K,\forall x\in\mathbb{R}$; which gives a multiplicative action by K on $K\otimes_{\mathbb{Q}}\mathbb{R}$. If we identify $K\otimes_{\mathbb{Q}}\mathbb{R}$ with $\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$ via σ then this multiplicative action can be equivalently defined by:

$$\theta.(\tilde{x}_1,\cdots,\tilde{x}_{r_1+r_2})=\big(\sigma_1(\theta)\tilde{x}_1,\cdots,\sigma_{r_1+r_2}(\theta)\tilde{x}_{r_1+r_2}\big). \tag{2.2}$$

By construction this multiplication action is compatible with σ in the sense that

$$\theta.\sigma(\mu)=\sigma(\theta\mu),\forall\theta,\mu\in K. \tag{2.3}$$

Definition 2.1. *A number field K is* **CM** *if it is a totally complex quadratic extension of a totally real number field.*

The following correspondence was known. For a more general statement, we refer to [EL03, Prop. 2.1] and [Sch95, §7 & §29].

Proposition 2.2. *A group action $\alpha:\mathbb{Z}^r\curvearrowright\mathbb{T}^d$ is a faithful $\mathbb{Z}^r$-Cartan action by automorphisms such that $\alpha(\mathbb{Z}^r)$ contains a totally irreducible toral automorphism, if and only if there exist*

- *a common eigenbasis $\{w_1,\cdots,w_d\}$ in $\mathbb{C}^d$ with respect to which $\forall\mathbf{n}\in\mathbb{Z}^r$, $\alpha^{\mathbf{n}}$ can be diagonalized as $\operatorname{diag}(\zeta_1^{\mathbf{n}},\zeta_2^{\mathbf{n}},\cdots,\zeta_d^{\mathbf{n}})$ where $\zeta_1^{\mathbf{n}},\cdots,\zeta_{r_1}^{\mathbf{n}}\in\mathbb{R}$; $\zeta_{r_1+1}^{\mathbf{n}},\cdots,\zeta_d^{\mathbf{n}}\in\mathbb{C}$ and $\zeta_{r_1+j}^{\mathbf{n}}=\overline{\zeta_{r_1+r_2+j}^{\mathbf{n}}},\forall j=1,\cdots,r_2$;*

- *a non-CM number field K of degree d with r_1 real embeddings $\sigma_1\cdots,\sigma_{r_1}$ and r_2 conjugate pairs of complex embeddings $(\sigma_{r_1+1},\sigma_{r_1+r_2+1})$, $(\sigma_{r_1+2},\sigma_{r_1+r_2+2})$, $\cdots$, $(\sigma_{r_1+r_2},\sigma_d)$ where $r_1+r_2=r+1$, $r_1+2r_2=d$;*
- *a group embedding $\zeta:\mathbf{n}\mapsto\zeta^{\mathbf{n}}$ of $\mathbb{Z}^r$ into the group of units U_K;*
- *a subgroup $\Gamma<\sigma(K)$ which is a cocompact lattice in $\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$, where σ is given in (2.1);*

such that:

- $\zeta_i^{\mathbf{n}}=\sigma_i(\zeta^{\mathbf{n}}),\forall i\in\{1,\cdots,d\},\forall\mathbf{n}\in\mathbb{Z}^r$*;*
- *the image $\zeta(\mathbb{Z}^r)=\{\zeta^{\mathbf{n}}:\mathbf{n}\in\mathbb{Z}^r\}$ is a finite-index subgroup of U_K;*
- *$\forall\mathbf{n}\in\mathbb{Z}^r$, the multiplication (2.2) by $\zeta^{\mathbf{n}}$ on $\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$ preserves Γ, and therefore induces a multiplicative $\mathbb{Z}^r$-action ζ on $(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})/\Gamma$:*

$$\zeta^{\mathbf{n}}.(\tilde{x}\bmod\Gamma)=(\zeta^{\mathbf{n}}.\tilde{x}\bmod\Gamma),\forall\tilde{x}\in\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2},\forall\mathbf{n}\in\mathbb{Z}^r;\tag{2.4}$$

- *α is algebraically conjugate to the multiplicative action $\zeta:\mathbb{Z}^r\curvearrowright(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})/\Gamma$, i.e., $\alpha^{\mathbf{n}}.y=\psi^{-1}(\zeta^{\mathbf{n}}.\psi(y))$, $\forall\mathbf{n}\in\mathbb{Z}^r$, $\forall y\in\mathbb{T}^d$ for some continuous group isomorphism $\psi:\mathbb{T}^d\xrightarrow{\sim}(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})/\Gamma$.*

Proof. For a proof of the "only if" direction, see for example [Wan10a, Thm. 2.12]. We now show the "if" part.

Suppose K, Γ, ζ are as described in the statement of theorem and α is algebraically conjugate to the multiplicative action ζ, we want to show α is a faithful $\mathbb{Z}^r$-Cartan action by automorphisms and contains a totally irreducible element.

First of all, notice the multiplication by any $\zeta^{\mathbf{n}}$ preserves the additive structure on $(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})/\Gamma$. As α is conjugate with the action $\zeta:\mathbb{Z}^r\curvearrowright(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})/\Gamma$ via a continuous group isomorphism ψ, the action $\alpha:\mathbb{Z}^r\curvearrowright\mathbb{T}^d$ is continuous and preserves the additive group structure on $\mathbb{T}^d$, hence $\alpha^{\mathbf{n}}\in\mathrm{Aut}(\mathbb{T}^d)=\mathrm{SL}(d,\mathbb{Z})$ for all $\mathbf{n}$.

Moreover to see α is faithful it suffices to show so is ζ. Suppose this is not true, then the multiplication by $\zeta^{\mathbf{n}}$ on $(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})/\Gamma$ is the identity map for some $\mathbf{n}\neq 0$, Then $\zeta^{\mathbf{n}}.\tilde{x}\equiv\tilde{x}\bmod\Gamma$ for all $\tilde{x}\in\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$, i.e. the map $\delta:\tilde{x}\mapsto\zeta^{\mathbf{n}}.\tilde{x}-\tilde{x}$ sends $\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$ into Γ. As Γ is discrete, $\delta(\tilde{x})\equiv 0$, or equivalently $\zeta^{\mathbf{n}}.\tilde{x}=\tilde{x},\forall\tilde{x}\in\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$. But this can happen only if for any $i\in I$, $\sigma_i(\zeta^{\mathbf{n}})=\zeta_i^{\mathbf{n}}=1$, so $\zeta^{\mathbf{n}}=1$. This contradicts the assumption that $\zeta:\mathbb{Z}^r\hookrightarrow U_K$ is a group embedding. Thus α is faithful.

Next we show $\zeta^{\mathbf{n}_0}$ is totally irreducible for some $\mathbf{n}_0\in\mathbb{Z}^r$. As K is not CM, it is a corollary to Dirichlet's Unit Theorem that for any proper subfield F in K, $\mathrm{rank}(U_F)<\mathrm{rank}(U_K)$ (cf. [Par75]). As $\zeta(\mathbb{Z}^r)$ is of finite index in U_K, for all subfield F the group $H_F = \{\mathbf{n} \in \mathbb{Z}^r : \zeta^{\mathbf{n}} \in U_F\}$ has rank strictly less than r and $\zeta(H_F)$ is of finite index in U_F. Embed $\mathbb{Z}^r$ in $\mathbb{R}^r$ and let P_F be the $\mathbb{R}$-span of H_F, then $\dim P_F < r$. Since there are only finitely many subfields, $\bigcup_F P_F \subsetneq \mathbb{R}^r$. It follows that $\exists \mathbf{n}_0 \in \mathbb{Z}^r$ such that $\mathbf{n}_0 \notin \bigcup_F P_F$. In particular for any non-zero integer l, $l\mathbf{n}_0$ doesn't belong to $\bigcup_F H_F$. So $\zeta^{l\mathbf{n}_0} \in U_K$ is not in any proper subfield of K and is therefore an algebraic number of degree d. Thus the eigenvalues $\zeta_1^{l\mathbf{n}_0}, \cdots, \zeta_d^{l\mathbf{n}_0}$ of $\alpha^{l\mathbf{n}_0}$ are all degree d algebraic numbers and $\alpha^{l\mathbf{n}_0} = (\alpha^{\mathbf{n}_0})^l$ is an irreducible toral automorphism. As $l \neq 0$ is arbitrary, $\alpha^{\mathbf{n}_0}$ is totally irreducible.

It remains to show $\alpha : \mathbb{Z}^r \hookrightarrow \mathrm{SL}(d, \mathbb{Z})$ satisfies the Cartan condition in Definition 1.3. Assume not, then α can be extended to an embedding $\tilde{\alpha}$ of $\mathbb{Z}^{r+1} = \mathbb{Z}^r \oplus \mathbb{Z}$ into $\mathrm{SL}(d, \mathbb{Z})$, i.e. $\tilde{\alpha}|_{\mathbb{Z}^r} = \alpha$. The action $\tilde{\alpha} : \mathbb{Z}^{r+1} \curvearrowright \mathbb{T}^d$ is a faithful $\mathbb{Z}^r$-action and contains a totally irreducible automorphism. By the "only if" part, there is a common eigenbasis in $\mathbb{C}^d$ with respect to which all elements in $\tilde{\alpha}(\mathbb{Z}^{r+1})$ can be diagonalized simultaneously. Since we already proved $\alpha(\mathbb{Z}^r) < \tilde{\alpha}(\mathbb{Z}^{r+1})$ contains irreducible elements, this eigenbasis must be the same $\{w_1, \cdots, w_d\}$ associated to α by the condition in the proposition. For all $1 \leq i \leq d$ and $\mathbf{n} \in \mathbb{Z}^{r+1}$, denote by $\zeta_i^{\mathbf{n}}$ the i-th eigenvalue of $\tilde{\alpha}^{\mathbf{n}}$, which is compatible with previous notations in case that $\mathbf{n} \in \mathbb{Z}^r \subset \mathbb{Z}^{r+1}$. By [Wan10a, Thm 2.12], there is an embedding $\tilde{\zeta} : \mathbb{Z}^{r+1} \hookrightarrow U_{\tilde{K}}$ where $\tilde{K}$ is a degree d number field and the image has finite index; and $\zeta_i^{\mathbf{n}} = \tilde{\sigma}_i(\tilde{\zeta}^{\mathbf{n}}), \forall \mathbf{n} \in \mathbb{Z}^{r+1}$ where $\tilde{\sigma}_1, \cdots, \tilde{\sigma}_d$ are all the archimedean embeddings of $\tilde{K}$. Let $\mathbf{n}_0 \in \mathbb{Z}^r$ be such that $\alpha^{\mathbf{n}_0}$ is an irreducible toral automorphism. Then the eigenvalue $\zeta_1^{\mathbf{n}_0}$ is an algebraic number of degree d in both $\sigma_1(K)$ and $\tilde{\sigma}_1(\tilde{K})$. As both of them are embeddings of degree d number fields, $\sigma_1(K) = \tilde{\sigma}_1(\tilde{K}) = \mathbb{Q}[\zeta_1^{\mathbf{n}_0}]$. Hence the group of units $\sigma_1(U_K)$ and $\tilde{\sigma}_1(U_{\tilde{K}})$ are equal, in particular $\mathrm{rank}(U_K) = \mathrm{rank}(U_{\tilde{K}})$. But by previous discussion, $\mathrm{rank}(U_{\tilde{K}}) \geq \mathrm{rank}\left(\zeta(\mathbb{Z}^{r+1})\right) = r + 1$ and $\mathrm{rank}(U_K) = r$, a contradiction. This completes the proof. □

Remark 2.3. *The proof of the proposition actually shows that $\exists \mathbf{n} \in \mathbb{Z}^r$ such that $\zeta^{l\mathbf{n}}$ doesn't belong to any proper subfield of K for all nonzero integer l.*

In particular, Proposition 2.2 implies Cartan actions are indeed covered by Theorem 1.1 as a special case.

Lemma 2.4. *Suppose $r \geq 2$, α is a faithful $\mathbb{Z}^r$-Cartan action on $\mathbb{T}^d$ by automorphisms and $\alpha(\mathbb{Z}^r)$ contains a totally irreducible element, then α satisfies the conditions in Theorem 1.1.*

Proof. Conditions (1) and (2) in Theorem 1.1 are already assumed. It suffices to check (3). By Proposition 2.2, for the i-th common eigenvector v_i, $\frac{|\alpha^{\mathbf{n}}.v_i|}{|v_i|} = |\zeta_i^{\mathbf{n}}|$. By Dirichlet's unit theorem, for any index i, there is $u \in U_K$ whose i-th embedding $\sigma_i(u)$ has norm greater than 1. As $\zeta(\mathbb{Z}^r)$ is of finite index in U_K, there is a positive integer q such that $u^q \in \zeta(\mathbb{Z}^r)$. In other words, $\exists \mathbf{n} \in \mathbb{Z}^r$, $\zeta^{\mathbf{n}} = u^q$; hence $|\zeta_i^{\mathbf{n}}| = |\sigma_i(\zeta^{\mathbf{n}})| = |\sigma_i(u^q)| = |\sigma_i(u)|^q > 1$. This verifies condition (3) in Theorem 1.1. ☐

2.2. **Notations.** Throughout this paper let α be a $\mathbb{Z}^r$-Cartan action on $\mathbb{T}^d$ by automorphisms such that $\alpha(\mathbb{Z}^r)$ contains a totally irreducible element.

Let K, Γ, ψ, ζ and σ_i's be as in Proposition 2.2. And construct multiplications by elements of K on $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ as in (2.2).

Write $I = \{1, \cdots, r_1 + r_2\}$ and to each $i \in I$ associate a subspace V_i of $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ as follows: V_i is the i-th copy of $\mathbb{R}$ in $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ if $1 \le i \le r_1$; and for $r_1 < i \le r_1 + r_2$, V_i is the $(i - r_1)$-th copy of $\mathbb{C}$. So $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2} = \bigoplus_{i \in I} V_i$. Set $d_i = \dim_{\mathbb{R}} V_i$.

Denote

$$X = (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})/\Gamma = (\bigoplus_{i \in I} V_i)/\Gamma, \tag{2.5}$$

and let π be the canonical projection from $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ to X.

For $x \in X$, let $\|x\| = \min_{\substack{\tilde{x} \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2} \\ \pi(\tilde{x}) = x}} |\tilde{x}|$. Then $(x, x') \mapsto \|x - x'\|$ is a distance on x and this makes X a locally Euclidean metric space.

In this paper, for a point ω in a metric space, let $B_\epsilon(\omega)$ and $\mathring{B}_\epsilon(\omega)$ respectively denote the closed and open balls of radius ϵ around ω.

In order to characterize how the action ζ expands or contracts different V_i's, for all $i \in I$ we introduce a group morphism $\lambda_i : \mathbb{Z}^r \mapsto \mathbb{R}$ by

$$\lambda_i(\mathbf{n}) = \log |\zeta_i^{\mathbf{n}}|, \tag{2.6}$$

and construct a map $\mathcal{L} : \mathbb{Z}^r \mapsto \mathbb{R}^I$ by

$$\mathcal{L}(\mathbf{n}) = \big(\lambda_i(\mathbf{n})\big)_{i \in I}. \tag{2.7}$$

Then $\mathcal{L}$ is a group morphism and its image $\mathcal{L}(\mathbb{Z}^r)$ lies in the subspace

$$W = \{w \in \mathbb{R}^I : \sum_{i \in I} d_i w_i = 0\}. \tag{2.8}$$

This is because $\zeta_i^{\mathbf{n}} = \sigma_i(\zeta^{\mathbf{n}})$ where $\zeta^{\mathbf{n}} \in U_K$, and $\forall u \in U_K$,

$$\begin{aligned}\sum_{i\in I} d_i \log|\sigma_i(u)| &= \sum_{i=1}^{r_1} \log|\sigma_i(u)| + \sum_{i=r_1+1}^{r_1+r_2} 2\log|\sigma_i(u)| \\ &= \sum_{i=1}^{r_1} \log|\sigma_i(u)| + \sum_{i=r_1+1}^{r_1+r_2} (\log|\sigma_i(u)| + \log|\sigma_{i+r_2}(u)|) \\ &= \sum_{i=1}^{d} \log|\sigma_d(u)| = \log|\mathrm{N}_{K/\mathbb{Q}}(u)| = 0.\end{aligned} \tag{2.9}$$

Remark 2.5. *The set* $\left\{\left(\log|\sigma_i(u)|\right)_{i\in I} : u \in U_K\right\}$ *is a full-rank lattice in* W *by Dirichlet's unit theorem. Since* $\{\zeta^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^r\}$ *has finite index in* U_K *by Proposition 2.2, for any finite index subgroup* H *in* $\mathbb{Z}^r$*,* $\mathcal{L}(H)$ *is a finite-index subgroup in the above lattice, and thus is itself a full-rank lattice in* W*.*

Write $X^2 = X \oplus X$, then

$$X^2 = (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2/\Gamma^2. \tag{2.10}$$

Denote by $\pi_\triangle$ the quotient map and equip X^2 with a distance given by $\|\mathbf{x}\| = \min_{\substack{\tilde{\mathbf{x}}\in(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})^2 \\ \pi_\triangle(\tilde{\mathbf{x}})=\mathbf{x}}} |\tilde{\mathbf{x}}|$. Then $\|\mathbf{x}\| = (\|x^{(1)}\|^2 + \|x^{(2)}\|^2)^{\frac{1}{2}}$ if $\mathbf{x}$ writes $(x^{(1)}, x^{(2)})$ where $x^{(1)}, x^{(2)} \in X$.

For $\tilde{x} = (x_1, \cdots, x_{r_1+r_2}) \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$, because the i-th coordinate of $\zeta^{\mathbf{n}}.\tilde{x}$ is $\zeta_i^{\mathbf{n}} x_i$ we have $|\zeta^{\mathbf{n}}.\tilde{x}| \leq \max_{i\in I} |\zeta_i^{\mathbf{n}}| \cdot |\tilde{x}| = e^{\max_{i\in I} \lambda_i(\mathbf{n})}|\tilde{x}|$. And it easily follows that

$$\|\zeta^{\mathbf{n}}.x\| \leq e^{\max_{i\in I}\lambda_i(\mathbf{n})}\|x\|, \forall x \in X, \forall \mathbf{n} \in \mathbb{Z}^r; \tag{2.11}$$

and

$$\|\zeta_\triangle^{\mathbf{n}}.\mathbf{x}\| \leq e^{\max_{i\in I}\lambda_i(\mathbf{n})}\|\mathbf{x}\|, \forall \mathbf{x} \in X^2, \forall \mathbf{n} \in \mathbb{Z}^r. \tag{2.12}$$

Now we set

$$\zeta_\triangle^{\mathbf{n}}.(x^{(1)}, x^{(2)}) = (\zeta^{\mathbf{n}}.x^{(1)}, \zeta^{\mathbf{n}}.x^{(2)}), \forall x^{(1)}, x^{(2)} \in X, \forall \mathbf{n} \in \mathbb{Z}^r, \tag{2.13}$$

where $\zeta^{\mathbf{n}}.x^{(1)}$, $\zeta^{\mathbf{n}}.x^{(2)}$ are defined in (2.4). Since the multiplication by different $\zeta^{\mathbf{n}}$'s commute, this yields a diagonal $\mathbb{Z}^r$-action $\zeta_\triangle$ on X^2.

For a subset $A \subset X^2$, define its stabilizer by

$$\mathrm{Stab}_{\zeta_\triangle}(A) = \{\mathbf{n} \in \mathbb{Z}^r : \zeta_\triangle^{\mathbf{n}}.A \subset A\}. \tag{2.14}$$

$(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2 \cong \mathbb{R}^{2d}$ decomposes as $\bigoplus_{i\in I, k=1,2} V_i^{(k)}$ where $V_i^{(k)}$ is the i-th component V_i in the k-th copy of $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2} = \bigoplus_{i\in I} V_i$. Write

$$V_i^\square = V_i^{(1)} \oplus V_i^{(2)}, \tag{2.15}$$

then

$$(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2 = \oplus V_i^{\square}. \tag{2.16}$$

Similar to (2.13), define a multiplicative $\mathbb{Z}^r$-action on $(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2$ by

$$\zeta_\triangle^{\mathbf{n}}.(\tilde{x}^{(1)}, \tilde{x}^{(2)}) = (\zeta^{\mathbf{n}}.\tilde{x}^{(1)}, \zeta^{\mathbf{n}}.\tilde{x}^{(2)}), \forall \tilde{x}^{(1)}, \tilde{x}^{(2)} \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}, \tag{2.17}$$

which actually projects to (2.17) via the quotient map $\pi_\triangle$.

Notice $V_i^{\square}$ is isomorphic to either $\mathbb{R}^2$ or $\mathbb{C}^2$ and under this identification we have

$$\zeta_\triangle^{\mathbf{n}}.\mathbf{v} = \zeta_i^{\mathbf{n}}\mathbf{v}, \forall \mathbf{v} \in V_i^{\square}, \forall \mathbf{n} \in \mathbb{Z}^r. \tag{2.18}$$

Construct a continuous group morphism $\psi_\triangle : (\mathbb{T}^d)^2 \xrightarrow{\sim} X^2$ by

$$\psi_\triangle(y^{(1)}, y^{(2)}) = \big(\psi(y^{(1)}), \psi(y^{(2)})\big). \tag{2.19}$$

.

It follows from Proposition 2.2 that the action $\alpha_\triangle : \mathbb{Z}^r \curvearrowright (\mathbb{T}^d)^2$ is algebraically conjugate to the multiplicative action $\zeta_\triangle : \mathbb{Z}^r \curvearrowright X^2$ in (2.13) via $\psi_\triangle$:

$$\alpha_\triangle^{\mathbf{n}}.\mathbf{y} = \psi_\triangle^{-1}\big(\zeta_\triangle^{\mathbf{n}}.\psi_\triangle(\mathbf{y})\big), \forall \mathbf{y} \in (\mathbb{T}^d)^2, \forall \mathbf{n} \in \mathbb{Z}^r. \tag{2.20}$$

2.3. **Rigidity results in X.** We now translate the special case of Theorem 1.1 regarding Cartan actions into the X setting.

Remark 2.6. *As ψ is a group isomorphism, $y \in \mathbb{T}^d$ is a rational point if and only if $\psi(y)$ is a torsion point in the compact abelian group X. In consequence, $\mathbf{y} \in (\mathbb{T}^d)^2$ is rational if and only if $\psi_\triangle(\mathbf{y})$ is a torsion point in X^2.*

Theorem 2.7. *Let X and ζ be as above, then the only infinite ζ-invariant closed subset of X is X itself. In fact, for all $x \in X$ and any finite index subgroup H of $\mathbb{Z}^r$, $\{\zeta^{\mathbf{n}}.x : \mathbf{n} \in H\}$ is dense in X unless x is a torsion point.*

Proof. Proposition 2.2 established a conjugation between the $\mathbb{Z}^r$-actions ζ and α. Thus by Lemma 2.4, we may apply Theorem 1.1 to ζ, this proves the theorem for $H = \mathbb{Z}^r$ where we used Remark 2.6. The statement extends to any finite index subgroup H by Remark 1.2. □

Actually, when $r \geq 3$, for a generic x there is no need to apply the full action by H to get a dense orbit in X. The next proposition is a special case from [Wan10b, Theorem 1.7].

Proposition 2.8. *In the setting of Theorem 2.7, if $r \geq 3$ then $\forall i \in I$, $\forall \epsilon > 0$, $\forall x \in X$, the set $\{\zeta^{\mathbf{n}}.x : \mathbf{n} \in H, |\lambda_i(\mathbf{n})| < \epsilon\}$ is dense in X unless x can be written as $x_0 + v$ where x_0 is a torsion point in X and $v \in V_i$.*

Proof. To avoid technical details, instead of explaining settings and notations from [Wan10b], we simply notify the reader that in order to verify the proposition is indeed a special case of Theorem 1.7 from that paper it suffices to check the collection of conditions (C1)-(C3) listed below. To begin with, remark for all $j \in I$, $\lambda_j : \mathbb{Z}^r \mapsto \mathbb{R}$ can be uniquely extended to a linear map from $\mathbb{R}^r$ to $\mathbb{R}$, still denoted by λ_j.

(C1) *Suppose $L_i \subset (\mathbb{R}^r)^*$ is the subspace of linear maps spanned by λ_i, then $\lambda_j \notin L_i$ for $j \neq i$;*
(C2) *$r \geq \dim L_i + 2$;*
(C3) *$\forall \epsilon > 0$, $\exists \mathbf{n} \in H$ such that $|\lambda_i(\mathbf{n})| < \epsilon$ and $\zeta^{\mathbf{n}} \in K$ is a totally irreducible element, i.e. $\zeta^{l\mathbf{n}}$ doesn't belong to any proper subfield of K for all non-zero integer l.*

Proof of (C1). Suppose (C1) fails then $\exists j \in I \backslash \{i\}$ such that $\lambda_j(\mathbf{n}) = c\lambda_i(\mathbf{n}), \forall \mathbf{n}$ for some constant c. And this would imply the image $\mathcal{L}(\mathbb{Z}^r)$ lies in $\{(w_i)_{i \in I} \in W : w_j = cw_i\}$, which is a proper linear subspace of W. This contradicts the fact that $\mathcal{L}(\mathbb{Z}^r)$ is a full-rank lattice in W. So (C1) is verified.

Proof of (C2). $r \geq 3$ but $\dim L_i = 1$.

Proof of (C3). Let $P_i = \{\eta \in \mathbb{R}^r : \lambda_i(\eta) = 0\}$, then $\dim P_i = r - 1$ because $\lambda_i \in (\mathbb{R}^r)^*$ is nonzero. In fact, if $\lambda_i \equiv 0$ then the image $\mathcal{L}(\mathbb{Z}^r)$ lies in the proper subspace $\{w \in \mathbb{R}^I : \sum_{j \in I} d_j w_j = 0, w_i = 0\}$ of W, which is a contradiction to Remark 2.5.

For any proper subfield F of K, let $H_F = \{\mathbf{n} \in \mathbb{Z}^r : \zeta^{\mathbf{n}} \in U_F\}$, which is a subgroup in $\mathbb{Z}^r \subset \mathbb{R}^r$, and call by P_F its linear span in $\mathbb{R}^r$. Following the proof of Proposition 2.2, we know $\dim P_F = \operatorname{rank}(H_F) = \operatorname{rank}(U_F) < r$. In particular $\zeta(H_F)$ has finite index in U_F.

We hope to show $P_i \not\subset P_F$. As $\dim P_i = r - 1 \geq \dim P_F$, it suffices to show $P_i \neq P_F$. Suppose for the moment that $P_i = P_F$, then $\forall \mathbf{n} \in H_F$, $\lambda_i(\mathbf{n}) = 0$ or equivalently $\sigma_i(\zeta^{\mathbf{n}}) = \zeta_i^{\mathbf{n}}$ has absolute value 1. Because $\zeta(H_F)$ has finite index in U_F, it follows that $|\sigma_i(u)| = 1$, $\forall u \in U_F$. Let $\tau = \sigma_i|_F$, then τ is an archimedean embedding of F and the above property rewrites $|\tau(u)| = 1$, $\forall u \in U_F$, which contradicts Dirichlet's Unit Theorem for F. Therefore $P_i \not\subset P_F$ and thus $P_i \cap P_F$ is a proper subspace in P_i.

In the rest of proof let F run over all proper subfields F of K, as there are only finitely many such, $\bigcup_F (P_i \cap P_F) \subsetneq P_i$ or

$$P_i \not\subset \bigcup_F P_F. \tag{2.21}$$

Take an arbitrary non-zero vector $\eta_0 \in P_i \backslash \big(\bigcup_F P_F\big)$. As $\bigcup_F P_F$ is closed, $\exists \delta > 0$ such that

$$B_\delta(\eta_0) \cap \big(\bigcup_F P_F\big) = \emptyset. \tag{2.22}$$

In particular, $\delta < |\eta_0|$.

For any $\eta \in \mathbb{R}^r$, let $\bar{\eta}$ denote its projection in the quotient $\mathbb{T}^r = \mathbb{R}^r/\mathbb{Z}^r$. By compactness of $\mathbb{T}^r$, there is an increasing sequence of positive integers $m_1, m_2, \cdots$ such that $\{\overline{m_k\eta_0}\}_{k=1}^\infty$ converges. In particular, $\exists m, m' \in \mathbb{N}$ such that

$$m - m' \geq \frac{\epsilon}{|\mathbb{Z}^r/H| \cdot \|\lambda_i\| \delta} \tag{2.23}$$

and the distance between $\overline{m\eta_0}$ and $\overline{m'\eta_0}$ is less than $\frac{\epsilon}{|\mathbb{Z}^r/H| \cdot \|\lambda_i\|}$, where $\|\lambda_i\|$ denotes the norm of the linear functional λ_i. This means $\exists \mathbf{n}_0 \in \mathbb{Z}^r$ such that

$$|(m - m')\eta_0 - \mathbf{n}_0| < \frac{\epsilon}{|\mathbb{Z}^r/H| \cdot \|\lambda_i\|}. \tag{2.24}$$

As $\delta < |\eta_0|$, by comparing (2.23) with (2.24) we see $\mathbf{n}_0 \neq 0$. Using pigeonhole principle one can easily check $\exists p \in \{1, \cdots, |\mathbb{Z}^r/H|\}$ such that $\mathbf{n} = p\mathbf{n}_0$ is a non-trivial element in H.

Because $\eta_0 \in P_i$, $\lambda_i\big((m - m')\eta_0\big) = 0$ and (2.24) implies

$$|\lambda_i(\mathbf{n}_0)| \leq \|\lambda_i\| \cdot \frac{\epsilon}{|\mathbb{Z}^r/H| \cdot \|\lambda_i\|} = \frac{\epsilon}{|\mathbb{Z}^r/H|}. \tag{2.25}$$

Therefore

$$|\lambda_i(\mathbf{n})| = p|\lambda_i(\mathbf{n}_0)| \leq |\mathbb{Z}^r/H| \cdot |\lambda_i(\mathbf{n}_0)| < \epsilon. \tag{2.26}$$

On the other hand, remark

$$\begin{aligned} \Big|\frac{\mathbf{n}_0}{m - m'} - \eta_0\Big| =& \frac{|\mathbf{n}_0 - (m - m')\eta_0|}{m - m'} \\ <& \frac{\epsilon}{|\mathbb{Z}^r/H| \cdot \|\lambda_i\|} \Big/ \frac{\epsilon}{|\mathbb{Z}^r/H| \cdot \|\lambda_i\| \delta} \\ =& \delta. \end{aligned} \tag{2.27}$$

Thus by (2.22), $\frac{\mathbf{n}_0}{m-m'} \notin \bigcup_F P_F$. So for any $l \in \mathbb{Z}\backslash\{0\}$, since $l\mathbf{n} = lp\mathbf{n}_0$ is proportional to $\frac{\mathbf{n}_0}{m-m'}$, it doesn't belong to $\bigcup_F P_F$; and thus by construction of P_F, $\zeta^{l\mathbf{n}}$ is not in any proper subfield F of K. This completes the verification of condition (C3) and finally establishes the proposition. □

2.4. **Characters in the new setting.** Now we study characters of X^2. Observe the Pontryagin dual group $\widehat{\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}}$ of $\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$ is isomorphic to $\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$ itself, where the duality relation is given by

$$\langle\xi,\tilde{x}\rangle=\Big(\sum_{i=1}^{r_1}\xi_i\tilde{x}_i+\sum_{i=r_1+1}^{r_1+r_2}2\,\mathrm{Re}(\xi_i\tilde{x}_i)\bmod\mathbb{Z}\Big),\tag{2.28}$$

$\forall\xi,\tilde{x}\in\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$. Hence there is a self-duality of $(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})^2$ given by

$$\begin{aligned}\langle\boldsymbol{\xi},\tilde{\mathbf{x}}\rangle=&\langle\xi^{(1)},\tilde{x}^{(1)}\rangle+\langle\xi^{(2)},\tilde{x}^{(2)}\rangle\\=&\Big(\sum_{i=1}^{r_1}\sum_{k=1,2}\xi_i^{(k)}\tilde{x}_i^{(k)}+\sum_{i=r_1+1}^{r_1+r_2}\sum_{k=1,2}2\,\mathrm{Re}(\xi_i^{(k)}\tilde{x}_i^{(k)})\bmod\mathbb{Z}\Big)\end{aligned}\tag{2.29}$$

for all $\boldsymbol{\xi}=(\xi^{(1)},\xi^{(2)})$ and $\tilde{\mathbf{x}}=(\tilde{x}^{(1)},\tilde{x}^{(2)})$ from $(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})^2$.

The Pontryagin dual of $X=\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}/\Gamma$ is the subgroup

$$\widehat{X}=\{\xi\subset\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}:\langle\xi,\gamma\rangle=0(\mathrm{mod}\ \mathbb{Z}),\forall\gamma\in\Gamma\}\tag{2.30}$$

with the duality relation

$$\langle\xi,x\rangle=\langle\xi,\tilde{x}\rangle\text{ if }x=\pi(\tilde{x}),\xi\in\hat{X},x\in X,\tilde{x}\in\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}.\tag{2.31}$$

And for X^2,

$$\widehat{X^2}=\{\boldsymbol{\xi}\in(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})^2:\langle\boldsymbol{\xi},\boldsymbol{\gamma}\rangle=0(\mathrm{mod}\ \mathbb{Z}),\forall\boldsymbol{\gamma}\in\Gamma^2\}=(\hat{X})^2,\tag{2.32}$$

where similarly,

$$\langle\boldsymbol{\xi},\mathbf{x}\rangle=\langle\boldsymbol{\xi},\tilde{\mathbf{x}}\rangle\text{ if }\mathbf{x}=\pi_\triangle(\tilde{\mathbf{x}}),\boldsymbol{\xi}\in\widehat{X^2},\mathbf{x}\in X^2,\tilde{\mathbf{x}}\in(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})^2.\tag{2.33}$$

The $\mathbb{Z}^r$-actions ζ and $\zeta_\triangle$ respectively induce dual actions on $\hat{X}$ and on $\widehat{X^2}$ in natural ways. To understand the dual actions, notice that

$$\langle\xi,\theta.\tilde{x}\rangle=\langle\theta.\xi,\tilde{x}\rangle,\forall\theta\in K,\forall\xi,\tilde{x}\in\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2},\tag{2.34}$$

where both multiplications by t are given by (2.2).

In particular, when $t=\zeta^{\mathbf{n}}\in U_K$ this establishes a duality

$$\langle\zeta^{\mathbf{n}}.\xi,\tilde{x}\rangle=\langle\xi,\zeta^{\mathbf{n}}.\tilde{x}\rangle.\tag{2.35}$$

Since ζ preserves the lattice Γ, by (2.30) it follows that $\hat{X}\subset\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$ is invariant under $\zeta^{\mathbf{n}}$ for all $\mathbf{n}\in\mathbb{Z}^r$. This yields a multiplicative action by $\mathbb{Z}^r$ on $\hat{X}$ which is the dual action of $\zeta:\mathbb{Z}^r\curvearrowright X$ in the sense that

$$\langle\zeta^{\mathbf{n}}.\xi,x\rangle=\langle\xi,\zeta^{\mathbf{n}}.x\rangle,\forall\xi\in\hat{X},\forall x\in X,\forall\mathbf{n}\in\mathbb{Z}^r.\tag{2.36}$$

In addition, $\widehat{X^2} = (\hat{X})^2 \subset (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2$ is stable under the action $\zeta_\triangle$ and $\langle \zeta_\triangle^{\mathbf{n}}.\boldsymbol{\xi}, \mathbf{x}\rangle = \langle \boldsymbol{\xi}, \zeta_\triangle^{\mathbf{n}}.\mathbf{x}\rangle$, $\forall \boldsymbol{\xi} \in \widehat{X^2}, \forall \mathbf{x} \in X^2, \forall n \in \mathbb{Z}^r$.

Before finishing this section we show an important property of the dual groups, namely every character on X arises from an algebraic number in K.

Proposition 2.9. *In the identification (2.30), $\hat{X}$ is actually contained in $\sigma(K) \subset \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$; namely $\forall \xi \in \hat{X}$, $\exists \theta \in K$ such that $\xi = \sigma(\theta)$.*

Proof. Fix $\xi \in \hat{X}$. By (2.28) and (2.30), $\forall \gamma \in \Gamma \subset \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$, $\sum_{i=1}^{r_1} \xi_i\gamma_i + \sum_{i=r_1+1}^{r_1+r_2} 2\operatorname{Re}(\xi_i\gamma_i) \in \mathbb{Z}$. Since Γ is a lattice of rank d and lies in the d-dimensional $\mathbb{Q}$-vector space $\sigma(K)$, its $\mathbb{Q}$-span is $\sigma(K)$. In consequence, $\forall \tilde{x} \in \sigma(K)$, $\sum_{i=1}^{r_1} \xi_i\tilde{x}_i + \sum_{i=r_1+1}^{r_1+r_2} 2\operatorname{Re}(\xi_i\tilde{x}_i) \in \mathbb{Q}$. Equivalently, if we denote $\xi_{r_1+r_2+j} = \xi_{r_1+j} \in \mathbb{C}$ for $j = 1, \cdots, r_2$ then $\forall \theta \in K$,

$$\begin{aligned}
\sum_{i=1}^{d} \xi_i\sigma_i(\theta) &= \sum_{i=1}^{r_1} \xi_i\sigma_i(\theta) + \sum_{i=r_1+1}^{r_1+r_2} (\xi_i\sigma_i(\theta) + \xi_{i+r_2}\sigma_{i+r_2}(\theta)) \\
&= \sum_{i=1}^{r_1} \xi_i\sigma_i(\theta) + \sum_{i=r_1+1}^{r_1+r_2} (\xi_i\sigma_i(\theta) + \overline{\xi_i} \cdot \overline{\sigma_i(\theta)}) \\
&= \sum_{i=1}^{r_1} \xi_i\sigma_i(\theta) + \sum_{i=r_1+1}^{r_1+r_2} 2\operatorname{Re}(\xi_i\sigma_i(\theta)) \in \mathbb{Q}.
\end{aligned} \tag{2.37}$$

So $(\xi_1, \cdots, \xi_d)$ is inside

$$F = \{(\xi_1, \cdots, \xi_d) \in \mathbb{C}^d : \sum_{i=1}^{d} \xi_i\sigma_i(\theta) \in \mathbb{Q}, \forall \theta \in K\}. \tag{2.38}$$

First of all, take a $\mathbb{Z}$-basis $\{\theta_1, \cdots, \theta_k\}$ of $\mathcal{O}_K$. Then $\forall \tilde{\xi} \in F$, $A\tilde{\xi} \in \mathbb{Q}^d$ where A is the $d \times d$ matrix $\big(\sigma_i(\theta_j)\big)_{i,j=1}^{d}$. Recall A is invertible (actually $(\det A)^2$ is the discriminant of K). Hence F is contained in $F^+ = A^{-1}.\mathbb{Q}^d$ which is a $\mathbb{Q}$-vector space of dimension d.

On the other hand let

$$F^- = \Big\{\big(\sigma_1(\theta), \cdots, \sigma_d(\theta)\big) : \theta \in K\Big\}. \tag{2.39}$$

Then it is easy to check F^- is also a d-dimensional $\mathbb{Q}$-vector space. Moreover, it follows from

$$\sum_{i=1}^{d} \sigma_i(\theta)\sigma_i(\mu) = \operatorname{Tr}_{K/\mathbb{Q}}(\theta\mu) \in \mathbb{Q}, \forall \theta, \mu \in K, \tag{2.40}$$

that $F^- \subset F \subset F$. As F^- and F^+ have the same dimension, we must have $F^- = F = F^+$. In particular, by (2.39) for any $(\xi_1, \cdots, \xi_d) \in F$, $\exists \theta \in K$ such that $\xi_i = \sigma_i(\theta)$ for all i, which implies the proposition. $\square$

3. Description of homogeneous invariant subsets

Three types of homogeneous $\alpha_\triangle$-invariant closed sets in $(\mathbb{T}^d)^2$ were described in the discussion preceding Theorem 1.6. In this part, we are going to discuss their counterparts in X^2.

Under the map $\psi_\triangle$, finite sets of rational points correspond to finite sets of torsion points by Remark 2.6, and $(\mathbb{T}^d)^2$ just becomes X^2.

Remark 3.1. *The orbit of any torsion point $\mathbf{z}$ of X^2 under the $\mathbb{Z}^r$-action $\zeta_\triangle$ is finite. This is because $\forall \mathbf{n}$, $\zeta_\triangle^{\mathbf{n}}.\mathbf{z}$ is a torsion point of the same order, and there are only finitely many such points in X^2.*

It remains to describe d-dimensional homogeneous invariant subsets in X^2.

3.1. Homogeneous invariant subtori in X^2.

0 and X^2 are respectively the only 0-dimensional and $2d$-dimensional subtori in X^2 and they are clearly $\zeta_\triangle$-invariant. We now construct d-dimensional $\zeta_\triangle$-invariant subtori.

Definition 3.2. *For any $\kappa \in K$, inside $\widehat{X^2}$ we construct a subgroup*

$$(\widehat{X^2})^\kappa = \{(\xi^{(1)}, \xi^{(2)}) \in \widehat{X^2},\ \xi^{(1)} + \kappa.\xi^{(2)} = 0\}, \tag{3.1}$$

where the expression $\xi^{(1)} + \kappa.\xi^{(2)}$ is calculated as an element in $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$. For $\kappa = \infty$, set

$$(\widehat{X^2})^\infty = \{(\xi^{(1)}, 0) : \xi^{(1)} \in \hat{X}\}. \tag{3.2}$$

*A d-**dimensional homogeneous $\zeta_\triangle$-invariant subtorus** is a subgroup in X^2 of the form*

$$T^\kappa = \big((\widehat{X^2})^\kappa\big)^\perp = \{\mathbf{x} \in X^2 : \langle \boldsymbol{\xi}, \mathbf{x} \rangle = 0 \ mod\ \mathbb{Z}, \forall \boldsymbol{\xi} \in (\widehat{X^2})^\kappa\} \tag{3.3}$$

where $\kappa \in K \cup \{\infty\}$.

Remark 3.3. *By exchanging coordinates $x^{(1)}$ and $x^{(2)}$, $(\widehat{X^2})^\infty$ becomes $(\widehat{X^2})^0$ and T^∞ becomes T^0.*

Now we justify that the subgroup T^κ is indeed a subtorus.

Lemma 3.4. $\forall \kappa \in K \cup \{\infty\}$, *$T^\kappa$ is a d-dimensional subtorus in X^2.*

Proof. T^κ is a closed subgroup in X^2. As $T^\kappa = \left((\widehat{X^2})^\kappa\right)^\perp$, its Pontryagin dual is the quotient $\widehat{X^2}/(\widehat{X^2})^\kappa$. In order to show $T^\kappa \cong \mathbb{T}^d$ it suffices to show

$$\widehat{X^2}/(\widehat{X^2})^\kappa \cong \mathbb{Z}^d. \tag{3.4}$$

As $X \cong \mathbb{T}^d$, $\hat{X} \cong \mathbb{Z}^d$ and by (2.30) it is a closed subgroup in $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$. Thus $\hat{X}$ is a full-rank lattice in the vector space $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ and so is $\widehat{X^2} = (\hat{X})^2$ in $(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2 \cong (\widehat{\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}})^2 \cong \mathbb{R}^{2d}$. When $\kappa = \infty$, (3.4) follows directly from construction of $(\widehat{X^2})^\infty$. Assume from now on $\kappa \in K$, then it is easy to verify that

$$F^\kappa = \{(\xi^{(1)}, \xi^{(2)}) \in (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2 : \xi^{(1)} + \kappa.\xi^{(2)} = 0\} \tag{3.5}$$

is a d-dimensional linear subspace of $(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2$. Moreover $(\widehat{X^2})^\kappa = \widehat{X^2} \cap F^\kappa$, hence is a sublattice of rank at most d. Notice the subset $\{(-\kappa.\xi, \xi) : \xi \in \hat{X}\} \subset F^\kappa$ is a lattice of rank d.

In order to show $(\widehat{X^2})^\kappa$ has rank d, it suffices to show that $\forall \xi \in \hat{X}$, $\exists q \in \mathbb{N}$ such that $q(-\kappa.\xi, \xi) \in \widehat{X^2}$ or equivalently $q\kappa.\xi \in \hat{X}$, which is furthermore equivalent to

$$\langle \xi, q\kappa.\gamma \rangle = \langle q\kappa.\xi, \gamma \rangle = 0 \bmod \mathbb{Z}, \forall \gamma \in \Gamma \tag{3.6}$$

by (2.30). Since ξ is in $\hat{X}$ itself, (3.6) would follow if we could find $q \in \mathbb{N}$ such that $q\kappa.\gamma \in \Gamma, \forall \gamma \in \Gamma$. Take a $\mathbb{Z}$-basis $\{\gamma^1, \cdots, \gamma^d\}$ of Γ, it suffices to find q such that $q\kappa.\gamma^j \in \Gamma, \forall j = 1, \cdots, d$. By Proposition 2.2 $\Gamma \subset \sigma(K)$, and thus $\kappa.\gamma^j \in \sigma(\kappa.K) = \sigma(K)$ by (2.3). Recall $\sigma(K)$ is a $\mathbb{Q}$-vector space which contains Γ as a lattice of full rank. Thus for each j, there is $q_j \in \mathbb{N}$ such that $q_j\kappa.\gamma^j \in \Gamma$. Then $q = \gcd(q_1, \cdots, q_d)$ satisfies (3.6). This shows $(\widehat{X^2})^\kappa$ is of rank d.

Therefore $\operatorname{rank}\left(\widehat{X^2}/(\widehat{X^2})^\kappa\right) = 2d - d = d$. In order to establish (3.4) it remains to prove $\widehat{X^2}/(\widehat{X^2})^\kappa$ is torsion-free. In fact, assume $\widehat{X^2}/(\widehat{X^2})^\kappa$ contains a torsion element, then there exist $\boldsymbol{\xi} \in \widehat{X^2}$ and a non-zero integer n such that $\boldsymbol{\xi} \notin (\widehat{X^2})^\kappa$ but $n\boldsymbol{\xi} \in (\widehat{X^2})^\kappa$. Because $(\widehat{X^2})^\kappa = \widehat{X^2} \cap F^\kappa$, this is equivalent to $\boldsymbol{\xi} \notin F^\kappa$, $n\boldsymbol{\xi} \in F^\kappa$, which cannot be true since F^κ is a vector space. Hence we proved (3.4) and in consequence T^κ is isomorphic to $\mathbb{T}^d$ as a topological subgroup in X^2 for all $\kappa \in K \cup \{\infty\}$. ☐

Corollary 3.5. *Suppose $\kappa, \kappa' \in K \cup \{\infty\}$, $\kappa \neq \kappa'$ and $\mathbf{x}, \mathbf{x}' \in X^2$, then $(\mathbf{x} + T^\kappa) \cap (\mathbf{x}' + T^{\kappa'})$ is a finite set.*

Proof. Suppose the intersection is non-empty and let $\mathbf{y}$ be a point from it. Then $\forall \mathbf{y}' \in (\mathbf{x}+T^\kappa)\cap(\mathbf{x}'+T^{\kappa'})$, $\mathbf{y}'-\mathbf{y} \in T^\kappa \cap T^{\kappa'}$. Hence it suffices to show $T^\kappa \cap T^{\kappa'}$ is finite.

Note because $T^\kappa \cap T^{\kappa'}$ is a closed subgroup of X^2, it is enough to prove the annihilator $(T^\kappa \cap T^{\kappa'})^\perp$ has finite index in $\widehat{X^2}$. Notice that this annihilator contains both $(\widehat{X^2})^\kappa$ and $(\widehat{X^2})^{\kappa'}$. It was proved in the proof of the previous lemma that $(\widehat{X^2})^\kappa, (\widehat{X^2})^{\kappa'} \cong \mathbb{Z}^d$. Therefore it suffices to show $(\widehat{X^2})^\kappa \cap (\widehat{X^2})^{\kappa'} = \{\mathbf{0}\}$. Actually if this true, then $(\widehat{X^2})^\kappa \oplus (\widehat{X^2})^{\kappa'} \cong \mathbb{Z}^{2d}$ and is a finite index subgroup in $\widehat{X^2} \cong \mathbb{Z}^{2d}$, so the larger subgroup $(T^\kappa \cap T^{\kappa'})^\perp$ is also of finite index and this would establish the corollary.

Suppose $\boldsymbol{\xi} = (\xi^{(1)}, \xi^{(2)}) \in (\widehat{X^2})^\kappa \cap (\widehat{X^2})^{\kappa'}$. Assume first $\kappa, \kappa' \in K$, then $\xi^{(1)} + \kappa.\xi^{(2)} = \xi^{(1)} + \kappa'.\xi^{(2)} = 0$ and thus $(\kappa - \kappa').\xi^{(2)} = 0$. But since $\kappa - \kappa' \neq 0$ this can be true only if $\xi^{(2)} = 0$. And furthermore $\xi^{(1)} = -\kappa'.\xi^{(2)} = 0$. Therefore $\boldsymbol{\xi} = \mathbf{0}$. Now assume one of κ and κ' is ∞, without loss of generality let $\kappa' = \infty$ and $\kappa \in K$. Then by definition of $(\widehat{X^2})^\infty$, $\xi^{(2)} = 0$. Moreover $\xi^{(1)} = \xi^{(1)} + \kappa.\xi^{(2)} = 0$. So again $\boldsymbol{\xi}$ vanishes. This fulfills the proof. ☐

Lemma 3.6. *$\forall \kappa \in K \cup \{\infty\}$, T^κ is a $\zeta_\triangle$-invariant subset.*

Proof. By construction of T^κ, to obtain the invariance it suffices to show $(\widehat{X^2})^\kappa$ is invariant under the dual action $\zeta_\triangle : \mathbb{Z}^r \curvearrowright \widehat{X^2}$. By Remark 3.3, we may assume $\kappa \in K$. Suppose $\boldsymbol{\xi} = (\xi^{(1)}, \xi^{(2)}) \in (\widehat{X^2})^\kappa$, then $\xi^{(1)} + \kappa.\xi^{(2)} = 0$ and $\forall \mathbf{n} \in \mathbb{Z}^r$,

$$\zeta^{\mathbf{n}}.\xi^{(1)} + \kappa.\zeta^{\mathbf{n}}.\xi^{(2)} = \zeta^{\mathbf{n}}.(\xi^{(1)} + \kappa.\xi^{(2)}) = 0. \tag{3.7}$$

Hence $\zeta_\triangle.\boldsymbol{\xi} = (\zeta^{\mathbf{n}}.\xi^{(1)}, \zeta^{\mathbf{n}}.\xi^{(2)})$belongs to $\widehat{X^2}$ as well. ☐

3.2. **Local structures of homogeneous invariant subtori.** Now we discuss the structure of T^κ in further detail.

Definition 3.7. *$\forall \kappa \in K$, set*

$$V^\kappa = \{(\tilde{x}^{(1)}, \tilde{x}^{(2)}) \in (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2 : \kappa.\tilde{x}^{(1)} - \tilde{x}^{(2)} = 0\}. \tag{3.8}$$

Furthermore, when $\tau = \infty$, let

$$V^\infty = \{(0, \tilde{x}^{(2)}) \in (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2 : \tilde{x}^{(2)} \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}\}. \tag{3.9}$$

Denote $V_i^\kappa = V^\kappa \cap V_i^\square$, $\forall i \in I, \forall \kappa \in K \cup \{\infty\}$, where $V_i^\square$ is defined in (2.15).

Clearly V^κ is a d-dimensional linear subspace for all $\kappa \in K \cup \{\infty\}$. Observe

$$V_i^\kappa = \{(v, \kappa.v) : v \in V_i\}, \forall \kappa \in K, \text{ and } V_i^\infty = \{(0, v) : v \in V_i\}, \quad (3.10)$$

so $\forall \kappa \in K \cup \{\infty\}$, V_i^κ is isomorphic to V_i and has real dimension 1 or 2 depending on whether $i \le r_1$ or not.

Furthermore, both V^κ and V_i^κ are invariant under the multiplicative action $\zeta^\triangle$.

It follows from equation (2.16) that

$$V^\kappa = \bigoplus_{i \in I} V_i^\kappa, \ \forall k \in K \cup \{\infty\}. \quad (3.11)$$

It turns out that V^κ is the tangent space of the subtorus T^κ:

Lemma 3.8. *$T^\kappa = \pi_\triangle(V^\kappa) \cong V^\kappa / (\Gamma^2 \cap V^\kappa)$, $\forall \kappa \in K \cup \{\infty\}$.*

Recall $\pi_\triangle$ denotes the projection from $(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2$ to X^2.

Proof. Consider T^κ as a Lie subgroup in the abelian Lie group X^2. To get the lemma, it suffices to show that the Lie algebra of T^κ is V^κ. Since $\dim T^\kappa = d = \dim V^\kappa$, we only need to prove $\pi_\triangle(\mathbf{v}) \in T^\kappa$ for all $v \in V^\kappa$. This is clear for $\kappa = \infty$ as $V^\infty = (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}) \times \{0\}$ and $T^\infty = X \times \{0\}$. Now let $\kappa \in K$ and assume $\mathbf{v} = (v, \kappa.v)$ where $v \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$. Take an arbitrary $\boldsymbol{\xi} \in (\widehat{X^2})^\kappa \subset (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2$, then $\langle \boldsymbol{\xi}, \pi_\triangle(\mathbf{v}) \rangle = (\langle \boldsymbol{\xi}, \mathbf{v} \rangle \bmod \mathbb{Z})$ but $\langle \boldsymbol{\xi}, \mathbf{v} \rangle = \langle \xi^{(1)}, v \rangle + \langle \xi_i^{(2)}, \kappa.v \rangle = \langle \xi^{(1)}, v \rangle + \langle \kappa.\xi_i^{(2)}, v \rangle = \langle \xi^{(1)} + \kappa.\xi_i^{(2)}, v \rangle = \langle 0, v \rangle = 0$. Thus $\pi_\triangle(\mathbf{v}) \in \big((\widehat{X^2})^\kappa\big)^\perp = T^\kappa$. □

The next two corollaries follow easily from the lemma.

Corollary 3.9. *If $\kappa, \kappa' \in K \cup \{\infty\}$ and $\kappa \neq \kappa'$ then $(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2 = V^\kappa \oplus V^{\kappa'}$.*

Proof. Since $\dim(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2 = 2d = \dim V^\kappa + \dim V^{\kappa'}$ it suffices to show $V^\kappa \cap V^{\kappa'} = \{\mathbf{0}\}$. Suppose not, then $V^\kappa \cap V^{\kappa'}$ has positive dimension. In consequence $T^\kappa \cap T^{\kappa'} \supset \pi_\triangle(V^\kappa \cap V^{\kappa'})$ has positive dimension and hence is infinite; this contradicts Corollary 3.5 by taking $\mathbf{x} = \mathbf{x}' = \mathbf{0}$. □

Corollary 3.10. *If $\kappa \in K$, then $\forall \mathbf{x} \in X^2$, $(\mathbf{x} + T^\kappa) \cap T^\infty \neq \emptyset$.*

Proof. It is equivalent to show $T^\kappa \cap (-\mathbf{x} + T^\infty) \neq \emptyset$. Suppose $\mathbf{x} = (x^{(1)}, x^{(2)}) \in X^2$ and choose $\tilde{x}^{(1)} \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ which projects to $x^{(1)}$. Let $\mathbf{x}' = \big(x^{(1)}, \pi(\kappa.\tilde{x}^{(1)})\big) \in X^2$. Then $-\mathbf{x}' = \pi_\triangle\big((-\tilde{x}^{(1)}, -\kappa.\tilde{x}^{(1)})\big) \in \pi(V^\kappa) = T^\kappa$ and $(-\mathbf{x}') - (-\mathbf{x}) = \big(0, x^{(2)} - \pi(\kappa.\tilde{x}^{(1)})\big) \in T^\infty$. Therefore $T^\kappa \cap (-\mathbf{x} + T^\infty) \supset \{-\mathbf{x}'\} \neq \emptyset$. □

Now we claim that Berend's theorem applies to the torus T^κ.

Lemma 3.11. *Let $H \leq \mathbb{Z}^r$ be a finite-index subgroup (so $H \cong \mathbb{Z}^r$) and $\kappa \in K \cup \{\infty\}$ then the action of H on T^κ by $\zeta_\triangle$ satisfies the conditions in Theorem 1.1. Any $\zeta_\triangle$-invariant closed subset of T^κ is either T^κ itself or a finite set of torsion points. In particular, for $\mathbf{x} \in T^\kappa$, the H-orbit $\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in H\}$ of x under $\zeta^\triangle$ is dense in T^κ unless $\mathbf{x}$ is a torsion point; and T^κ is topologically transitive under the action of H by $\zeta_\triangle$.*

Proof. We only need to prove the lemma for $\kappa \in K$ because of Remark 3.3.

By (3.10), V^κ can be identified with $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ where $(v, \kappa.v)$ corresponds to $v \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$. Lemma 3.8 implies $T^\kappa \cong (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})/\Gamma'$ where $\Gamma' = \{v \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2} : (\kappa.v, v) \in \Gamma^2\} = \{\gamma \in \Gamma : \kappa.\gamma \in \Gamma\} \subset \Gamma$. As we know $T^\kappa \cong \mathbb{T}^d$, by Lemma 3.8 Γ' is a full-rank sublattice of Γ; in particular $\Gamma' \subset \sigma(K)$. Because $\zeta_\triangle^{\mathbf{n}}$ sends $(\kappa.v, v)$ to $\big(\kappa.(\zeta^{\mathbf{n}}.v), \zeta^{\mathbf{n}}.v\big)$, when we make the identification above, the restriction of $\zeta_\triangle$ to the invariant hyperplane V^κ is conjugate to the action $\zeta : \mathbb{Z}^r \curvearrowright \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ in (2.2). In consequence, $\Gamma' \subset \sigma(K)$ is a ζ-invariant lattice.

K, ζ and Γ' satisify the conditions listed in Proposition 2.2. Therefore the action $\zeta : \mathbb{Z}^r \curvearrowright (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})/\Gamma'$ is algebraically conjugate to a faithful $\mathbb{Z}^r$-Cartan action by d-dimensional toral automorphisms that contains a totally irreducible element. Therefore by Lemma 2.4 $\zeta_\triangle : \mathbb{Z}^r \curvearrowright T^\kappa$ satisfies the conditions in Theorem 1.1, and by Remark 1.2 so does the restriction $\zeta_\triangle|_H$. By Theorem 1.1 the only $\zeta_\triangle$-invariant infinite closed subset of T^κ is T^κ itself. The density and topological transitivity statements follow from this and Remark 1.5. ☐

Corollary 3.12. *For any finite-index subgroup H in $\mathbb{Z}^r$, $\forall i \in I$, $\forall \epsilon > 0$, $\forall \kappa \in K \cup \{\infty\}$, for a point $\mathbf{x} \in T^\kappa$,*

$$\overline{\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in H, \lambda_i(\mathbf{n}) \in (-\epsilon, \epsilon)\}} = T^\kappa \tag{3.12}$$

unless $\mathbf{x}$ can be written as $\mathbf{x}_0 + \mathbf{v}$ where $\mathbf{x}_0 \in T^\kappa$ is of torsion and $\mathbf{v} \in V_i^\kappa$.

Proof. Following the proof of the previous lemma, we obtain the corollary by applying Proposition 2.8 to the multiplicative action $\zeta : \mathbb{Z}^r \curvearrowright (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})/\Gamma'$, which is conjugate to the restriction of $\zeta_\triangle$ on T^κ. ☐

3.3. Homogeneous invariant subsets in X^2.

Definition 3.13. *A d-**dimensional homogeneous** $\zeta_\triangle$-**invariant subset** in X^2 is a subset of the form*

$$L = \{\zeta_\triangle^{\mathbf{n}}.\mathbf{z}' : \ \mathbf{n} \in \mathbb{Z}^r, \mathbf{z}' \in \mathbf{z} + T^\kappa\}, \tag{3.13}$$

where $\mathbf{z}$ *is a given torsion point in* X^2 *and* κ *is a given element in* $K \cup \{\infty\}$.

Such a subset is said to be "homogeneous" because it is a disjoint union of several parallel subtori.

Proposition 3.14. *Let* $L \subset X^2$ *be a* d*-dimensional homogeneous* $\zeta_\triangle$*-invariant subset of the form (3.13), then:*

(1) L *is a finite disjoint union* $\bigsqcup_{t=1}^{s} L_t$ *where each component* L_t *is a translate of the invariant subtorus* T^κ *by some torsion element of* X^2*,* κ *being the same as in (3.13);*

(2) $\mathrm{Stab}_{\zeta_\triangle}(L_1) = \mathrm{Stab}_{\zeta_\triangle}(L_2) = \cdots = \mathrm{Stab}_{\zeta_\triangle}(L_d)$*, which is a subgroup of finite index* d *in* $\mathbb{Z}^r$.

(3) L *is invariant and topologically transitive under the* $\mathbb{Z}^r$*-action* $\zeta_\triangle$*. Actually,* $\overline{\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\}} = L$ *if* $\mathbf{x} \in L$ *is not a torsion element of* X^2 .

Proof. (1) Since T^κ is $\zeta_\triangle$-invariant, the stabilizer $H = \mathrm{Stab}_{\zeta_\triangle}(\mathbf{z} + T^\kappa)$ contains $H_0 = \mathrm{Stab}_{\zeta_\triangle}(\mathbf{z})$, which is a finite-index subgroup of $\mathbb{Z}^r$. It follows that H itself, which is obviously a semigroup, must be a subgroup of finite index.

Suppose the finite quotient group $\mathbb{Z}^r/H$ consists of s different cosets $\mathbf{n}_1 + H, \cdots, \mathbf{n}_s + H$. Then for any t, $\forall \mathbf{n} \in \mathbf{n}_t + H$, $\zeta_\triangle^{\mathbf{n}}.(\mathbf{z} + T^\kappa) = \zeta_\triangle^{\mathbf{n}_t}.\mathbf{z} + \zeta_\triangle^{\mathbf{n}_t}.T^\kappa = \zeta_\triangle^{\mathbf{n}_t}.\mathbf{z} + T^\kappa$ is the translate of T_k by another torsion element $\zeta_\triangle^{\mathbf{n}_t}.\mathbf{z}$, which we denote by L_t (here we used the fact $\zeta_\triangle^{\mathbf{n}_t}.T^\kappa = T^\kappa$). Moreover the L_t's are all different, since if $\zeta_\triangle^{\mathbf{n}_t}.(\mathbf{z} + T^\kappa)$ and $\zeta_\triangle^{\mathbf{n}_{t'}}.(\mathbf{z} + T^\kappa)$ are equal, then $\zeta_\triangle^{\mathbf{n}_t - \mathbf{n}_{t'}}.(\mathbf{z} + T^\kappa) = (\mathbf{z} + T^\kappa)$ and thus $\mathbf{n}_t - \mathbf{n}_{t'} \in H$, which contradicts the assumption that $\mathbf{n}_t + H \neq \mathbf{n}_{t'} + H$. So

$$L = \bigcup_{t=1}^{s} \bigcup_{\mathbf{n} \in \mathbf{n}_t + H} \left(\zeta_\triangle^{\mathbf{n}}.(\mathbf{z} + T^\kappa)\right) = \bigcup_{t=1}^{s} \bigcup_{\mathbf{n} \in \mathbf{n}_t + H} L_t = \bigsqcup_{t=1}^{s} L_t. \tag{3.14}$$

This proves part (1).

(2) Remark $\zeta_\triangle^{\mathbf{n}}.L_{t'} = \zeta_\triangle^{\mathbf{n}}.\zeta_\triangle^{\mathbf{n}_{t'}}.(\mathbf{z} + T^\kappa) = \zeta_\triangle^{\mathbf{n} + \mathbf{n}_{t'}}.(\mathbf{z} + T^\kappa)$ is equal to L_t if and only if $\mathbf{n} \in \mathbf{n}_t - \mathbf{n}_{t'} + H$. Hence the action $\zeta_\triangle^{\mathbf{n}}$ stabilizes L_t if and only if $\mathbf{n} \in H$, i.e. for all $t = 1, \cdots, s$ the stabilizer $\mathrm{Stab}_{\zeta_\triangle}(L_t)$ is exactly H, which is of index l in $\mathbb{Z}^r$.

(3) The invariance is clear. To see L is topologically transitive, because under the continuous group action $\zeta_\triangle$, the sets $L_1, \cdots, L_s$ are permuted in a transitive way among themselves, it suffices to show for

any L_t and any $\mathbf{x} \in L_t$ that is not a torsion element of X^2 (remark such a point always exists as L_t is uncountable but there are only countably many torsion points in X^2), $\overline{\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in H\}} = L_t$.

Recall that $L_t = \zeta^{\mathbf{n}_t}.\mathbf{z}+T^\kappa$, and that $H_0 = \mathrm{Stab}_{\zeta_\triangle}(\mathbf{z}) = \mathrm{Stab}_{\zeta_\triangle}(\zeta^{\mathbf{n}_t}.\mathbf{z})$ is inside H and has finite index in $\mathbb{Z}^r$. Observe $\mathbf{x} - \zeta^{\mathbf{n}_t}.\mathbf{z}$ is not a torsion element since otherwise so is $\mathbf{x}$ because $\zeta^{\mathbf{n}_t}.\mathbf{z}$ is of torsion. By Lemma 3.11, $\overline{\{\zeta_\triangle^{\mathbf{n}}.(\mathbf{x} - \zeta^{\mathbf{n}_t}.\mathbf{z}) : \mathbf{n} \in H_0\}} = T^\kappa$. Remark for $\mathbf{n} \in H_0$, $\zeta_\triangle^{\mathbf{n}}.(\mathbf{x}-\zeta^{\mathbf{n}_t}.\mathbf{z}) = \zeta_\triangle^{\mathbf{n}}.\mathbf{x}-\zeta^{\mathbf{n}_t}.\mathbf{z}$, and thus $\overline{\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in H_0\}} = \zeta^{\mathbf{n}_t}.\mathbf{z}+T^\kappa = L_t$. This completes the proof. □

A d-dimensional homogeneous $\zeta_\triangle$-invariant subset is decided by a torsion point $\mathbf{z}$ (which is not unique) and a slope $\kappa \in K \cup \{\infty\}$, as a consequence we will see that there are only finitely many such sets of low complexity.

Lemma 3.15. $\forall\epsilon > 0$, *there are only finitely many d-dimensional homogeneous $\zeta_\triangle$-invariant subsets that are not ϵ-dense in X^2.*

Proof. **Step 1.** We claim first that $\forall\epsilon > 0$, there are only finitely many $\kappa \in K$ such that T^κ is not ϵ-dense in X^2. Suppose the opposite, then there is a sequence of $\kappa_h \in K \cup \{\infty\}$, all different from each other, and points $\mathbf{y}_h \in X^2$ such that $T^{\kappa_h} \cap B_\epsilon(\mathbf{y}_h) = \emptyset, \forall h$. By passing to a subsequence we may assume that $\exists \mathbf{y} \in X^2$ such that $\mathbf{y}_h$ is within distance $\frac{\epsilon}{2}$ from $\mathbf{y}$ for all h. Then $B_{\frac{\epsilon}{2}}(\mathbf{y}) \subset B_\epsilon(\mathbf{y}_h)$ and therefore by assumption $T^{\kappa_h} \cap B_{\frac{\epsilon}{2}}(\mathbf{y}) = \emptyset, \forall h$. So the subsets T^{κ_h} do not converge to X^2 in the Hausdorff metric as h tends to ∞. Because T^{κ_h} is a closed subgroup of X^2, by [Ber83, Lemma 4.7] there exists a nonzero character $\boldsymbol{\xi} \in \widehat{X^2}$ which lies in the annihilator of T^{κ_h} for an infinite number of h. In other words, there are infinitely many different slopes κ_h such that $\boldsymbol{\xi} \in (\widehat{X^2})^{\kappa_h}$. Without loss of generality assume all these κ_h's are in K.

By (2.32) and Proposition 2.9, $\boldsymbol{\xi} = (\sigma(\theta^{(1)}), \sigma(\theta^{(2)}))$. By definition of $(\widehat{X^2})^{\kappa_h}$, $\sigma(\theta^{(1)}) + \kappa_h.\sigma(\theta^{(2)}) = 0$ for infinitely many κ_h's. However it is easy to check that $\sigma(\theta^{(1)}) + \kappa_h.\sigma(\theta^{(2)}) = \sigma(\theta^{(1)} + \kappa_h\theta^{(2)})$ and this expression vanishes if and only if $\theta^{(1)} + \kappa_h\theta^{(2)} = 0$. In order to make this happen for more than one $\kappa_h \in K$, $\theta^{(1)}$ and $\theta^{(2)}$ must be both 0; hence $\boldsymbol{\xi} = \mathbf{0}$ and a contradiction is obtained. This shows the claim.

Step 2. Notice if T^κ is not ϵ-dense, then neither is any translate of it. Therefore given the claim in Step 1, to prove the lemma it suffices to show that for any given $\epsilon > 0$ and T^κ, there are only finitely many d-dimensional homogeneous $\zeta_\triangle$-invariant subsets that are not ϵ dense and decompose into a disjoint union of translates of T^κ. Without loss of generality, we may assume $\kappa \in K$ following Remark 3.3.

Suppose the opposite is true, namely, there are infinitely many d-dimensional d-dimensional homogeneous $\zeta_\triangle$-invariant subsets in T^κ direction that are not ϵ-dense. By the same argument as in Step 1, we may find a sequence of different d-dimensional homogeneous $\zeta_\triangle$-invariant subsets $L_h = \bigsqcup_{t=1}^{s_h} L_{h,t}$ that avoid a fixed ball $B_{\frac{\epsilon}{2}}(\mathbf{y})$ of radius $\frac{\epsilon}{2}$, where each $L_{h,t}$ is a translate of T^κ. It should be remarked that the $L_{h,t}$'s are all different. In fact, $L_{h,t} \neq L_{h,t'}$ for $t \neq t'$ by construction; and if $L_{h,t} = L_{h',t'}$ for $h \neq h'$ then by topological transitivity of L_h and $L_{h'}$ this would imply $L_h = L_{h'}$ which contradicts the way the L_h's are chosen.

By Corollary 3.10, $\forall h$, $\forall t \leq s_h$, one can find a point $\mathbf{x}_{h,t} \in L_{h,t} \cap T^\infty$. Since $L_{h,t} = \mathbf{x}_{h,t} + T^\kappa$, the $\mathbf{x}_{h,t}$'s are all different and thus $\bigcup_{h=1}^\infty (L_h \cap T^\infty)$ is an infinite set. However by invariance of L_h and T^∞, this set is a $\zeta_\triangle$-invariant subset of T^∞. Hence it follows from Lemma 3.11 that $\bigcup_{h=1}^\infty (L_h \cap T^\infty)$ is dense in T^∞.

Consider the center $\mathbf{y}$ of the ball $B_{\frac{\epsilon}{2}}(\mathbf{y})$, by construction $B_{\frac{\epsilon}{2}}(\mathbf{y})$ is disjoint from L_h for all h. But by Corollary 3.10, $\exists \mathbf{x} \in (\mathbf{y} + T^\kappa) \cap T^\infty$. By density of $\bigcup_{h=1}^\infty (L_h \cap T^\infty)$, there are $h \in \mathbb{N}$ and $\mathbf{x}' \in L_h \cap T^\infty$ such that the distance between $\mathbf{x}'$ and $\mathbf{x}$ is less than $\frac{\epsilon}{2}$. Then $\mathbf{x}' + (\mathbf{y} - \mathbf{x}) \in B_{\frac{\epsilon}{2}}(\mathbf{y})$. Thus $\mathbf{x}' + (\mathbf{y} - \mathbf{x}) \notin L_h$. But on the other hand $\mathbf{y} - \mathbf{x} \in T^\kappa$ and $\mathbf{x}' + (\mathbf{y} - \mathbf{x}) \in \mathbf{x}' + T^\kappa \subset L_h$, a contradiction. The proof is completed. $\square$

3.4. **Restatement of the main results.** Finally we can restate the main result of this paper in the X^2 setting:

Theorem 3.16. *Suppose K, Γ, X and ζ satisfy the conditions in Proposition 2.2 and $\zeta_\triangle$ is constructed by (2.13), then:*

(1) If $r \geq 3$ and an infinite proper closed subset A of X^2 is invariant and topologically transitive under the $\mathbb{Z}^r$-action $\zeta_\triangle$, then A is a d-dimensional homogeneous $\zeta_\triangle$-invariant subset.

(2) If $r = 2$, then there exist a point $\mathbf{x} \in X^2$ and three different d-dimensional homogeneous $\zeta_\triangle$-invariant subtori $T^{\kappa_1}, T^{\kappa_2}, T^{\kappa_3} \subset (\mathbb{T}^d)^2$ such that the $\zeta_\triangle$-orbit closure of $\mathbf{x}$ is a disjoint union:

$$\overline{\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\}} = \{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\} \sqcup \big(\bigcup_{i=1}^3 T^{\kappa_i}\big). \tag{3.15}$$

Theorem 3.17. *In the same setting as above, suppose $r \geq 3$. If $\epsilon > \epsilon' > 0$, let $L_1, \cdots, L_q$ be all the d-dimensional homogeneous $\zeta_\triangle$-invariant subsets failing to be ϵ'-dense in X^2. Then among all the* *torsion points whose $\zeta_\triangle$-orbits are not ϵ-dense, there are only finitely many which are not contained in $\bigcup_{h=1}^{q} L_h$.*

In particular, $\forall\epsilon > 0$, there is a finite union of d-dimensional homogeneous $\zeta_\triangle$-invariant subsets that contains any torsion point whose $\zeta_\triangle$ -orbit is not ϵ-dense in X^2.

Here Lemma 3.15 is implicitly used to assert that there are only finitely many L_h's which are not ϵ'-dense.

Lemma 3.18. *Theorem 3.16 implies Theorem 1.6; and Theorem 3.17 implies Theorem 1.7.*

Proof. Because the $\mathbb{Z}^r$-actions $\zeta_\triangle$ on X and $\alpha_\triangle$ on $\mathbb{T}^d$ are equivalent via the continuous group isomorphism $\psi_\triangle$, to deduce Theorem 1.6 from Theorem 3.16 it suffices to observe that the preimage under ψ of a d-dimensional homogeneous $\zeta_\triangle$-invariant subtorus is a d-dimensional $\alpha_\triangle$-invariant subtorus in $\mathbb{T}^d$ by Lemma 3.4, and thus the preimage of a d-dimensional homogeneous $\zeta_\triangle$-invariant subset is a disjoint union of finitely many parallel translates of a d-dimensional $\alpha_\triangle$-invariant subtorus by Proposition 3.14.

In addition to these ingredients, to get the implication from Theorem 3.17 to Theorem 1.7 it is enough to observe that under $\psi_\triangle$, any ϵ-dense subset in $(\mathbb{T}^d)^2$ becomes $C\epsilon$-dense in X^{2d} , where C is a positive constant depending only on $\psi_\triangle$. □

The proofs of Theorems 3.16 and 3.17 are going to occupy the rest of paper.

4. Any infinite invariant set has a homogeneous subset

In order to get the first half of Theorem 3.16, our strategy is to prove first that the set A contains a homogeneous invariant subset and then show that there are no other points in it. The aim of this section is to show the following:

Proposition 4.1. *Suppose $r \geq 2$ and $A \subset X^2$ is an infinite $\zeta_\triangle$-invariant closed subset, then there is a d-dimensional homogeneous $\zeta_\triangle$-invariant subset L of X^2 such that $L \subset A$.*

It should be emphasized that the proposition applies in both $r \geq 3$ and $r = 2$ cases.

The proof of Proposition 4.1 borrows ideas from [Ber83, §4]

4.1. **Translate of a torsion point along an eigenspace.** In this part, we are going to show A contains a torsion point $\mathbf{z}$ as well as another point in the V_i-foliation through $\mathbf{z}$ for some $i \in I$.

Lemma 4.2. *If $A \subset X^2$ is an infinite $\zeta_\triangle$-closed subset then it has an accumulation point which is a torsion point in X^2.*

Proof. A is compact as a closed subset of the compact set X^2. Hence as it is infinite there must be an accumulation point $\mathbf{y} = (y^{(1)}, y^{(2)}) \in A$. If $\mathbf{y}$ is a torsion point then we are done, so we assume $\mathbf{y}$ is not of torsion.

Let $A' = \overline{\{\zeta_\triangle^{\mathbf{n}}.\mathbf{y} : \mathbf{n} \in \mathbb{Z}^r\}}$. Then $A' \subset A$ and is also $\zeta_\triangle$-invariant. By Theorem 2.7 either the set $\{\zeta^{\mathbf{n}}.y^{(1)} : \mathbf{n} \in \mathbb{Z}^r\}$ is dense in X or $y^{(1)}$ is a torsion point. In both cases, there is a sequence of $\mathbf{n}_k$ such that $\zeta^{\mathbf{n}_k}.y^{(1)}$ converges to a torsion point $z^{(1)}$ (when $y^{(1)}$ is a torsion point, simply take $\mathbf{n}_k = \mathbf{0}$ for all k). As X is compact, by passing to a subsequence if necessary we may assume $\zeta^{\mathbf{n}_k}.y^{(2)} \to z_*^{(2)}$ for some $z_*^{(2)} \in X$. Then $\mathbf{z}_* = (z^{(1)}, z_*^{(2)}) = \lim_{k\to\infty} \zeta_\triangle^{\mathbf{n}_k}.\mathbf{y} \in A'$. Let H be the stabilizer $\{\mathbf{n} \in \mathbb{Z}^r : \zeta^{\mathbf{n}}.z^{(1)} = z^{(1)}\}$, which has finite index in $\mathbb{Z}^r$ as $z^{(1)}$ is of torsion. It follows from Theorem 2.7 that $\overline{\{\zeta^{\mathbf{n}}.z_*^{(2)} : \mathbf{n} \in H\}} \subset X$ contains a torsion point, which we denote by $z^{(2)}$. Then there is a new sequence of $\mathbf{m}_k \in H$ such that $\zeta^{\mathbf{m}_k}.z_*^{(2)}$ converges to $z^{(2)}$ as $k \to \infty$. Then as A' is a $\zeta_\triangle$-invariant closed set, it contains $\mathbf{z} := (z^{(1)}, z^{(2)}) = \lim_{k\to\infty} \zeta_\triangle^{\mathbf{m}_k}.\mathbf{z}_*$, which is a torsion point in X^2.

Observe as $\mathbf{z} \in A'$ it is a limit point of the $\zeta_\triangle$-orbit of $\mathbf{y}$. But since $\mathbf{z}$ is of torsion and $\mathbf{y}$ is not, none of the points from the orbit of $\mathbf{y}$ coincide with $\mathbf{z}$. Thus $\mathbf{z}$ is an accumulation point of $A' \subset A$. This completes the proof. □

In particular, the lemma gives the following corollary.

Corollary 4.3. *When $r \geq 2$, any minimal $\zeta_\triangle$-invariant closed subset of X^2 is a finite set of torsion points.*

Proof. Let M be a minimal invariant set. By lemma, M contains a torsion point $\mathbf{z}$. By minimality M is the $\zeta_\triangle$-orbit of $\mathbf{z}$, which consists of finitely many torsion points. □

Lemma 4.2 tells us that there is a sequence of points in A converging to a torsion point. We wish to have some control on the direction along which such convergence takes place.

Lemma 4.4. *For any finite-index subgroup $H \leq \mathbb{Z}^r$, there is a positive number $C = C(H)$ such that $\forall i \in I$, $\exists \mathbf{n} \in \mathbb{Z}^r$ such that $0 < \lambda_i(\mathbf{n}) \leq C$ and $\lambda_j(\mathbf{n}) < 0$ for all $j \in I \backslash \{i\}$.*

Proof. It follows from Remark 2.5 that any cone of non-empty interior in W contains points from $\mathcal{L}(H)$. In particular for any given index

$i \in I$, the cone $\{(\lambda_j)_{j\in I} \in W : \lambda_i > 0 \text{ and } \lambda_j < 0, \forall j \neq i\}$, which has open interior in W, contains a point $\mathcal{L}(\mathbf{n}_i)$ where $\mathbf{n}_i \in H$. It suffices to take $C = \max_{i\in I} \lambda_i(\mathbf{n}_i)$. □

Corollary 4.5. *If A is an infinite $\zeta_\triangle$-invariant closed subset then $\exists i \in I$ such that there are a torsion point $\mathbf{z}$ in X^2 and a nonzero vector $\mathbf{v} \in V_i^\square$ such that A contains both $\mathbf{z}$ and $\mathbf{z}+\mathbf{v}$.*

Proof. Let $\mathbf{z} \in A$ be given by Lemma 4.2. Then there is a sequence of nonzero vectors $\mathbf{v}_k \in (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2$ converging to 0 as $k \to \infty$, such that $\mathbf{z} + \mathbf{v}_k \in A$. There is a unique decomposition $\mathbf{v}_k = \sum_{i\in I}(\mathbf{v}_k)_i$ where $(\mathbf{v}_k)_i \in V_i^\square$. As I is finite, by reducing to a subsequence, we can assume $\exists i \in I$ such that $|(\mathbf{v}_k)_i| \geq |(\mathbf{v}_k)_j|, \forall j \neq i, \forall k \in \mathbb{N}$. In particular, $|(\mathbf{v}_k)_i| > 0$.

Now let $H = \mathrm{Stab}_{\zeta_\triangle}(\mathbf{z})$. As H has finite index in $\mathbb{Z}^r$, Lemma 4.4 applies and we obtain $\mathbf{n} \in H$ such that $1 < |\zeta_i^{\mathbf{n}}| = e^{\lambda_i(\mathbf{n})} \leq e^C$ and $|\zeta_j^{\mathbf{n}}| < 1, \forall j \in I\backslash\{i\}$ where $C = C(H)$. For all sufficiently large k, there is a positive integer l_k such that $|\zeta_i^{\mathbf{n}}|^{l_k} \cdot |(\mathbf{v}_k)_i| \in [1, e^C]$. Hence since $\zeta^{\mathbf{n}}$ acts as the multiplication by $\zeta_i^{\mathbf{n}}$ on $V_i^\square$, which is isomorphic to $\mathbb{R}^2$ or $\mathbb{C}^2$,

$$\zeta^{l_k\mathbf{n}}.(\mathbf{v}_k)_i = (\zeta_i^{\mathbf{n}})^{l_k}(\mathbf{v}_k)_i \in \{\mathbf{v} \in V_i^\square : |\mathbf{v}| \in [1, e^C]\}. \tag{4.1}$$

Because this is a compact subset in $V_i^\square$, by passing to a subsequence of $k \in \mathbb{N}$, it is alright to assume the sequence $\{\zeta^{l_k\mathbf{n}}.(\mathbf{v}_k)_i\}$ converges to a limit vector $\mathbf{v}$. Then $\mathbf{v} \in V_i^\square$ and $|\mathbf{v}| \in [1, e^C]$.

Furthermore,

$$\begin{aligned}|\zeta^{l_k\mathbf{n}}.\mathbf{v}_k - \zeta^{l_k\mathbf{n}}.(\mathbf{v}_k)_i| =& \Big|\sum_{j\in I\backslash\{i\}} \zeta^{l_k\mathbf{n}}.(\mathbf{v}_k)_j\Big| \leq \sum_{j\in I\backslash\{i\}} |(\zeta_j^{\mathbf{n}})^{l_k}(\mathbf{v}_k)_j| \\ &\leq \sum_{j\in I\backslash\{i\}} |(\mathbf{v}_k)_j|.\end{aligned} \tag{4.2}$$

Since $\mathbf{v}_k \to 0$, $\lim_{k\to\infty} |(\mathbf{v}_k)_j| = 0, \forall j$. Hence (4.2) decays to 0 as $k \to \infty$ and $\lim_{k\to\infty} \zeta^{l_k\mathbf{n}}.\mathbf{v}_k = \lim_{k\to\infty} \zeta^{l_k\mathbf{n}}.(\mathbf{v}_k)_i = \mathbf{v}$.

Notice since $l_k\mathbf{n} \in H$ stabilizes $\mathbf{z}$ under $\zeta_\triangle$, $\zeta_\triangle^{l_k\mathbf{n}}.(\mathbf{z} + \mathbf{v}_k)$ writes $\mathbf{z} + \zeta^{l_k\mathbf{n}}.\mathbf{v}_k$ and converges to $\mathbf{z} + \mathbf{v}$. Since $\mathbf{z} + \mathbf{v}_k$ is in the $\zeta_\triangle$-invariant closed subset A, this implies $\mathbf{z} + \mathbf{v} \in A$ and completes the proof. □

4.2. **Construction of an infinitely long line.** We showed A contains both $\mathbf{z}$ and $\mathbf{z}+\mathbf{v}$. Using invariance under the group action, a large subset of $V_i^\square$ can be constructed from the difference vector $\mathbf{v} \in V_i^\square$.

Recall $V_i^\square = V_i^{(1)} \oplus V_i^{(2)}$, so any non-zero vector $\mathbf{v} \in V_i^\square$ can be accordingly written as $(v^{(1)}, v^{(2)})$ where $v^{(1)}, v^{(2)} \in V_i$, which is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. As at least one of $v^{(1)}$, $v^{(2)}$ is not equal to 0, for the moment we are going to assume $v^{(1)} \neq 0$. Then there is a number κ_*, from either $\mathbb{R}$ or $\mathbb{C}$, whichever V_i is isomorphic to, such that $v^{(2)} = \kappa_* v^{(1)}$.

Lemma 4.6. *Suppose $\mathbf{z}$ is a torsion point and $\mathbf{v} = (v, \kappa_* v)$ is a nonzero vector from $V_i^\square$ for some $i \in I$ where $v \in V_i$ and κ_* is a number from $\mathbb{R}$ or $\mathbb{C}$, whichever V_i is isomorphic to. If a $\zeta_\triangle$-invariant closed subset A contains both $\mathbf{z}$ and $\mathbf{z}+\mathbf{v}$, then there are $\mathbf{y} \in A$ and a nonzero vector $w \in V_i$ such that $\mathbf{y} + \rho\mathbf{w} \in A, \forall \rho \in \mathbb{R}$ where $\mathbf{w} = (w, \kappa_* w) \in V_i^\square$.*

We need the following fact to prove Lemma 4.6.

Lemma 4.7. *Suppose $r \geq 2$. Then for any finite-index subgroup $H \leq \mathbb{Z}^r$, $\forall i \in I$, $\forall \delta > 0$, there exists $\mathbf{n} \in H \backslash \{0\}$ such that $|\operatorname{Log} \zeta_i^{\mathbf{n}}| < \delta$ where $\operatorname{Log} \zeta_i^{\mathbf{n}} \in \mathbb{C}$ denotes the principal value of logarithm of $\zeta_i^{\mathbf{n}}$.*

Proof. As $\operatorname{rank}(H) = r$, we can fix two linearly independent non-trivial elements $\mathbf{m}_1, \mathbf{m}_2 \in H$. Set $\alpha(p_1, p_2) = \big(\lambda_i(p_1\mathbf{m}_1 + q_2\mathbf{m}_2), \arg \zeta_i^{p_1\mathbf{m}_1 + q_2\mathbf{m}_2}\big)$, then α is a group morphism from $\mathbb{Z}^2$ to $\mathbb{R} \oplus (\mathbb{R}/2\pi\mathbb{Z})$. For all $p_1, p_2 \in \{-P, P-1, \cdots, P-1, P\}$, $\alpha(p_1, p_2) \in [-MP, MP] \oplus (\mathbb{R}/2\pi\mathbb{Z})$ where $M = \max(|\lambda_i(\mathbf{m}_1)|, |\lambda_i(\mathbf{m}_2))|$. Because $[-MP, MP] \oplus (\mathbb{R}/2\pi\mathbb{Z})$ can be covered by $O_{M,\delta}(P)$ balls of radius δ but we are considering $(2P+1)^2$ pairs of (p_1, p_2), there are two distinct pairs $(p_1, p_2), (p_1', p_2') \in \{-P, P-1, \cdots, P-1, P\}^2$ such that $\alpha(p_1, p_2)$ and $\alpha(p_1', p_2')$ are in the same ball if P is sufficiently large with respect to M and δ. Let $\mathbf{n} = (p_1 - p_1')\mathbf{m}_1 + (p_2 - p_2')\mathbf{m}_2$, then as $\mathbf{m}_1$ and $\mathbf{m}_2$ are linearly independent elements in H, $\mathbf{n}$ is a nonzero element of H. Moreover $(\lambda_i(\mathbf{n}), \arg \zeta_i^{\mathbf{n}}) = \alpha(p_1, p_2) - \alpha(p_1', p_2')$ is of distance δ from the origin in $\mathbb{R} \oplus (\mathbb{R}/2\pi\mathbb{Z})$. In other words, if $\operatorname{Arg} \zeta_i^{\mathbf{n}}$ denotes the principal value of complex argument of $\zeta_i^{\mathbf{n}}$, then $|\lambda_i(\mathbf{n}) + \mathrm{i} \operatorname{Arg} \zeta_i^{\mathbf{n}}| \leq \delta$. However $\lambda_i(\mathbf{n}) + \mathrm{i} \operatorname{Arg} \zeta_i^{\mathbf{n}}$ is just $\operatorname{Log} \zeta_i^{\mathbf{n}}$ as $\lambda_i(\mathbf{n}) = \log|\lambda_i(\mathbf{n})|$. □

We show first a finitary statement, then take limit to get Lemma 4.6.

Lemma 4.8. *Let A, i, $\mathbf{z}$, $\mathbf{v}$ and κ_* be as in Lemma 4.6. Then $\forall R > 0$, $\forall \epsilon > 0$, $\exists \mathbf{y} \in A$, $\exists w \in V_i$ such that $|w| = 1$ and $A \cap B_\epsilon(\mathbf{y} + \rho\mathbf{w}) \neq \emptyset, \forall \rho \in [-R, R]$ where $\mathbf{w} = (w, \kappa_* w) \in V_i^\square$.*

Proof. Without loss of generality, assume $R > \epsilon$.

Again let H be the finite-index subgroup $\operatorname{Stab}_{\zeta_\triangle}(\mathbf{z}) \leq \mathbb{Z}^r$. By Lemma 4.4, $\exists \mathbf{m} \in H$ such that $\lambda_i(\mathbf{m}) \in (0, C]$ where $C = C(H)$. Then as $v \in V_i$, for all integer l, $\zeta^{l\mathbf{m}}.v = \zeta_i^{l\mathbf{m}}.v$ and has length $e^{l\lambda_i(\mathbf{m})}|v|$. We may choose $l \in \mathbb{Z}$ such that

$$\frac{2e^C\sqrt{1+|\kappa_*|^2} \cdot R^2}{\epsilon} < |\zeta_i^{l\mathbf{m}} v| \leq \frac{2e^{2C}\sqrt{1+|\kappa_*|^2} \cdot R^2}{\epsilon}. \tag{4.3}$$

Applying Lemma 4.7, pick a nontrivial element $\mathbf{n} \in H$ such that

$$|\operatorname{Log} \zeta_i^{\mathbf{n}}| < \frac{\epsilon^2}{4e^{2C}\sqrt{1+|\kappa_*|^2} \cdot R^2}. \tag{4.4}$$

Since ζ is an embedding of $\mathbb{Z}^r$ into U_K, $\zeta_i^{\mathbf{n}} = \sigma_i(\zeta^{\mathbf{n}}) \neq 1$ and $\operatorname{Log} \zeta_i^{\mathbf{n}} \neq 0$.

Define $\mathbf{y} = \zeta_\triangle^{l\mathbf{m}}.(\mathbf{z}+\mathbf{v}) = \mathbf{z} + \zeta_i^{l\mathbf{m}}\mathbf{v}$, $w = \frac{(\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}v}{|(\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}v|}$ and denote $(w, \kappa_* w)$ by $\mathbf{w}$. Then $w \in V_i$ has length 1 and

$$\mathbf{w} = \frac{(\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}\mathbf{v}}{|(\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}v|}. \tag{4.5}$$

For all $\rho \in [-R, R]$, take the integer

$$t = \operatorname{sign}(\rho) \cdot \left\lfloor \frac{|\rho|}{|(\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}v|} \right\rfloor. \tag{4.6}$$

We claim

$$\left\|\zeta_\triangle^{l\mathbf{m}+t\mathbf{n}}.(\mathbf{z}+\mathbf{v}) - (\mathbf{y}+\rho\mathbf{w})\right\| \leq \epsilon. \tag{4.7}$$

To see this, first observe that as $\mathbf{m}, \mathbf{n} \in H$,

$$\begin{aligned}
\zeta_\triangle^{l\mathbf{m}+t\mathbf{n}}.(\mathbf{z}+\mathbf{v}) =& \mathbf{z} + \zeta_i^{l\mathbf{m}+t\mathbf{n}}\mathbf{v} \\
=& \mathbf{z} + e^{t \operatorname{Log}\zeta_i^{\mathbf{n}}}\zeta_i^{l\mathbf{m}}\mathbf{v} \\
=& \mathbf{z} + \zeta_i^{l\mathbf{m}}\mathbf{v} + t(\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}\mathbf{v} \\
&+ \sum_{k=2}^{\infty} \frac{(t\operatorname{Log}\zeta_i^{\mathbf{n}})^k}{k!} \cdot \zeta_i^{l\mathbf{m}}\mathbf{v} \\
=& \mathbf{y} + \rho\mathbf{w} + \left(t - \frac{\rho}{|(\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}v|}\right)(\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}\mathbf{v} \\
&+ \sum_{k=2}^{\infty} \frac{(t\operatorname{Log}\zeta_i^{\mathbf{n}})^k}{k!} \cdot \zeta_i^{l\mathbf{m}}\mathbf{v},
\end{aligned} \tag{4.8}$$

where the last step used (4.5). Therefore it suffices to show

$$\left|\left(t - \frac{\rho}{|(\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}v|}\right) \cdot (\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}\mathbf{v}\right| \leq \frac{\epsilon}{2}. \tag{4.9}$$

and

$$\left|\sum_{k=2}^{\infty} \frac{(t\operatorname{Log}\zeta_i^{\mathbf{n}})^k}{k!} \cdot \zeta_i^{l\mathbf{m}}\mathbf{v}\right| \leq \frac{\epsilon}{2} \tag{4.10}$$

By choice of t, $\left|t - \frac{\rho}{|(\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}v|}\right| \leq 1$. Moreover by (4.3) and (4.4), $|(\operatorname{Log}\zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}}\mathbf{v}| \leq \frac{\epsilon^2}{4e^{2C}\sqrt{1+|\kappa_*|^2}\cdot R^2} \cdot \frac{2e^{2C}\sqrt{1+|\kappa_*|^2}\cdot R^2}{\epsilon} = \frac{\epsilon}{2}$, (4.9) follows.

On the other hand, note

$$
\begin{aligned}
|t \operatorname{Log} \zeta_i^{\mathbf{n}}| \leq & \frac{|\rho|}{|(\operatorname{Log} \zeta_i^{\mathbf{n}})\zeta_i^{l\mathbf{m}} v|} \cdot |\operatorname{Log} \zeta_i^{\mathbf{n}}| = \frac{|\rho|}{|\zeta_i^{l\mathbf{m}} v|} \\
\leq & R \left(\frac{2e^C \sqrt{1+|\kappa_*|^2} \cdot R^2}{\epsilon} \right)^{-1} \\
= & \frac{\epsilon}{2e^C \sqrt{1+|\kappa_*|^2} \cdot R}
\end{aligned} \tag{4.11}
$$

As we assumed $R > \epsilon$, this expression is bounded by 1. Thus

$$
\begin{aligned}
\left| \sum_{k=2}^{\infty} \frac{(t \operatorname{Log} \zeta_i^{\mathbf{n}})^k}{k!} \right| = & \left| \sum_{k=2}^{\infty} \frac{(t \operatorname{Log} \zeta_i^{\mathbf{n}})^{k-2}}{k!} \right| \cdot |t \operatorname{Log} \zeta_i^{\mathbf{n}}|^2 \\
\leq & \left(\sum_{k=2}^{\infty} \frac{1}{k!} \right) \cdot |t \operatorname{Log} \zeta_i^{\mathbf{n}}|^2 \\
\leq & |t \operatorname{Log} \zeta_i^{\mathbf{n}}|^2 \leq \left(\frac{\epsilon}{2e^C \sqrt{1+|\kappa_*|^2} \cdot R} \right)^2 \\
\leq & \frac{\epsilon^2}{4e^{2C} \sqrt{1+|\kappa_*|^2} R^2}.
\end{aligned} \tag{4.12}
$$

(4.10) is obtained by taking the product of (4.3) and (4.12). This shows the claim (4.7). The lemma follows as $\zeta_{\triangle}^{l\mathbf{m}+t\mathbf{n}}.(\mathbf{z}+\mathbf{v}) \in A$. $\square$

Proof of Lemma 4.6. By Lemma 4.8, $\forall n \in \mathbb{N}$, $\exists \mathbf{y}_n \in X^2$ and $w_n \in V_i$ with $|w_n| = 1$ such that if we denote $\mathbf{w}_n = (w_n, \kappa_* w_n) \in V_i^{\square}$ then there exists a point $\mathbf{y}_{n,\rho}$ in $A \cap B_{\frac{1}{n}}(\mathbf{y}_n + \rho \mathbf{w}_n)$ for all $\rho \in [-n, n]$.

Since both X^2 and the unit circle in V_i (which is just $\{\pm 1\}$ if $V_i \cong \mathbb{R}$) are compact, there is a subsequence $\{n_k\}_{k=1}^{\infty}$ such that as k tends to ∞, $\mathbf{y}_{n_k} \to \mathbf{y}$ and $w_{n_k} \to w$ for some limits $\mathbf{y} \in X^2$ and $w \in V_i$ with $|w| = 1$.

Set $\mathbf{w} = (w, \kappa_* w) \in V_i^{\square}$. Now for any $\rho \in \mathbb{R}$ and $\epsilon > 0$, choose k such that $n_k \geq \max(\rho, \frac{3}{\epsilon})$, $\|\mathbf{y}_{n_k} - \mathbf{y}\| \leq \frac{\epsilon}{3}$ and $|w_{n_k} - w| \leq \frac{\epsilon}{3|\rho|\sqrt{1+|\kappa_*|^2}}$. Then $|\mathbf{w}_{n_k} - \mathbf{w}| = |(w_{n_k} - w, \kappa_*(w_{n_k} - w))| = \sqrt{1+|\kappa_*|^2} \cdot |w_{n_k} - w| \leq \frac{\epsilon}{3|\rho|}$. By assumption, there exists a point $\mathbf{y}_{n_k,\rho} \in A$ within distance $\frac{1}{n_k}$ from $\mathbf{y}_{n_k} + \rho \mathbf{w}_{n_k}$. Then the distance between $\mathbf{y}_{n_k,\rho}$ and $\mathbf{y} + \rho \mathbf{w}$ is bounded by

$$\begin{aligned} &\|\mathbf{y}_{n_k,\rho} - (\mathbf{y}_{n_k} + \rho\mathbf{w}_{n_k})\| + \|(\mathbf{y}_{n_k} + \rho\mathbf{w}_{n_k}) - (\mathbf{y} + \rho\mathbf{w})\| \\ \leq &\frac{1}{n_k} + \big(\|\mathbf{y}_{n_k} - \mathbf{y}\| + |\rho| \cdot |\mathbf{w}_{n_k} - \mathbf{w}|\big) \\ \leq &(\frac{3}{\epsilon})^{-1} + \frac{\epsilon}{3} + |\rho| \cdot \frac{\epsilon}{3|\rho|} = \epsilon. \end{aligned} \tag{4.13}$$

The proof is completed. □

4.3. **Construction of a homogeneous invariant set.**

Lemma 4.9. *Suppose a closed subset $A \subset X^2$ is $\zeta_\triangle$-invariant and contains two points $\mathbf{z}$ and $\mathbf{z}+\mathbf{v}$ where $\mathbf{z}$ is a torsion point and $\mathbf{v}$ is a nonzero vector from $V_i^\square$ for some $i \in I$, then:*

(1) If $\exists \kappa \in K \cup \{\infty\}$ such that $\mathbf{v} \in V_i^\kappa$ then the d-dimensional homogeneous $\zeta_\triangle$-invariant subset $L = \{\zeta_\triangle^{\mathbf{n}}.\mathbf{z}' : \mathbf{n} \in \mathbb{Z}^r, \mathbf{z}' \in \mathbf{z} + T^\kappa\}$, which contains both $\mathbf{z}$ and $\mathbf{z}+\mathbf{v}$, is a subset in A.

(2) If there doesn't exist such a slope κ then $A = X^2$.

For the construction of V_i^κ, see Definition 3.7.

Proof. (1) If $\mathbf{v} \in V_i^\kappa$ where $\kappa \in K$ or $\kappa = \infty$, construct L as in the statement of lemma. Then L is a d-dimensional homogeneous $\zeta_\triangle$-invariant subset and obviously $\mathbf{z} \in L$. By Lemma 3.8, $\mathbf{z}+\mathbf{v} \in \mathbf{z}+T^\kappa \subset L$ as $\mathbf{v} \in V^\kappa$.

We want to show from the fact $\mathbf{z}+\mathbf{v} \in L$ that $L \subset A$. As A is closed and $\zeta_\triangle$-invariant, using Proposition 3.14.(3) it suffices to prove $\mathbf{z}+\mathbf{v}$ is not a torsion point. As $\mathbf{z}$ is of torsion, this reduces to prove that $\pi_\triangle(\mathbf{v}) = \big(\pi(v^{(1)}), \pi(v^{(2)})\big)$ is not a torsion point. Because $\mathbf{v} \neq 0$, there is at least one of $v^{(1)}$, $v^{(2)}$ that doesn't vanish, so it is enough to show that $\pi(v) \in X$ is not a torsion point of X for any nonzero vector $v \in V_i$. Suppose this is not true then there is a nonzero integer q such that $qv \in \Gamma$. But $\Gamma \subset \sigma(K)$, hence $qv = \sigma(\gamma)$ where $\gamma \in K$. In particular, as the V_j coordinate of $qv \in V_i$ vanishes, $\sigma_j(\gamma) = 0$ for all $j \in I\backslash\{i\}$ (because $r_1 + r_2 = r + 1 \geq 3$, there is always another index $j \neq i$ in I). But this can happen only if $\gamma = 0$, in consequence $qv = 0$ and thus $v = 0$, which provides a contradiction. This shows part (1) of the lemma.

(2) Suppose $\mathbf{v} \notin V_i^\kappa, \forall \kappa \in K \cup \{\infty\}$. As before, let $\mathbf{v} = (v^{(1)}, v^{(2)})$ where $v^{(1)}, v^{(2)} \in V_i$, which is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. As $\mathbf{v} \notin V_i^\infty = \{0\} \times V_i$, $v^{(1)} \neq 0$. So there is a number κ_* from either $\mathbb{R}$ or $\mathbb{C}$, whichever V_i is isomorphic to, such that $v^{(2)} = \kappa_* v^{(1)}$.

Hence by the assumption of $\mathbf{v}$, $\kappa_* \notin \sigma_i(K)$. In fact by Definition 3.7, if $\kappa_* = \sigma_i(\kappa)$ for some $\kappa \in K$ then $\mathbf{v} \in V_i^\kappa$.

By Lemma 4.6, A contains an infinitely long line $\{\mathbf{y} + \rho\mathbf{w} : \rho \in \mathbb{R}\}$ where $\mathbf{y} \in X^2$ and $\mathbf{w}$ can be written as $(w, \kappa_* w)$ for some nonzero vector $w \in V_i$. Furthermore by Remark 2.3, $\exists \mathbf{n} \in \mathbb{Z}^r$ such that $\forall l \in \mathbb{Z}\backslash\{0\}$, $\zeta^{l\mathbf{n}}$ is not in any proper subfield of K. It follows from Lemma 4.10, which is stated and proved below, that the set $\bigcup_{l=1}^{\infty}\{\zeta_\triangle^{l\mathbf{n}}.(\mathbf{y} + \rho\mathbf{w}) : \rho \in \mathbb{R}\}$ is dense in X^2. Because this union is contained in the $\zeta_\triangle$-invariant subset A, A must be X^2. □

Lemma 4.10. *Suppose $\kappa_* \in \mathbb{R}$ if $1 \leq i \leq r_1$, $\kappa_* \in \mathbb{C}$ if $r_1 < i \leq r_1 + r_2$, and $\kappa_* \notin \sigma_i(K)$. For $\mathbf{y} \in X^2$ and $w \in V_i\backslash\{0\}$, denote $\mathbf{w} = (w, \kappa_* w)$. Suppose in addition $\mathbf{n} \in \mathbb{Z}^r$ satisfies that $\zeta^{l\mathbf{n}}$ is not in any proper subfield of K for all nonzero integer l, then $\bigcup_{l=1}^{\infty}\{\zeta_\triangle^{l\mathbf{n}}.(\mathbf{y} + \rho\mathbf{w}) : \rho \in \mathbb{R}\}$ is dense in X^2.*

Proof. Assume the union is not dense, then there is a ball $\mathring{B}_\epsilon(\mathbf{x}) \subset X^2$ which is disjoint from $\{\zeta_\triangle^{l\mathbf{n}}.(\mathbf{y} + \rho\mathbf{w}) : \rho \in \mathbb{R}\}$ for all $l \in \mathbb{N}$. Since X^2 is compact, it can be covered by finitely many balls of radius $\frac{\epsilon}{2}$. In consequence there is a ball $\mathring{B}_{\frac{\epsilon}{2}}(\mathbf{x}')$ that contains $\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{y}$ for a subsequence l_k. Since $\zeta_\triangle^{l_k\mathbf{n}}.(\mathbf{y} + \rho\mathbf{w}) = \zeta_\triangle^{l_k\mathbf{n}}.\mathbf{y} + \zeta_\triangle^{l_k\mathbf{n}}.\rho\mathbf{w}$, it follows that for each l_k, $\{\zeta_\triangle^{l_k\mathbf{n}}.\rho\mathbf{w} : \rho \in \mathbb{R}\}$ is disjoint from $\mathring{B}_{\frac{\epsilon}{2}}(\mathbf{x} - \mathbf{x}')$. Hence the closure $P_{l_k} := \overline{\{\zeta_\triangle^{l_k\mathbf{n}}.\rho\mathbf{w} : \rho \in \mathbb{R}\}}$ avoids $\mathring{B}_{\frac{\epsilon}{2}}(\mathbf{x} - \mathbf{x}')$, in particular the sequence of subsets $\{P_{l_k}\}_{k=1}^{\infty}$ does not converge to X^2 in the Hausdorff metric.

On the other hand, it should be remarked that the P_{l_k} are all closed subgroups of X^2. Hence by [Ber83, Lemma 4.7], there is a non-trivial character $\boldsymbol{\xi} \in \widehat{X^2}$ which is in the annihilator of P_{l_k} for sufficiently large k. In particular

$$\langle \boldsymbol{\xi}, \pi_\triangle(\zeta_\triangle^{l_k\mathbf{n}}.\rho\mathbf{w})\rangle = 0 \pmod{\mathbb{Z}}, \forall \rho \in \mathbb{R} \tag{4.14}$$

for all k large enough.

Since $w \in V_i$, when regarded as a vector in $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$, its i-th coordinate is a nonzero number $w_i \in \mathbb{R}$ or $\mathbb{C}$ while all other coordinates w_j's are equal to 0.

Write $\boldsymbol{\xi} = (\xi^{(1)}, \xi^{(2)})$. We claim that $\xi_i^{(1)} + \kappa_* \xi_i^{(2)} = 0$. To prove this we distinguish between real and complex V_i's.

Case 1. When $i \le r_1$, $V_i \cong \mathbb{R}$. Fix an arbitrary l_k which is large enough, then (4.14) holds. By the duality formulas (2.33) and (2.29),

$$\begin{aligned}\langle \boldsymbol{\xi}, \pi_\triangle(\zeta_\triangle^{l_k\mathbf{n}}.\rho\mathbf{w})\rangle =&\langle \xi^{(1)}, \pi(\zeta^{l_k\mathbf{n}}.\rho w)\rangle + \langle \xi^{(2)}, \pi(\zeta^{l_k\mathbf{n}}.\rho\kappa_* w)\rangle \\ =&(\xi_i^{(1)}\zeta_i^{l_k\mathbf{n}}\rho w_i + \xi_i^{(2)}\zeta_i^{l_k\mathbf{n}}\rho\kappa_* w_i \bmod \mathbb{Z}) \\ =&\left(\rho\big(\xi_i^{(1)} + \kappa_*\xi_i^{(2)}\big)\zeta_i^{l_k\mathbf{n}} w_i \bmod \mathbb{Z}\right).\end{aligned} \tag{4.15}$$

Since $w_i \neq 0$, $\zeta_i^{l_k\mathbf{n}} \neq 0$ (as it is an algebraic unit) and ρ is an arbitrary real number, (4.14) cannot be true unless $\xi_i^{(1)} + \kappa_*\xi_i^{(2)} = 0$.

Case 2. Suppose now $r_1 < i \le r_1 + r_2$, in which case σ_i is a complex embedding and $V_i \cong \mathbb{C}$. Then by (2.33) and (2.29),

$$\begin{aligned}\langle \boldsymbol{\xi}, \pi_\triangle(\zeta_\triangle^{l_k\mathbf{n}}.\rho\mathbf{w})\rangle =&\big(\operatorname{Re}(\xi_i^{(1)}\zeta_i^{l_k\mathbf{n}}\rho w_i) + \operatorname{Re}(\xi_i^{(2)}\zeta_i^{l_k\mathbf{n}}\rho\kappa_* w_i) \bmod \mathbb{Z}\big) \\ =&\Big(\rho \operatorname{Re}\big((\xi_i^{(1)} + \kappa_*\xi_i^{(2)})\zeta_i^{l_k\mathbf{n}} w_i\big) \bmod \mathbb{Z}\Big).\end{aligned} \tag{4.16}$$

Hence for any sufficiently large k, it follows from (4.14) that the expression $\operatorname{Re}\big((\xi_i^{(1)} + \kappa_*\xi_i^{(2)})\zeta_i^{l_k\mathbf{n}} w_i\big)$ must vanish.

Assume first $\xi_i^{(1)} + \kappa_*\xi_i^{(2)} \neq 0$, then there is a nonzero number $u \in \mathbb{C}$ such that for $z \in \mathbb{C}$, $\operatorname{Re}\big((\xi_i^{(1)} + \kappa_*\xi_i^{(2)})z\big) = 0$ if and only if $\frac{z}{u} \in \mathbb{R}$. Hence $\zeta_i^{l_k\mathbf{n}} w_i \in \mathbb{R}u$ for large k. In consequence, if we fix two different large terms l_{k_1} and l_{k_2} from the subsequence $\{l_k\}$, then $\zeta_i^{(l_{k_1}-l_{k_2})\mathbf{n}} = \dfrac{\zeta_i^{l_{k_1}\mathbf{n}} w_i}{\zeta_i^{l_{k_2}\mathbf{n}} w_i} \in \mathbb{R}$. In other words, $\zeta^{(l_{k_1}-l_{k_2})\mathbf{n}} \in \sigma_i^{-1}(\mathbb{R})$. But since σ_i is a complex embedding, $\sigma_i^{-1}(\mathbb{R})$ is a proper subfield of K. Because $l_{k_1} \neq l_{k_2}$, this contradicts the assumption on $\mathbf{n}$. Therefore $\xi_i^{(1)} + \kappa_*\xi_i^{(2)} = 0$.

So we proved that $\xi_i^{(1)} + \kappa_*\xi_i^{(2)} = 0$ always holds. Recall $\xi^{(1)}, \xi^{(2)} \in \hat{X}$, and thus, by Proposition 2.9, can be respectively represented by vectors $\sigma(\epsilon^{(1)}), \sigma(\epsilon^{(2)}) \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ where $\epsilon^{(1)}, \epsilon^{(2)} \in K$.

Notice $\epsilon^{(2)} \neq 0$. In fact, otherwise $\xi_i^{(2)} = \sigma_i(\epsilon^{(2)})$ and $\xi_i^{(1)} = -\kappa_*\sigma(\epsilon^{(2)})$ are both zero; since $\xi_i^{(1)} = \sigma_i(\epsilon^{(1)})$ we know $\epsilon^{(1)} = 0$ as well. Thus $\boldsymbol{\xi} = \big(\sigma(\epsilon^{(1)}), \sigma(\epsilon^{(2)})\big)$ is the trivial character on X^2, which contradicts the choice of $\boldsymbol{\xi}$.

So $\xi_i^{(2)} = \sigma_i(\epsilon^{(2)}) \in \sigma_i(K)\backslash\{0\}$. And $\kappa_* = -\frac{\xi_i^{(1)}}{\xi_i^{(2)}} = \sigma_i(-\frac{\epsilon^{(1)}}{\epsilon^{(2)}}) \in \sigma_i(K)$. This produces a contradiction to the hypothesis on κ_*, hence establishes the lemma. □

Now we show the main proposition of this section.

Proof of Proposition 4.1. Given an infinite $\zeta_\triangle$-invariant closed subset A, by Corollary 4.5 there is an index $i \in I$ satisfying the condition in Lemma 4.9. By applying the lemma, we see A always contains a d-dimensional homogeneous $\zeta_\triangle$-invariant subset. □

5. Actions of rank 3 or higher

We now establish the first half of Theorem 1.6 as well as Theorem 1.7. By Lemma 3.18 it suffices to prove Theorem 3.16.(1) and Theorem 3.17.

Throughout this section, assume $r \geq 3$ and $A \subset X^2$ satisfies the following assumption:

Condition 5.1. *A is an infinite $\zeta_\triangle$-invariant subset and for any d-dimensional homogeneous $\zeta_\triangle$-invariant subset L contained in A, $A \backslash L$ is dense in A.*

We are going to show that:

Proposition 5.2. *When $r \geq 3$, if A satisfies Condition 5.1 then $A = X^2$.*

5.1. A self-returning property.

To begin with, we claim an inductive fact:

Lemma 5.3. *Suppose $r \geq 3$ and A satisfies Condition 5.1. If A contains a finite but non-empty collection of d-dimensional homogeneous $\zeta_\triangle$-invariant subsets $L_0, \cdots, L_q$, then it contains at least one more d-dimensional homogeneous $\zeta_\triangle$-invariant set L_{q+1} which is different from any of these.*

The proof of the lemma is the topic of subsections 5.1 and 5.2. For now we prove Proposition 5.2 assuming the lemma.

Proof of Proposition 5.2. By Proposition 4.1, A contains at least one d-dimensional homogeneous $\zeta_\triangle$-invariant subset. Fix such a subset L_0. By applying Lemma 5.3 repeatedly, we see A contains infinitely many d-dimensional homogeneous $\zeta_\triangle$-invariant subsets. By Lemma 3.15, A is ϵ-dense in X^2 for any $\epsilon > 0$. Since A is closed, it must be X^2. □

Before showing Lemma 5.3, let us fix a few additional notations.

From now on, suppose $q \geq 0$ and $L_0, L_1, L_2, \cdots, L_q$ are d-dimensional homogeneous $\zeta_\triangle$-invariant subsets, all different from each other and contained in A. Then there are $\kappa_0, \kappa_1, \cdots, \kappa_q \in K \cup \{\infty\}$ such that $L_h = \bigsqcup_{t=1}^{s_h} L_{h,t}, \forall h = 0, .., q$ where $s_h \geq 1$ and each $L_{h,t}$ is a translate of T^{κ_h} by a torsion element of X^2.

Set $E = \big(\bigcup_{h=1}^q L_h\big) \cap L_{0,1}$.

Lemma 5.4. *E is a finite set of torsion points.*

Proof. Since $E = \bigcup_{h=1}^q \bigcup_{t=1}^{s_h} (L_{h,t} \cap L_{0,1})$, it suffices to show $L_{h,t} \cap L_{0,1}$ is finite for all pairs (h,t) with $h \geq 1$. Recall $L_{h,t}$ is a translate of T^{κ_h}. If $\kappa_h \neq \kappa_0$ then the finiteness is claimed by Corollary 3.5. If $\kappa_h = \kappa_0$ then $L_{h,t}$ and $L_{0,1}$ are parallel, hence have a non-empty intersection if and only if they coincide with each other. But this would imply $L_h = L_0$ as L_h and L_0 are respectively the full $\zeta_\triangle$-orbits of $L_{h,t}$ and $L_{0,1}$. By choice $L_h \neq L_0$, hence $L_{h,t} \cap L_{0,1} = \emptyset$ when $\kappa_h = \kappa_0$.

Fix a torsion point $\mathbf{z}$ such that $L_{0,1} = \mathbf{z} + T^{\kappa_0}$. Then as T^{κ_0} is $\zeta_\triangle$-invariant, $L_{0,1} = \mathbf{z} + T^{\kappa_0}$ is preserved by $\mathrm{Stab}_{\zeta_\triangle}(\mathbf{z})$ under $\zeta_\triangle$.

As all the L_h's are $\zeta_\triangle$-invariant, $\mathrm{Stab}_{\zeta_\triangle}(\mathbf{z})$ stabilizes E under $\zeta_\triangle$. Thus it stabilizes the subset $-\mathbf{z} + E \subset -\mathbf{z} + L_{0,1} = T^{\kappa_0}$ as well. By applying Lemma 3.11 to $-\mathbf{z} + E$ and the finite-index subgroup $\mathrm{Stab}_{\zeta_\triangle}(\mathbf{z}) < \mathbb{Z}^r$, we see that $-\mathbf{z} + E$ consists of torsion points. Since $\mathbf{z}$ is of torsion, so is any point in E; which finishes the proof. $\square$

It should be remarked that E may be empty. In the rest of this section, write

$$H = \begin{cases} \bigcap_{\mathbf{x} \in E} \mathrm{Stab}_{\zeta_\triangle}(\mathbf{x}), & \text{if } E \neq \emptyset; \\ \mathrm{Stab}_{\zeta_\triangle}(\mathbf{z}_{0,1}), & \text{if } E = \emptyset. \end{cases} \tag{5.1}$$

where $\mathbf{z}_{0,1}$ is any torsion point in $L_{0,1}$.

Then H preserves $L_{0,1}$ under $\zeta_\triangle$ in any case because as long as H stabilizes any $\mathbf{x} \in L_{0,1}$ it stabilizes $L_{0,1} = \mathbf{x} + T^{\kappa_0}$ too. Moreover, H is a finite index subgroup of $\mathbb{Z}^r$. When E is empty this is clear. Suppose $E \neq \emptyset$, then as each $\mathbf{x}$ is of torsion, all the $\mathrm{Stab}_{\zeta_\triangle}(\mathbf{x})$'s have finite index in $\mathbb{Z}^r$ and so does H.

Clearly E is preserved by the subgroup H.

Remark 5.5. *Since $L_{0,1}$ is a translate of the d-dimensional subtorus T^{κ_0} in $X^2 \cong \mathbb{T}^{2d}$, for any sufficiently small positive number $\rho > 0$, if $\mathbf{x} \in L_{0,1}$ and $\mathbf{v} \in (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2$ satisfies $\mathbf{x} + \mathbf{v} \in L_{0,1}$ and $|\mathbf{v}| < \rho$ then $\mathbf{v}$ must be in V^{κ_0}, which is the tangent space of $L_{0,1}$.*

We define the **injective radius** $\mathrm{rad}_{X^2}(L_{0,1})$ *to be the largest such number ρ.*

So for any $\theta \in \big(0, \mathrm{rad}_{X^2}(L_{0,1})\big)$, the set

$$E_\theta = \bigcup_{\mathbf{x} \in E} (\mathring{B}_\theta(\mathbf{x}) \cap L_{0,1}) \tag{5.2}$$

is equal to $\{\mathbf{x}+\mathbf{v} : \mathbf{x} \in E, \mathbf{v} \in V^{\kappa_0}, |\mathbf{v}| < \theta\}$, the open θ-neighborhood of E in $L_{0,1}$. What we are trying to show is that when θ is sufficiently small, the complement $L_{0,1}\backslash E_\theta$ has the following self-returning property:

Lemma 5.6. *There are positive constants C and θ_0, which may depend on $L_{0,1}$ and E, such that $\forall i \in I$, $\forall \mathbf{x} \in L_{0,1}\backslash E_\theta$, $\forall \theta \in (0, \theta_0)$, $\exists \mathbf{n} \in H$ such that:*
(1) $C < \lambda_i(\mathbf{n}) < 2C$ and $\lambda_j(\mathbf{n}) < \frac{C}{2}$ for all $j \in I\backslash\{i\}$;
(2) $\zeta^{\mathbf{n}}.\mathbf{x} \in L_{0,1}\backslash E_\theta$.

Proof. The proof goes in two steps.

Step 1. We show that there exists $C > 2\log(r_1 + r_2)$ and $\mathbf{n}_{ik} \in H$ for all pairs $i, k \in I$ with $i \neq k$ such that

$$\begin{cases} C < \lambda_i(\mathbf{n}_{ik}) < 2C, \\ \dfrac{C}{4} < \lambda_k(\mathbf{n}_{ik}) < \dfrac{C}{2}, \\ -4C < \lambda_j(\mathbf{n}_{ik}) < \dfrac{C}{2}, \forall k \in I\backslash\{i,j\}. \end{cases} \tag{5.3}$$

This claim follows from two facts. The first fact is that, because $\mathcal{L}(H)$ is a full-rank lattice in W (see Remark 2.5) where $\mathcal{L}$ and W were constructed in §2.2, there exists $b > 0$ such that any ball of radius b in W contains at least one point from $\mathcal{L}(H)$.

Second, consider the subset

$$\begin{aligned}\Omega_{ik} =& \{(w_i)_{i\in I} : \sum_{i\in I} d_i w_i = 0, w_i \in (1,2), \\ & w_k \in (\frac{1}{4}, \frac{1}{2}), w_j \in (-4, \frac{1}{2}) \text{ for any other } j\}\end{aligned} \tag{5.4}$$

of W. Recall that d_i is the real dimension of V_i, which is either 1 or 2 depending on whether $i \le r_1$ or not. Then Ω_{ik} is open, as it is defined by open conditions, and non-empty, as it contains the point given by $w_i = \frac{4}{3}$, $w_k = \frac{1}{3}$ and $w_j = -\frac{4d_i+d_k}{3(d-d_i-d_k)}$ for all the other j's (note this is possible because $d_i, d_k \le 2$ and $d - d_i - d_k = \sum_{j\in I\backslash\{i,k\}} d_j \ge |I| - 2 = (r+1) - 2 \ge 1$ as long as $r \ge 2$).

When the constant C is chosen to be large enough, $C > 2\log(r_1+r_2)$ and $C\Omega_{ik}$ contains a ball of radius b for all pairs (i,k). Thus by the choice of b, $\exists \mathbf{n}_{ik}$ such that $\mathcal{L}(\mathbf{n}_{ik}) \in C\Omega_{ik}$. By using the definition of $\mathcal{L}$, it turns out this is equivalent to the inequalities (5.3).

Step 2. The next claim is that:

($\star$) *$\exists\theta_0 > 0$ such that $\forall\theta \in (0, \theta_0)$, $\forall \mathbf{x} \in L_{0,1} \backslash E_\theta$, $\forall i \in I$, there exists $k \in I \backslash \{i\}$ such that $\zeta^{\mathbf{n}_{ik}}.\mathbf{x} \in L_{0,1} \backslash E_\theta$.*

Clearly, together with (5.3) this would imply the lemma.

Remark if the length of a nonzero vector $\mathbf{v} \in (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2$ is less than $\min_{\substack{\boldsymbol{\gamma}\in\Gamma^2\\ \boldsymbol{\gamma}\neq\mathbf{0}}} |\boldsymbol{\gamma}|$ then $\pi_\triangle(\mathbf{v}) \neq \mathbf{0}$. On the other hand E is a finite subset of $L_{01} = \mathbf{z}_{01} + T^{\kappa_0}$, so the number $\min_{\substack{\mathbf{y},\mathbf{y}'\in E\\ \mathbf{y}\neq\mathbf{y}'}} \|\mathbf{y} - \mathbf{y}'\|$ is strictly positive. Define a positive number

$$\rho = \min\Big(\min_{\substack{\boldsymbol{\gamma}\in\Gamma^2\\ \boldsymbol{\gamma}\neq\mathbf{0}}} |\boldsymbol{\gamma}|, \min_{\substack{\mathbf{y},\mathbf{y}'\in E\\ \mathbf{y}\neq\mathbf{y}'}} \|\mathbf{y} - \mathbf{y}'\|\Big). \tag{5.5}$$

It is easy to verify that if $\mathbf{v} \in V^{\kappa_0}$ satisfies $0 < |\mathbf{v}| < \rho$ then $\forall \mathbf{y} \in E, \mathbf{y} + \mathbf{v} \notin E$.

We prove the claim ($\star$) for

$$\theta_0 = \frac{\rho}{2e^{6C}}. \tag{5.6}$$

Suppose the claim fails for some $\mathbf{x} \in L_{0,1} \backslash E_\theta$ and $i \in I$ where $\theta < \theta_0$. Take an arbitrary $k \in I$ which is different from i, then $\zeta^{\mathbf{n}_{ik}}.\mathbf{x} \notin L_{0,1} \backslash E_\theta$. Since $\mathbf{n}_{ik} \in H$, $\zeta_\triangle^{\mathbf{n}_{ik}}.L_{0,1} = L_{0,1}$. So $\zeta^{\mathbf{n}_{ik}}.\mathbf{x} \in L_{0,1}$, and in consequence $\zeta^{\mathbf{n}_{ik}}.\mathbf{x} \in E_\theta$, in other words $\exists \mathbf{y}_k \in E$ such that $\|\zeta^{\mathbf{n}_{ik}}.\mathbf{x} - \mathbf{y}_k\| < \theta$. In this case $E \neq \emptyset$, hence because $\mathbf{n}_{ik} \in H$, $\zeta_\triangle^{-\mathbf{n}_{ik}}.\mathbf{y}_k = \mathbf{y}_k$ by construction of H. Thus

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}_k\| =& \|\zeta_\triangle^{-\mathbf{n}_{ik}}.(\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{x} - \mathbf{y}_k)\| \\ \leq& e^{\max_{j\in I}(-\lambda_j(\mathbf{n}_{ik}))}\|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{x} - \mathbf{y}_k\| \quad \text{(by (2.12))} \\ <& e^{4C}\theta < e^{4C}\theta_0 \\ =& \frac{\rho}{2e^{2C}} \end{aligned} \tag{5.7}$$

So for $k \neq k'$,

$$\|\mathbf{y}_k - \mathbf{y}_{k'}\| \leq \|\mathbf{x} - \mathbf{y}_k\| + \|\mathbf{x} - \mathbf{y}_{k'}\| < \rho \leq \min_{\substack{\mathbf{y},\mathbf{y}'\in E\\ \mathbf{y}\neq\mathbf{y}'}} \|\mathbf{y} - \mathbf{y}'\|. \tag{5.8}$$

Therefore all the $\mathbf{y}_k$'s are the same, which we write as $\mathbf{y}$. Then $\|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{x} - \mathbf{y}\| < \theta$ for all $k \in I \backslash \{i\}$.

By (5.7), there is a vector $\mathbf{v}$ from $(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2$ (actually from V^{κ_0}) such that $\mathbf{x} - \mathbf{y} = \pi_\triangle(\mathbf{v})$ and $|\mathbf{v}| < \frac{\rho}{2e^{4C}}$.

Note $\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{x} = \mathbf{y} + \zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}$. By (2.12), $|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}| \leq e^{\max_{j\in I}\lambda_j(\mathbf{n}_{ik})}\|\mathbf{v}\|$. Plug in (5.3); we get $|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}| \leq e^{2C} \cdot \frac{\rho}{2e^{2C}} = \frac{\rho}{2}$. On the other hand, we know $\|\pi_\triangle(\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v})\| = \|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{x} - \mathbf{y}\| < \theta$, in other words, $\exists \boldsymbol{\gamma} \in \Gamma^2$ such that $|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}-\boldsymbol{\gamma}|<\theta$. So $|\boldsymbol{\gamma}|<|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}-\boldsymbol{\gamma}|+|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}|<\frac{\rho}{2}+\theta<\frac{\rho}{2}+\frac{\rho}{2}=\rho$. By construction of ρ, $\boldsymbol{\gamma}$ must be the zero vector. Hence

$$|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}|<\theta, \forall k\in I\backslash\{i\} \tag{5.9}$$

Write $\mathbf{v}$ as $\sum_{j\in I}\mathbf{v}_j$ where $\mathbf{v}_j\in V_j^{\square}$ (actually $\mathbf{v}_j\in V_j^{\kappa_0}$). Then by (2.18) the length of $\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}_k$ is $|\zeta_i^{\mathbf{n}_{ik}}|\cdot|\mathbf{v}_k|=e^{\lambda_k(\mathbf{n}_{ik})}|\mathbf{v}_k|\geq e^{\frac{C}{4}}|\mathbf{v}_k|$, where the last inequality follows from (5.3). Similarly $|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}_i|=e^{\lambda_i(\mathbf{n}_{ik})}|\mathbf{v}_i|\geq e^C|\mathbf{v}_i|$. Thus we see

$$\begin{aligned}|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}| =&\Big(\sum_{j\in I}|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}_j|^2\Big)^{\frac12}\\ \geq&\Big(|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}_i|^2+|\zeta_\triangle^{\mathbf{n}_{ik}}.\mathbf{v}_k|^2\Big)^{\frac12}\\ \geq&\Big((e^C|\mathbf{v}_i|)^2+(e^{\frac{C}{4}}|\mathbf{v}_k|)^2\Big)^{\frac12}\\ \geq&\Big((r_1+r_2)|\mathbf{v}_i|^2+(r_1+r_2)|\mathbf{v}_k|^2\Big)^{\frac12},\end{aligned} \tag{5.10}$$

where in the last step we used the fact $C>2\log(r_1+r_2)$.

It follows from (5.9) and (5.10) that $|\mathbf{v}_i|<\frac{\theta^2}{r_1+r_2}$ as well as $|\mathbf{v}_k|<\frac{\theta^2}{r_1+r_2}, \forall k\in I\backslash\{i\}$. Thus $|\mathbf{v}|=(\sum_{j\in I}|\mathbf{v}_j|^2)^{\frac12}<\big((r_1+r_2)\cdot\frac{\theta^2}{r_1+r_2}\big)^{\frac12}=\theta$. Recall $\mathbf{x}=\mathbf{y}+\mathbf{v}$ with $\mathbf{y}\in E$, so $\mathbf{x}\in E_\theta$; which contradicts the assumption on the point x. Thus the claim ($\star$) is proved and the lemma follows. ☐

Remark 5.7. *As the assumption $r\geq 3$ was not used in the proof, Lemma 5.6 actually holds for all $r\geq 2$.*

5.2. **Construction of an extra homogeneous invariant subset.** In this part let A, $L_1,\cdots,L_q$, E and E_θ be as in §5.1. Now we construct a new homogeneous invariant subset L_{q+1} contained in A.

Recall that $L_{0,1}$ is parallel to T^{κ_0} where $\kappa_0\in K$, hence it tangent space is V^{κ_0}, which is transversal to V^∞. When $A\supsetneq L_0$ we create points that deviate from $L_{0,1}$ along V^∞ direction.

Lemma 5.8. *For all sufficiently small $\theta>0$ and all $\delta>0$, if $\tau\in K\cup\{\infty\}$ is different from κ_0 then there are $i\in I$, $\mathbf{y}\in L_{0,1}\backslash E_\theta$ and a nonzero vector $\mathbf{w}\in V_i^\tau$ with $|\mathbf{w}|\leq\delta$, such that $\mathbf{y}+\mathbf{w}\in A$.*

Proof. Let C and θ_0 be as in Lemma 5.6 and choose $\theta\in(0,\theta_0)$ so that the difference set $L_{0,1}\backslash E_{2\theta}$ is not empty, in which we fix a point $\mathbf{x}$.

By Condition 5.1, since $\mathbf{x}\in L_0$ there is a sequence of points $\mathbf{x}'_m\in A\backslash L_0$ converging to $\mathbf{x}$. We can write each $\mathbf{x}'_m$ as $\mathbf{x}+\mathbf{w}'_m$ where $\mathbf{w}'_m\in(\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2})^2$ and $\lim_{m\to\infty}\mathbf{w}'_m=\mathbf{0}$.

As $\tau \neq \kappa_0$, by Corollary 3.9, $(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2 = V^{\kappa_0} \oplus V^{\kappa}$. Thus the vector $\mathbf{w}'_m$ can be uniquely decomposed into $(\mathbf{w}'_m - \mathbf{w}_m) + \mathbf{w}_m$ where $\mathbf{w}_m \in V^\tau$ and $\mathbf{w}'_m - \mathbf{w}_m \in V^{\kappa_0}$, moreover both $|\mathbf{w}'_m - \mathbf{w}_m|$ and $|\mathbf{w}_m|$ converge to $\mathbf{0}$ as $m \to \infty$.

Note $\mathbf{w}_m \neq \mathbf{0}$ because otherwise $\mathbf{w}'_m \in V^{\kappa_0}$ and $\mathbf{x}'_m$ would belong to $L_{0,1} = \mathbf{x} + T^{\kappa_0}$. By neglecting finitely many terms at the beginning of the sequence, we may assume

$$|\mathbf{w}'_m - \mathbf{w}_m| < \theta, 0 < |\mathbf{w}_m| < \delta, \ \forall m. \tag{5.11}$$

$\mathbf{w}_m$ has an unique decomposition $\mathbf{w}_m = \sum_{j\in I} (\mathbf{w}_m)_j$ where $(\mathbf{w}_m)_j \in V_j^\tau = V^\tau \cap V_j^\square$. Then $|\mathbf{w}_m| = \big(\sum_{i\in I} |(\mathbf{w}_m)_j|^2\big)^{\frac{1}{2}}$ and $\lim_{m\to\infty} (\mathbf{w}_m)_j = 0, \forall j$. Passing to a subsequence if neccessary, we may suppose without loss of generality that there is an index $i \in I$ such that

$$|(\mathbf{w}_m)_i| \geq |(\mathbf{w}_m)_j|, \ \forall m \in \mathbb{N}, \forall j \in I\backslash\{i\}. \tag{5.12}$$

In particular $|(\mathbf{w}_m)_i| > 0$.

Let $\mathbf{x}_m = \mathbf{x} + (\mathbf{w}'_m - \mathbf{w}_m)$, which is in $L_{0,1}$ because $\mathbf{w}'_m - \mathbf{w}_m \in V^\kappa$. As $|\mathbf{w}'_m - \mathbf{w}_m| < \theta$ it follows from $\mathbf{x} \notin E_{2\theta}$ that $\mathbf{x}_m \notin E_\theta$.

Starting from such a point $\mathbf{x}_m$, Lemma 5.6 allows us to construct a sequence of points $\{\mathbf{x}_{m,l}\}_{l=1}^\infty \subset L_{0,1}\backslash E_\theta$ such that $\mathbf{x}_{m,0} = \mathbf{x}_m$ and $\mathbf{x}_{m,l+1} = \zeta_\triangle^{\mathbf{n}_{m,l+1}}.\mathbf{x}_{m,l} = \zeta_\triangle^{\sum_{k=1}^l \mathbf{n}_{m,k}}.\mathbf{x}_m, \forall m, l \in \mathbb{N}$ where $\mathbf{n}_{m,l} \in \mathbb{Z}^r$ satisfies $C < \lambda_i(\mathbf{n}_{m,l}) < 2C$ and $\lambda_j(\mathbf{n}_{m,l}) < \frac{C}{2}, \forall j \in I\backslash\{i\}$.

Construct a corresponding sequence $\{\mathbf{w}_{m,l}\}_{l=1}^\infty$ by $\mathbf{w}_{m,0} = \mathbf{w}_m$ and $\mathbf{w}_{m,l+1} = \zeta_\triangle^{\mathbf{n}_{m,l+1}}.\mathbf{w}_{m,l}$. Then $\mathbf{w}_{m,l} = \zeta_\triangle^{\sum_{k=1}^l \mathbf{n}_{m,k}}.\mathbf{w}_m$.

For $j \in I$ let $(\mathbf{w}_{m,l})_j \in V_i^\tau$ denote the j-th coordinate of $\mathbf{w}_{m,l}$ then by (2.18),

$$\begin{aligned} |(\mathbf{w}_{m,l+1})_i| =& |\zeta_i^{\mathbf{n}_{m,l+1}}| \cdot |(\mathbf{w}_{m,l})_i| = |e^{\lambda_j(\mathbf{n}_{m,l+1})}(\mathbf{w}_{m,l})_i| \\ \in& \big(e^C |(\mathbf{w}_{m,l})_i|, e^{2C}|(\mathbf{w}_{m,l})_i|\big); \end{aligned} \tag{5.13}$$

similarly

$$|(\mathbf{w}_{m,l+1})_j| \leq e^{\frac{C}{2}} |(\mathbf{w}_{m,l})_j|, \forall j \in I\backslash\{i\}. \tag{5.14}$$

For any $m \in \mathbb{N}$, as it was previously assumed that $|\mathbf{w}_{m,0}| = |\mathbf{w}_m|$ is bounded by δ. (5.13) implies that $\exists l_m \geq 0$ such that

$$|(\mathbf{w}_{m,l_m})_i| \in [e^{-2C}\delta, \delta]. \tag{5.15}$$

Furthermore,.it follows from (5.12), (5.13), (5.14) and (5.15) that for all $j \neq i$ from I,

$$\begin{aligned} |(\mathbf{w}_{m,l_m})_j| \le & \left(\frac{|(\mathbf{w}_{m,l_m})_i|}{|(\mathbf{w}_m)_i|}\right)^{\frac{1}{2}} \cdot |(\mathbf{w}_m)_j| \\ \le & \left(\frac{|(\mathbf{w}_{m,l_m})_i|}{|(\mathbf{w}_m)_i|}\right)^{\frac{1}{2}} \cdot |(\mathbf{w}_m)_i| \\ \le & \delta^{\frac{1}{2}}|(\mathbf{w}_m)_i|^{\frac{1}{2}} \le \delta^{\frac{1}{2}}|\mathbf{w}_m|^{\frac{1}{2}}. \end{aligned} \tag{5.16}$$

Let $m \to \infty$, by (5.15) there is a subsequence of $\{(\mathbf{w}_{m,l_m})_i\}_{m=1}^{\infty}$ that converges to a vector $\mathbf{w} \in V_i^{\tau}$ with $|\mathbf{w}| \in [e^{-2C}\delta, \delta]$. And by (5.16) and the fact that $\mathbf{w}_m$ converges to 0 as $m \to \infty$, $\lim_{m\to\infty}(\mathbf{w}_{m,l_m})_j = \mathbf{0}$ for all $j \in I\backslash\{0\}$. Therefore

$$\lim_{m\to\infty} \mathbf{w}_{m,l_m} = \lim_{m\to\infty} (\mathbf{w}_{m,l_m})_i = \mathbf{w}. \tag{5.17}$$

On the other hand, recall since $\mathbf{x}_{m,l_m} \in L_{0,1}\backslash E_\theta$ is at least of distance θ from E, any accumulation point cannot be in E. Thus by compactness of $L_{0,1}\backslash E_\theta$, by passing to a subsequence, we may suppose that $\lim_{m\to\infty} \mathbf{x}_{m,l_m} = \mathbf{y}$ for some $\mathbf{y} \in L_{0,1}\backslash E_\theta$. Then

$$\begin{aligned} \mathbf{y} + \mathbf{w} &= \lim_{m\to\infty} \mathbf{x}_{m,l_m} + \lim_{m\to\infty} \mathbf{w}_{m,l_m} \\ &= \lim_{m\to\infty} \left(\zeta_\triangle^{\sum_{l=1}^{l_m} \mathbf{n}_{m,l}}.\mathbf{x}_m\right) + \lim_{m\to\infty} \left(\zeta_\triangle^{\sum_{l=1}^{l_m} \mathbf{n}_{m,l}}.\mathbf{w}_m\right) \\ &= \lim_{m\to\infty} \left(\zeta_\triangle^{\sum_{l=1}^{l_m} \mathbf{n}_{m,l}}.(\mathbf{x}_m + \mathbf{w}_m)\right) \\ &= \lim_{m\to\infty} \left(\zeta_\triangle^{\sum_{l=1}^{l_m} \mathbf{n}_{m,l}}.\big(\mathbf{x} + (\mathbf{w}'_m - \mathbf{w}_m) + \mathbf{w}_m\big)\right) \\ &= \lim_{m\to\infty} \left(\zeta_\triangle^{\sum_{l=1}^{l_m} \mathbf{n}_{m,l}}.\mathbf{x}'_m\right). \end{aligned} \tag{5.18}$$

As all the $\mathbf{x}'_m$'s are in the $\zeta_\triangle$-invariant closed set A, this implies $\mathbf{y} + \mathbf{w} \in A$, which concludes the proof. □

To finish the proof of Theorem 3.16.(1) we need the following fact.

Lemma 5.9. *If* $\mathbf{y} \in L_{0,1}$, $\epsilon > 0$ *and* $i \in I$ *then*

$$\overline{\{\zeta_\triangle^{\mathbf{n}}.\mathbf{y} : \mathbf{n} \in H, |\lambda_i(\mathbf{n})| < \epsilon\}} = L_{0,1} \tag{5.19}$$

unless $\mathbf{y}$ *can be written as* $\mathbf{y}_0 + \mathbf{w}_0$ *where* $\mathbf{y}_0 \in L_{0,1}$ *is a torsion point and* $\mathbf{w} \in V_i^{\kappa_0}$.

Proof. By definition of H in 5.1, it stabilizes under $\zeta_\triangle$ at least one torsion point $\mathbf{x}$ from $L_{0,1}$. If $\mathbf{y}$ cannot be written in the particular form given above then $\mathbf{y}-\mathbf{x}\in T^{\kappa_0}$ cannot be written as $\mathbf{y}'+\mathbf{w}'$ where $\mathbf{y}'$ is a torsion point in T^{κ_0} and $\mathbf{w}'\in V_i^{\kappa_0}$.

By Corollary 3.12, $\overline{\{\zeta_\triangle^{\mathbf{n}}.(\mathbf{y}-\mathbf{x}):\mathbf{n}\in H,|\lambda_i(\mathbf{n})|<\epsilon\}}=T^{\kappa_0}$. As $\zeta_\triangle^{\mathbf{n}}.\mathbf{y}=\mathbf{x}+\zeta_\triangle^{\mathbf{n}}.(\mathbf{y}-\mathbf{x})$ for all $\mathbf{n}\in H$ and $L_{0,1}=\mathbf{x}+T^{\kappa_0}$, (5.19) follows. $\square$

Corollary 5.10. *If $r\geq 3$ then there exist an index $i\in I$, a torsion point $\mathbf{z}\in L_{0,1}$ and a nonzero vector $\mathbf{v}\in V_i^{\square}$ such that $\mathbf{z}+\mathbf{v}\in A\backslash(\bigcup_{h=0}^q L_h)$.*

Proof. Let θ be sufficiently small then $L_{0,1}\backslash E_\theta$ is non-empty and has open interior with respect to the relative topology in $L_{0,1}$. In particular, $L_{0,1}\backslash E_\theta$ contains torsion points because torsion points are dense in $L_{0,1}$.

Notice $L_{0,1}\backslash E_\theta$ is disjoint from all the $L_{0,t}$'s for $1<t\leq s_0$ and all the L_h's for $1\leq h\leq r$. As each of these sets are compact, it follows

$$\operatorname{dist}\Big(L_{0,1}\backslash E_\theta,(\bigcup_{t=2}^{s_0}L_{0,t})\cup(\bigcup_{h=1}^{q}L_h)\Big)>0. \tag{5.20}$$

Define a positive number

$$\delta=\frac{1}{2}\min\left(\operatorname{dist}\Big(L_{0,1}\backslash E_\theta,(\bigcup_{t=2}^{s_0}L_{0,t})\cup(\bigcup_{h=1}^{q}L_h)\Big),\operatorname{rad}_{X^2}(L_{0,1})\right). \tag{5.21}$$

Apply Lemma 5.8, we obtain $i\in I$, $\mathbf{y}\in L_{0,1}\backslash E_\theta$ and $\mathbf{w}\in V_i^\tau$ where $\tau\in K\cup\{\infty\}$ is distinct from κ_0, such that $0<|\mathbf{w}|<\delta$ and $\mathbf{y}+\mathbf{w}\in A$.

We distinguish between two cases:

Case 1. If $\mathbf{y}$ can be decomposed as $\mathbf{z}+\mathbf{w}'$ where $\mathbf{z}$ is a torsion point from $L_{0,1}$ and $\mathbf{w}'\in V_i^{\kappa_0}$. Then $\mathbf{y}+\mathbf{w}$ rewrites as $\mathbf{z}+\mathbf{v}$ where $\mathbf{v}=\mathbf{w}'+\mathbf{w}$ is in $V_i^\square$ since both $\mathbf{w}'$ and $\mathbf{w}$ are.

Case 2. Suppose $\mathbf{y}$ cannot be decomposed as above, then by Corollary 5.9,

$$\{\zeta_\triangle^{\mathbf{n}}.\mathbf{y}:\mathbf{n}\in H,|\lambda_i(\mathbf{n})|<\frac{1}{2}\}$$

is dense in $L_{0,1}$. In particular, we fix an arbitrary torsion point $\mathbf{z}\in L_{0,1}\backslash E_\theta$, then there is a sequence $\{\mathbf{n}_m\in H\}_{m=1}^\infty$ such that $|\mathbf{n}_m|<\frac{1}{2},\forall m$ and $\lim_{m\to\infty}\zeta_\triangle^{\mathbf{n}_m}.\mathbf{y}=\mathbf{z}$. Then for each m, $\zeta_\triangle^{\mathbf{n}_m}.\mathbf{w}=\zeta_i^{\mathbf{n}_m}\mathbf{w}\in V_i^\tau$ and $|\zeta_\triangle^{\mathbf{n}_m}.\mathbf{w}|=e^{\lambda_i(\mathbf{n}_m)}|\mathbf{w}|\in[e^{-\frac{1}{2}}\delta,e^{\frac{1}{2}}\delta]$. Since $\{\mathbf{v}\in V_i^\tau:|\mathbf{v}|\in[e^{-\frac{1}{2}}\delta,e^{\frac{1}{2}}\delta]\}$ is compact, without loss of generality we may assume $\zeta_\triangle^{\mathbf{n}_m}.\mathbf{w}$ converges to a vector $\mathbf{v}\in V_i^\tau$ with $|\mathbf{v}|\in[e^{-\frac{1}{2}}\delta,e^{\frac{1}{2}}\delta]$ as m tends to ∞. In particular $0<|\mathbf{v}|<2\delta$. Then $\mathbf{z}+\mathbf{v}$ is in A as it is the limit of $\zeta_\triangle^{\mathbf{n}_m}.(\mathbf{y}+\mathbf{w})$.

So in both cases A contains $\mathbf{z}+\mathbf{v}$ where $\mathbf{z}\in L_{0,1}$ is of torsion and $\mathbf{v}\in V_i^{\square}$ is non-trivial. Moreover, $\mathbf{z}+\mathbf{v}$ can always be written as $\mathbf{z}'+\mathbf{v}'$ where $\mathbf{z}'\in L_{0,1}\backslash E_\theta$ and $\mathbf{v}$ is a nonzero vector from V_i^τ with $|\mathbf{v}'|<2\delta$ (in Case 1, $\mathbf{z}'=\mathbf{y}$ and $\mathbf{v}'=\mathbf{w}$; while in Case 2 $\mathbf{z}'=\mathbf{z}$ and $\mathbf{v}'=\mathbf{v}$.)

It remains to show $\mathbf{z}+\mathbf{v}\notin L_h$ for all $h=0,\cdots,q$.

By the definition of δ, on the one hand

$$|\mathbf{v}'|<\operatorname{dist}\Big(L_{0,1}\backslash E_\theta,(\bigcup_{t=2}^{s_0}L_{0,t})\cup(\bigcup_{h=1}^{q}L_h)\Big), \tag{5.22}$$

so $\mathbf{z}+\mathbf{v}\notin(\bigcup_{t=2}^{s_0}L_{0,t})\cup(\bigcup_{h=1}^{q}L_h)$; on the other hand because $0<|\mathbf{v}'|<\operatorname{rad}_{X^2}(L_{0,1})$ and $\mathbf{v}'\notin V^{\kappa_0}$, $\mathbf{z}'+\mathbf{v}'\notin L_{0,1}$ by Remark 5.5. Therefore $\mathbf{z}+\mathbf{v}\notin\bigcup_{h=1}^{q}L_h$, which concludes the proof. □

Now we are ready to prove Lemma 5.3.

Proof of Lemma 5.3. As $r\geq 3$, by Corollary 5.10, A contains $\mathbf{z}$ and $\mathbf{z}+\mathbf{v}$ where $\mathbf{z}$ is a torsion point and $\mathbf{v}$ is a nonzero vector from one of the $V_i^{\square}$'s such that $\mathbf{z}+\mathbf{v}\notin L_h,\forall h=0,\cdots q$. By Lemma 4.9 it contains a d-dimensional homogeneous $\zeta_\triangle$-invariant subset L_{q+1} such that $\mathbf{z}+\mathbf{v}\in L_{q+1}$. It follows that L_{q+1} is different from all the previous L_h's. □

Hence so far we have completed the proof of Proposition 5.2, which is going to be used to establish both Theorems 3.16.(1) and 3.17.

5.3. Proof of rigidity results.

Proof of Theorem 3.16.(1). Let A be an infinite proper closed subset of X^2, invariant and topologically transitive under the action $\zeta_\triangle$.

Assuming A is not a d-dimensional homogeneous $\zeta_\triangle$-invariant subset, we try to deduce a contradiction.

We claim A meets Condition 5.1. To see this, assume a d-dimensional homogeneous $\zeta_\triangle$-invariant subset L is contained in A. Then $A\backslash L$ is non-empty and relatively open in L. Furthermore as both A and L are $\zeta_\triangle$-invariant, so is $A\backslash L$. Let U be any relatively open subset in A, by topological transitivity, $\exists\mathbf{n}\in\mathbb{Z}^r$ such that $\big(\zeta_\triangle^{\mathbf{n}}.(A\backslash L)\big)\cap U$ is non empty. But $\zeta_\triangle^{\mathbf{n}}.(A\backslash L)$ is just $A\backslash L$. This shows $\overline{A\backslash L}=A$.

As $r\geq 3$, Proposition 5.1 applies and $A=X^2$. But A is supposed to be proper and we get a contradiction. □

Proof of Theorem 3.17. Fix $\epsilon>\epsilon'>0$ and $\mathbf{x}\in X^2$. Let $C_{\mathbf{x}}\subset X^2$ be the set of all torsion points whose $\zeta_\triangle$-orbits are disjoint from $\mathring{B}_{\epsilon'}(\mathbf{x})$ and notice it is $\zeta_\triangle$-invariant. We want to show $C_{\mathbf{x}}$ is covered by a finite union of d-dimensional homogeneous $\zeta_\triangle$-invariant sets.

Take all the d-dimensional homogeneous $\zeta_\triangle$-invariant subsets $L_1, \cdots, L_q$ which are not ϵ'-dense. Write

$$A_{\mathbf{x}} = \overline{C_{\mathbf{x}} \backslash \bigcup_{h=1}^{q} L_h}. \tag{5.23}$$

Then as $C_{\mathbf{x}}$ and the L_h's are all $\zeta_\triangle$-invariant, so are both $C_{\mathbf{x}} \backslash \bigcup_{h=1}^{q} L_h$ and A. Moreover, A is disjoint from $\mathring{B}_{\epsilon'}(\mathbf{x})$.

We claim $A_{\mathbf{x}}$ is finite. Suppose not, assume $L \subset A$ is a d-dimensional homogeneous $\zeta_\triangle$-invariant subset (if such exists). Then as L avoids $\mathring{B}_{\epsilon'}(\mathbf{x})$ it is one of $L_1, \cdots, L_q$. In particular,

$$A_{\mathbf{x}} \backslash L \supset (C_{\mathbf{x}} \backslash \bigcup_{h=1}^{q} L_h) \backslash L_h = C_{\mathbf{x}} \backslash \bigcup_{h=1}^{q} L_h \tag{5.24}$$

and it follows that $\overline{A_{\mathbf{x}} \backslash L} \supset \overline{C_{\mathbf{x}} \backslash \bigcup_{h=1}^{q} L_h} = A_{\mathbf{x}}$. So $A_{\epsilon', \mathbf{x}}$ satisfies Condition 5.1. Because $r \geq 3$, $A_{\epsilon', \mathbf{x}} = X^2$ by Proposition 5.2, which contradicts the fact that A is disjoint from $\mathring{B}_{\epsilon'}(\mathbf{x})$. It follows

$$\left| C_{\mathbf{x}} \backslash \bigcup_{h=1}^{q} L_h \right| < \infty, \forall \mathbf{x} \in X^2. \tag{5.25}$$

By compactness of X^2, there are $\mathbf{x}_1, \cdots, \mathbf{x}_k$ such that $\bigcup_{k=1}^{l} \mathring{B}_{\epsilon-\epsilon'}(\mathbf{x}_k)$ covers X^2. That is, $\forall \mathbf{x} \in X$, $\exists k \in \{1, \cdots, l\}$ such that $\mathbf{x} \in \mathring{B}_{\epsilon-\epsilon'}(\mathbf{x}_k)$, in which case $\mathring{B}_{\epsilon'}(\mathbf{x}_k) \subset \mathring{B}_{\epsilon}(\mathbf{x})$.

Now let $\mathbf{y} \in X^2$ be a torsion point whose orbit $\{\zeta_\triangle^{\mathbf{n}}.\mathbf{y} : \mathbf{n} \in \mathbb{Z}^r\}$ is not ϵ-dense, or equivalently, is disjoint from $\mathring{B}_{\epsilon}(\mathbf{x})$ for some $\mathbf{x} \in X$. Then this orbit is also disjoint from $\mathring{B}_{\epsilon'}(\mathbf{x}_k)$ for at least one $k \in \{1, \cdots, l\}$ by the above remark. So by construction the set $C_{\mathbf{x}_k}$ contains $\mathbf{y}$.

Therefore the collection of all torsion points whose orbits are not ϵ-dense, which we denote by C, is contained in the set

$$\bigcup_{k=1}^{l} C_{\mathbf{x}} \subset (\bigcup_{h=1}^{q} L_h) \sqcup \Big(\bigcup_{k=1}^{l} (C_{\mathbf{x}} \backslash \bigcup_{h=1}^{q} L_h)\Big). \tag{5.26}$$

This proves the first part of theorem because $\left(\bigcup_{k=1}^{l} (C_{\mathbf{x}} \backslash \bigcup_{h=1}^{q} L_h)\right)$ is finite by (5.25).

To show the last claim in Theorem 3.17, i.e. C is covered by finitely many d-dimensional homogeneous $\zeta_\triangle$-invariant subsets, it suffices to note that every torsion point in X^2 is contained in some d-dimensional homogeneous $\zeta_\triangle$-invariant subset. So if we assign to each point in the finite set $\left(\bigcup_{k=1}^{l} (C_{\mathbf{x}} \backslash \bigcup_{h=1}^{q} L_h)\right)$ a d-dimensional homogeneous $\zeta_\triangle$-invariant subset that contains it, all these subsets together with $L_1, \cdots, L_q$ cover C. □

So far we showed Theorem 3.16.(1) as well as Theorem 3.17. Thus by Lemma 3.18, the proofs of Theorem 1.6.(1) and Theorem 1.7 are also accomplished.

6. Counterexample in rank 2

We now construct the counterexample required by the second part of Theorem 3.16. From now on let $r = 2$.

In this case, $r_1 + r_2 = r + 1 = 3$ so $I = \{1, 2, 3\}$ and $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2} = V_1 \oplus V_2 \oplus V_3$. Fix a non-trivial lattice point $\gamma \in \Gamma \backslash \{0\}$ and decompose it as $\sum_{i=1}^{3} \gamma_i$ where $\gamma_i \in V_i$. Recall $\gamma \in \sigma(K)$ so $\gamma_i = \sigma_i(\theta), i = 1, 2, 3$ for some $\theta \in K$. Note $\theta \neq 0$ as otherwise $\gamma = 0$. Hence $\gamma_i \neq 0$, $\forall i = 1, 2, 3$.

Take the point $\tilde{\mathbf{x}} = (\gamma_1, \gamma_2) \in (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})^2$. And let $\mathbf{x} = \pi_\triangle(\tilde{\mathbf{x}}) = \big(\pi(\gamma_1), \pi(\gamma_2)\big)$. We will prove in this section that (3.15) holds with

$$\kappa_1 = \infty,\ \kappa_2 = 0 \text{ and } \kappa_3 = -1. \tag{6.1}$$

From Lemma 3.8 one can easily check:

$$T^\infty = \pi_\triangle(V^\infty) = \pi_\triangle\big(\{0\} \times (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})\big) = \{0\} \times X; \tag{6.2}$$

$$T^0 = \pi_\triangle(V^0) = \pi_\triangle\big((\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}) \times \{0\}\big) = X \times \{0\}; \tag{6.3}$$

and

$$\begin{aligned} T^{-1} =& \pi_\triangle(V^{-1}) = \pi_\triangle\big(\{(\tilde{x}, -\tilde{x}) : \tilde{x} \in (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})\}\big) \\ =& \{(x, -x) : x \in X\}. \end{aligned} \tag{6.4}$$

The point $\mathbf{x}$ is special in the following sense:

Lemma 6.1. *$\forall i \in I$, there is a permutation (i, j, h) of $\{1, 2, 3\}$ such that $\mathbf{x}$ can be decomposed as $\pi_\triangle(\tilde{\mathbf{x}}_\parallel) + \pi_\triangle(\tilde{\mathbf{x}}_\top)$ where $\tilde{\mathbf{x}}_\parallel$ and $\tilde{\mathbf{x}}_\top$ are nonzero vectors, respectively from $V_j^{\kappa_i}$ and $V_i^{\kappa_j}$.*

Note that by Lemma 3.8, $\pi_\triangle(\tilde{\mathbf{x}}_\parallel) \in T^{\kappa_0}$.

Proof. We choose j, h, $\tilde{\mathbf{x}}_\parallel$ and $\tilde{\mathbf{x}}_\top$ in the following way :

- if $i = 1$, let $j = 2$, $h = 3$, $\tilde{\mathbf{x}}_\parallel = (0, \gamma_2)$ and $\tilde{\mathbf{x}}_\top = (\gamma_1, 0)$;
- if $i = 2$, let $j = 1$, $h = 3$, $\tilde{\mathbf{x}}_\parallel = (\gamma_1, 0)$ and $\tilde{\mathbf{x}}_\top = (0, \gamma_2)$;
- if $i = 3$, let $j = 1$, $h = 2$, $\tilde{\mathbf{x}}_\parallel = (\gamma_1, -\gamma_1)$ and $\tilde{\mathbf{x}}_\top = (0, -\gamma_3)$.

The lemma is obviously true for $i = 1, 2$ as $\tilde{\mathbf{x}}_\parallel = \tilde{\mathbf{x}}_\parallel + \tilde{\mathbf{x}}_\top$. For $i = 3$, notice $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_\parallel + \tilde{\mathbf{x}}_\top + (0, \gamma)$. Thus $\mathbf{x} = \pi_\triangle(\tilde{\mathbf{x}}) = \pi_\triangle(\tilde{\mathbf{x}}_\parallel) + \pi_\triangle(\tilde{\mathbf{x}}_\top)$ as $(0, \gamma) \in \Gamma^2$. ∎

Corollary 6.2. *For each $i = 1, 2, 3$, suppose a sequence $\{\mathbf{n}_l\}_{l=1}^\infty \subset \mathbb{Z}^2$ satisfies that $\lim_{l\to\infty} \lambda_i(\mathbf{n}_l) = -\infty$ and $\lim_{l\to\infty} \zeta_\triangle^{\mathbf{n}_l}.\mathbf{x} = \mathbf{y}$, then $\mathbf{y} \in T^{\kappa_i}$.*

Proof. Take the decomposition given by Lemma 6.1, then $\zeta_\triangle^{\mathbf{n}_l}.\mathbf{x} = \zeta_\triangle^{\mathbf{n}_l}.\big(\pi_\triangle(\tilde{\mathbf{x}}_\parallel) + \pi_\triangle(\tilde{\mathbf{x}}_\top)\big) = \zeta_\triangle^{\mathbf{n}_l}.\pi_\triangle(\tilde{\mathbf{x}}_\parallel) + \pi_\triangle(\zeta_\triangle^{\mathbf{n}_l}.\tilde{\mathbf{x}}_\top)$. Furthermore, by (2.18), $|\zeta_\triangle^{\mathbf{n}_l}.\tilde{\mathbf{x}}_\top| = e^{\lambda_i(\mathbf{n}_l)}|\tilde{\mathbf{x}}_\top|$. When l go to ∞, $e^{\lambda_i(\mathbf{n}_l)} \to 0$ and thus $\pi_\triangle(\zeta_\triangle^{\mathbf{n}_l}.\tilde{\mathbf{x}}_\top)$ approaches $\mathbf{0}$. Hence $\lim_{l\to\infty} \zeta_\triangle^{\mathbf{n}_l}.\mathbf{x} = \lim_{l\to\infty} \zeta_\triangle^{\mathbf{n}_l}.\pi_\triangle(\tilde{\mathbf{x}}_\parallel)$ and as $\pi_\triangle(\tilde{\mathbf{x}}_\parallel) \in T^{\kappa_i}$ this limit must be in T^{κ_i}. ∎

Corollary 6.3. *Any accumulation point of $\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\}$ is contained in at least one of T^∞, T^0 and T^{-1}.*

Proof. For each $i = 1, 2, 3$, we define a subset Φ_i of $\mathbb{Z}^2$ by:

$$\Phi_i = \left\{\mathbf{n} \in \mathbb{Z}^2 : \lambda_i(\mathbf{n}) < 0, \lambda_k(\mathbf{n}) \geq \lambda_k(\mathbf{n}), \forall k \in \{1,2,3\}\backslash\{i\}\right\}; \quad (6.5)$$

As $\sum_{i=1}^3 d_i\lambda_i(\mathbf{n}) = 0$, the union of Φ_i's covers $\mathbb{Z}^2\backslash\{\mathbf{0}\}$ This is because there is no nonzero element $\mathbf{n} \in \mathbb{Z}^2$ such that $\lambda_i(\mathbf{n}) = 0, \forall i$ in light of Remark 2.5.

Therefore for any accumulation point $\mathbf{y}$ of $\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\}$, $\exists i \in \{1, 2, 3\}$ and a sequence $\{\mathbf{n}_l\}_{l=1}^\infty \subset \Phi_i$ where all the $\mathbf{n}_l$'s are different from each other, such that $\zeta_\triangle^{\mathbf{n}_l}.\mathbf{x}$ converges to $\mathbf{y}$ as $l \to \infty$.

By Remark 2.5, for any $R > 0$, there are only finitely many $\mathbf{n} \in \mathbb{Z}^r$ such that $\mathcal{L}(\mathbf{n}) = \big(\lambda_1(\mathbf{n}), \lambda_2(\mathbf{n}), \lambda_3(\mathbf{n})\big)$ belongs to the compact set $\{(w_1, w_2, w_3) \in W : \max_{j=1}^3 |w_j| \leq R\}$. Therefore for sufficiently large l, $\max_{j=1}^3 |\lambda_j(\mathbf{n}_l)| \geq R$.

If $\mathbf{n} \in \Phi_i$, write $\{1, 2, 3\}\backslash\{i\}$ as $\{k_1, k_2\}$, then $\lambda_{k_1}(\mathbf{n}) > -\lambda_i(\mathbf{n})$ and $\lambda_{k_2}(\mathbf{n}) > -\lambda_i(\mathbf{n})$, moreover $\lambda_{k_1}(\mathbf{n}) = \frac{-d_i\lambda_i(\mathbf{n}) - d_{k_2}\lambda_{k_2}(\mathbf{n})}{d_{k_1}} \leq \frac{2|\lambda_i(\mathbf{n})| + 2|\lambda_i(\mathbf{n})|}{1} = 4|\lambda_i(\mathbf{n})|$. So $|\lambda_{k_1}(\mathbf{n})| \leq 4|\lambda_i(\mathbf{n})|$ and similarly $|\lambda_{k_2}(\mathbf{n})| \leq 4|\lambda_i(\mathbf{n})|$.

Hence $\max_{k=1}^3 |\lambda_k(\mathbf{n})| \leq 4|\lambda_i(\mathbf{n})|$, and for all $R > 0$ if l is large enough then $|\lambda_i(\mathbf{n}_l)| \geq \frac{R}{4}$. Moreover, as $\mathbf{n}_l \in \Phi_i$, $\lambda_i(\mathbf{n}_l) < -\frac{R}{4}$. Since R is arbitrary,

$$\lim_{l\to\infty} \lambda_i(\mathbf{n}_l) = -\infty. \quad (6.6)$$

Hence by Corollary 6.2, $\mathbf{y} \in T^{\kappa_i}$. □

Lemma 6.4. *For each $i = 1, 2, 3$, $T^{\kappa_i} \subset \overline{\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\}}$.*

Proof. Let (i, j, h) be the permutation given by Lemma 6.1. Because $\mathcal{L}(\mathbb{Z}^2)$ is a full-rank lattice in W, $\exists \mathbf{n} \in \mathbb{Z}^2$ such that $\lambda_i(\mathbf{n}) < 0$ and $\lambda_j(\mathbf{n}), \lambda_h(\mathbf{n}) > 0$. It follows from Lemma 6.2 that $\forall l \in \mathbb{N}$, $\zeta_\triangle^{l\mathbf{n}}.\mathbf{x}$ can be written as $\pi_\triangle(\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\parallel) + \zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\top$ where $\tilde{\mathbf{x}}_\parallel \in V_j^{\kappa_i}$, $\tilde{\mathbf{x}}_\top \in V_i^{\kappa_j}$ are nonzero vectors.

Thus

$$|\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\parallel| = |\zeta_j^{l\mathbf{n}}| \cdot |\tilde{\mathbf{x}}_\parallel| = e^{l\lambda_j(\mathbf{n})}|\tilde{\mathbf{x}}_\parallel| \quad (6.7)$$

and similarly

$$|\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\top| = e^{l\lambda_i(\mathbf{n})}|\tilde{\mathbf{x}}_\top|. \quad (6.8)$$

Since $\lambda_i(\mathbf{n}) < 0$, $\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\top$ converges to $\mathbf{0}$ as l tends to ∞.

As X^2 is compact, for a subsequence of integers $\{l_k\}_{k=1}^\infty$, $\pi_\triangle(\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\parallel)$ converges to a limit $\mathbf{y}$. Then $\lim_{k\to\infty} \zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x} = \mathbf{y}$ as well. Because $\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\parallel \in V_j^{\kappa_i}$, by Lemma 3.8 $\mathbf{y} \in T^{\kappa_i}$. In particular, there is at least one accumulation point of $\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\}$ in T^{κ_i}.

So we can consider the non-empty intersection $E = T^{\kappa_i} \cap \overline{\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\}}$. As both T^κ and $\overline{\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\}}$ are $\zeta_\triangle$-invariant, E is a $\zeta_\triangle$-invariant closed subset in T^{κ_i}.

To establish the lemma it suffices to show $E = T^{\kappa_i}$.

In order to deduce a contradiction, from now on we assume that $E \subsetneq T^{\kappa_i}$, then by Lemma 3.11 E is a finite collection of torsion points.

Thus the point $\mathbf{y}$ constructed above is of torsion and therefore the subgroup $H = \mathrm{Stab}_{\zeta_\triangle}(\mathbf{y})$ has finite index in $\mathbb{Z}^2$. In particular, $\exists p \in \mathbb{N}$ such that $p\mathbf{n} \in H$. Denote

$$C = \max\big(p\lambda_j(\mathbf{n}), p\lambda_h(\mathbf{n})\big). \tag{6.9}$$

Since E is a finite set, similar to (5.5), there is a constant ρ such that for all nonzero vector $\mathbf{v} \in V^{\kappa_i}$ with $|\mathbf{v}| < \rho$ we have $\mathbf{y} + \mathbf{v} \in T^{\kappa_i} \backslash E$.

Define a shell-shaped set

$$D = \big\{\mathbf{y} + \mathbf{v} : \mathbf{v} \in V^{\kappa_i}, |\mathbf{v}| \in \big[\frac{\rho}{2e^C}, \frac{\rho}{2}\big]\big\}, \tag{6.10}$$

which is a compact subset of T^{κ_i} and is disjoint from E. In particular, any point $\mathbf{z}$ from D is not an accumulation point of $\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\}$, and thus has an neighborhood $B_{\delta_{\mathbf{z}}}(\mathbf{z})$ that doesn't intersect $\{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\}$ where $\delta_{\mathbf{z}} > 0$. As D is compact, there exist $\mathbf{z}_1, \cdots, \mathbf{z}_M \in D$ such that $\bigcup_{m=1}^M \mathring{B}_{\frac{\delta_{\mathbf{z}_m}}{2}}(\mathbf{z}_m)$ covers D. Set

$$\delta = \frac{1}{2}\min(\delta_{\mathbf{z}_1}, \cdots, \delta_{\mathbf{z}_M}, \rho). \tag{6.11}$$

For any $\mathbf{z} \in D$, because $\mathbf{z} \in \mathring{B}_{\frac{\delta_{\mathbf{z}_m}}{2}}(\mathbf{z}_m)$ for some m, $B_\delta(\mathbf{z})$ is contained in $B_{\delta+\frac{\delta_{\mathbf{z}_m}}{2}}(\mathbf{z}_m) \subset B_{\delta_{\mathbf{z}_m}}(\mathbf{z}_m)$. Thus

$$B_\delta(\mathbf{z}) \cap \{\zeta_\triangle^{\mathbf{n}}.\mathbf{x} : \mathbf{n} \in \mathbb{Z}^r\} = \emptyset, \forall \mathbf{z} \in D \tag{6.12}$$

In light of earlier discussions, we know for any sufficiently large integer k,

$$\|\pi_\triangle(\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\parallel) - \mathbf{y}\| < \delta, \ |\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\top| < \delta. \tag{6.13}$$

Fix such a large k, then there is a vector $\mathbf{w} \in V^{\kappa_i}$ such that $|\mathbf{w}| < \frac{\delta}{2}$ and $\pi_\triangle(\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\parallel) = \mathbf{y} + \mathbf{w}$. Write $\mathbf{w}$ as $\mathbf{w}_i + \mathbf{w}_j + \mathbf{w}_h$ where $\mathbf{w}_i \in V_i^{\kappa_i}$, $\mathbf{w}_j \in V_j^{\kappa_i}$ and $\mathbf{w}_h \in V_h^{\kappa_i}$. Then $|\mathbf{w}|^2 = |\mathbf{w}_i|^2 + |\mathbf{w}_j|^2 + |\mathbf{w}_h|^2$.

Since $\mathbf{y} = \pi_\triangle(\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\parallel - \mathbf{w})$ is a torsion point, there is a nonzero integer q such that $q\mathbf{y} = \pi_\triangle\big(q(\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\parallel - \mathbf{w})\big) = \mathbf{0}$ or equivalently $q(\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\parallel - \mathbf{w}) \in \Gamma^2$. Since we may choose l_k to be arbitrarily large, by (6.7) and the assumption $\lambda_j(\mathbf{n}) > 0$, $\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\parallel$ can be made arbitrarily large. But $|\mathbf{w}| < \frac{\delta}{2}$, so $q(\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\parallel - \mathbf{w}) \in \Gamma^2 \backslash \{\mathbf{0}\}$. Recall $\Gamma \subset \sigma(K)$ in the setting of this paper. Thus $q(\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\parallel - \mathbf{w}) = \big(\sigma(\theta^{(1)}), \sigma(\theta^{(2)})\big)$ where $\theta^{(1)}, \theta^{(2)}$ are from K and are not simultaneously zero. In particular, the $V_h^{(1)}$ and $V_h^{(2)}$ coordinates of $q(\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\parallel - \mathbf{w})$ are respectively $\sigma_h(\theta^{(1)})$ and $\sigma_h(\theta^{(2)})$, and are not simultaneously zero. In other words, in the decomposition

$$q(\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\parallel - \mathbf{w}) = q(\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x}_\parallel - \mathbf{w}_j) + q\mathbf{w}_i + q\mathbf{w}_h, \tag{6.14}$$

the $V_h^\square$-component $q\mathbf{w}_h$ doesn't vanish. Thus $\mathbf{w}_h \neq \mathbf{0}$.

Now consider the sequence of vectors $\mathbf{v}_s = \zeta_\triangle^{sp\mathbf{n}}.\mathbf{w} \in V^{\kappa_i}$ where $s = 1, 2, \cdots$. Note $\mathbf{v}_s$ decomposes into $(\mathbf{v}_s)_i + (\mathbf{v}_s)_j + (\mathbf{v}_s)_h$ where $(\mathbf{v}_s)_f = \zeta_\triangle^{sp\mathbf{n}}.\mathbf{w}_f \in V_f^{\kappa_i}$ for $f = i, j, h$. By (2.18), $|(\mathbf{v}_s)_f| = e^{sp\lambda_i(\mathbf{n})}|\mathbf{w}_f|$, in particular $|(\mathbf{v}_s)_f| \leq e^{p\lambda_f(\mathbf{n})}|(\mathbf{v}_{s-1})_f| \leq e^C|(\mathbf{v}_{s-1})_f|, \forall s \geq 1, \forall f = i, j, h$ because of (6.9) and the fact that $\lambda_i(\mathbf{n}) < 0$. Recall $|\mathbf{v}_s|^2 = |(\mathbf{v}_s)_i|^2 + |(\mathbf{v}_s)_i|^2 + |(\mathbf{v}_s)_i|^2$ and it follows that

$$|\mathbf{v}_s| \leq e^C|\mathbf{v}_{s-1}|, \forall s \geq 1. \tag{6.15}$$

But on the other hand because $\mathbf{w}_h \neq \mathbf{0}$ and $\lambda_h(\mathbf{n}) > 0$, as s increases $|(\mathbf{v}_s)_h| = e^{sp\lambda_i(\mathbf{n})}|\mathbf{w}_h|$ grows to ∞, in consequence $|\mathbf{v}_s| \to \infty$ as well. Since $|\mathbf{v}_0| = |\mathbf{w}| \leq \delta \leq \frac{\rho}{2}$, the above facts imply that there is an integer $s_0 \geq 0$ such that

$$|\mathbf{v}_{s_0}| \in \Big[\frac{\rho}{2e^C}, \frac{\rho}{2}\Big]. \tag{6.16}$$

We investigate the point

$$\zeta_\triangle^{(l_k+s_0p)\mathbf{n}}.\mathbf{x} = \zeta_\triangle^{s_0p\mathbf{n}}.\zeta_\triangle^{l_k\mathbf{n}}.\mathbf{x} = \zeta_\triangle^{s_0p\mathbf{n}}.\pi_\triangle(\zeta_\triangle^{l_k\mathbf{n}}.\tilde{\mathbf{x}}_\parallel) + \zeta_\triangle^{s_0p\mathbf{n}}.(\zeta_\triangle^{l_k\mathbf{n}}.\tilde{\mathbf{x}}_\top). \tag{6.17}$$

Observe $\zeta_\triangle^{s_0p\mathbf{n}}.\pi_\triangle(\zeta_\triangle^{l_k\mathbf{n}}.\tilde{\mathbf{x}}_\parallel) = \zeta_\triangle^{s_0p\mathbf{n}}.\mathbf{y} + \zeta_\triangle^{s_0p\mathbf{n}}.\mathbf{w}$.

First, $\zeta_\triangle^{s_0p\mathbf{n}}.\mathbf{y} = \mathbf{y}$ because $s_0p\mathbf{n}$ belongs to H and thus stabilizes $\mathbf{y}$ under $\zeta_\triangle$. Furthermore, $\zeta_\triangle^{s_0p\mathbf{n}}.\mathbf{w} = \mathbf{v}_{s_0} \in V^{\kappa_i}$. So it follows from (6.10) and (6.16) that

$$\zeta_\triangle^{s_0p\mathbf{n}}.\pi_\triangle(\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\parallel) \in D. \tag{6.18}$$

On the other hand, since $\lambda_i(\mathbf{n}) < 0$ and $\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\top \in V_i^{\kappa_j}$, by (6.13)

$$|\zeta_\triangle^{s_0p\mathbf{n}}.(\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\top)| \leq e^{s_0p\lambda_i(\mathbf{n})}|\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\top| \leq |\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\top| \leq \delta. \tag{6.19}$$

From (6.16) and (6.17) we deduce that $\zeta_\triangle^{(l_k+s_0p)\mathbf{n}}.\mathbf{x} \in B_\delta\big(\zeta_\triangle^{s_0p\mathbf{n}}.\pi_\triangle(\zeta_\triangle^{l\mathbf{n}}.\tilde{\mathbf{x}}_\parallel)\big)$. But, together with (6.18), this contradicts the disjointness relation (6.12).

So E cannot be a proper subset of T^{κ_i} and the lemma follows. $\square$

Proof of Theorem 3.16.(2). The second half of the theorem directly follows from Corollary 6.3 and Lemma 6.4. $\square$

So finally we established Theorem 3.16 and, by Lemma 3.18, the main result Theorem 1.6 as well.

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References

[Ber83] D. Berend, *Multi-invariant sets on tori*, Trans. Amer. Math. Soc. **280** (1983), no. 2, 509–532. MR **716835 (85b:**11064)

[EL03] M. Einsiedler and E. Lindenstrauss, *Rigidity properties of $\mathbb{Z}^d$-actions on tori and solenoids*, Electron. Res. Announc. Amer. Math. Soc. **9** (2003), 99–110 (electronic). MR **2029471 (2005d:**37007)

[Fur67] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49. MR 0213508 (35 #4369)

[Gla03] E. Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, Providence, RI, 2003. MR **1958753 (2004c:**37011)

[KK02] B. Kalinin and A. Katok, *Measurable rigidity and disjointness for $\mathbb{Z}^k$ actions by toral automorphisms*, Ergodic Theory Dynam. Systems **22** (2002), no. 2, 507–523. MR **1898802 (2003b:**37046)

[Mau10] F. Maucourant, *A nonhomogeneous orbit closure of a diagonal subgroup*, Ann. of Math. (2) **171** (2010), no. 1, 557–570. MR 2630049

[Par75] C. J. Parry, *Units of algebraic number fields*, J. Number Theory **7** (1975), no. 4, 385–388. MR 0384752 (52 #5625)

[Sch95] K. Schmidt, *Dynamical systems of algebraic origin*, Progress in Mathematics, vol. 128, Birkhäuser Verlag, Basel, 1995. MR **1345152 (97c:**28041)

[Wan10a] Z. Wang, *Quantitatitve density under higher rank abelian algebraic toral actions*, Int. Math. Res. Not., posted on 2010, DOI 10.1093/imrn/rnq222, (to appear in print).

[Wan10b] Z. Wang, *Rigidity of commutative non-hyperbolic actions by toral automorphisms* (2010), preprint.

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