Duality in Finite Element Exterior Calculus

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Finite element exterior calculus

Triangulate the domain into simplices. On a simplex T, we have spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ of k-forms on T with polynomial coefficients of degree at most r.

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- scalar fields
  - Lagrange
  - Discontinuous Galerkin

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  - Raviart–Thomas elements
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In three dimensions, $\mathcal{P}_r\Lambda^1(T)$ and $\mathcal{P}_r^-\Lambda^1(T)$ are Nédélec $H(\mathrm{curl})$ elements of the 2nd and 1st kinds, respectively.

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See (Arnold, Falk, Winther, 2006).

Duality: a motivating example

Let Ω be an 3-dimensional domain. Given $\alpha \in \Lambda^1(\Omega)$ and $\beta \in \Lambda^2(\Omega)$, we can compute

$$\int_\Omega \alpha \wedge \beta.$$

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In this setting, given α, it is easy to construct such a dual β. If $\alpha = \alpha_x \, dx + \alpha_y \, dy + \alpha_z \, dz$, then we can set

$$\beta = \alpha_x \, dy \wedge dz + \alpha_y \, dz \wedge dx + \alpha_z \, dx \wedge dy = *\alpha.$$

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- $\int_\Omega \alpha \wedge \beta = \int_\Omega \left(\alpha_x^2 + \alpha_y^2 + \alpha_z^2\right)\, d\mathrm{vol} > 0$.
- β only depends on α pointwise.

Duality in finite element exterior calculus

Let T be a simplex. Given $\alpha \in \Lambda^k(T)$ and $\beta \in \Lambda^{n-k}(T)$, we consider the pairing

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Arnold, Falk, and Winther show that integration is a perfect pairing in the two settings

$$\mathcal{P}_r^- \Lambda^k(T) \times \mathring{\mathcal{P}}_{r+k} \Lambda^{n-k}(T) \to \mathbb{R},$$
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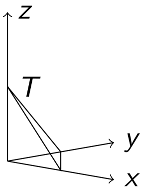
Problem

Given $\alpha \in \mathcal{P}_r \Lambda^k(T)$, find a dual $\beta \in \mathring{\mathcal{P}}_{r+k+1}^- \Lambda^{n-k}(T)$ such that

- $\int_T \alpha \wedge \beta > 0$, and
- β only depends on α pointwise.

The simplex

To illustrate, focus on dim $T = 2$. The standard simplex T sits inside the first orthant $\mathbf{O}$ as those points that satisfy $x + y + z = 1$.



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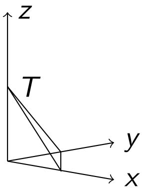

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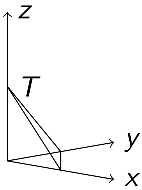

Key ideas

- Identify $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ with spaces $\mathbf{P}_r\Lambda^k(\mathbf{O})$ and $\mathbf{P}_r^-\Lambda^k(\mathbf{O})$ of differential forms on $\mathbf{O}$.

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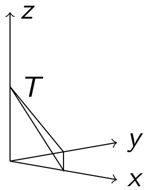

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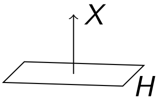
- Exploit a natural duality relationship between the $\mathbf{P}$ and $\mathbf{P}^-$ spaces.

Vertical and horizontal antisymmetric tensors

Let E be a vector space, let $H \subset E$ be a hyperplane, and let X be a vector not in the hyperplane. To illustrate, focus on $\dim E = 3$.



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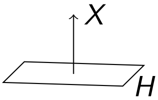

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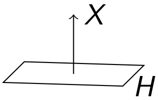

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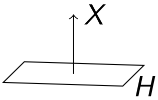

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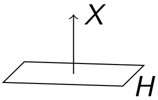

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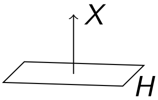

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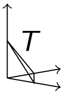
Let $\mathbf{x} = (x, y, z) \in T$. Apply the above discussion $E = \mathbb{R}^3 = T_{\mathbf{x}}\mathbf{O}$, $H = T_{\mathbf{x}}T$, $e^3 = dx + dy + dz$, and $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$.



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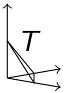

Definition

Let $\mathbf{P}_r\Lambda^k(\mathbf{O})$ denote those $(k+1)$-forms on $\mathbf{O}$ that

- are vertical at every point $\mathbf{x} \in T$, and
- whose coefficients are homogeneous polynomials of degree r.

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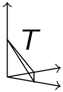

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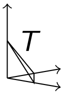

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Theorem

$$\mathcal{P}_r\Lambda^k(T) \cong \mathbf{P}_r\Lambda^k(\mathbf{O}), \qquad \mathcal{P}_r^-\Lambda^k(T) \cong \mathbf{P}_r^-\Lambda^k(\mathbf{O})$$

Duality

Problem (reframed)

Given $\alpha \in \mathbf{P}_r\Lambda^k(\mathbf{O})$, find a dual $\beta \in \mathring{\mathbf{P}}^-_{r+k+1}\Lambda^{n-k}(\mathbf{O})$ such that

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We explictly construct such a map $\mathbf{P}_r\Lambda^k(\mathbf{O}) \to \mathring{\mathbf{P}}^-_{r+k+1}\Lambda^{n-k}(\mathbf{O})$.

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Example

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Example

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- Write $\alpha = \alpha_x\, dy \wedge dz + \alpha_y\, dz \wedge dx + \alpha_z\, dx \wedge dy$.

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- Set $\beta = \alpha_x yz\, dx + \alpha_y zx\, dy + \alpha_z xy\, dz$.

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- Then β is horizontal, has vanishing tangential trace on the boundary, and has coefficients of degree $r + 2$.

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- Set $\beta = \alpha_x yz\, dx + \alpha_y zx\, dy + \alpha_z xy\, dz$.
- Then β is horizontal, has vanishing tangential trace on the boundary, and has coefficients of degree $r + 2$.
- $\alpha \wedge \beta = \left(\alpha_x^2 yz + \alpha_y^2 zx + \alpha_z^2 xy\right)\, d\mathbf{vol}$, a positive multiple of $d\mathbf{vol}$ on the interior.

Thank you

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Characterizations of α being vertical.

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$\Lambda^2 E^*$	$\langle e^1 \wedge e^3, e^2 \wedge e^3 \rangle$	$\langle e^1 \wedge e^2 \rangle$
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Characterizations of α being vertical.

- $\alpha \wedge e^3 = 0$.

Vertical and horizontal antisymmetric tensors

	vertical	horizontal
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Characterizations of β being horizontal.

- $i_X \beta = 0$.
- $\beta = i_X \gamma$ for some γ.
- β is orthogonal to all vertical tensors.