Supplementary material to
"Testing for risk aversion in first-price sealed-bid auctions"

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Abstract. The purpose of this supplementary material is twofold. In the first section, we provide the auxiliary lemmas that are used in the Appendix of the paper. In the second, we discuss an extension of our test to the case where we have a binding reserve price.

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S.1. Auxiliary lemmas

S.1.1. **Mathematical results.** The first two auxiliary lemmas follow by arguments similar to the ones in Guerre, Perrigne, and Vuong (2009, Lemma 1) and Campo, Guerre, Perrigne, and Vuong (2011, Supplementary material). We provide them for the sake of completeness.

Lemma S.1. *The following statements hold.*

(1) $\mathcal{G}_0$ can be characterized as in expression (A.1).

(2) Pick any $\delta \in \mathcal{D}$ and $G \in \mathcal{G}_0$. Then,

$$\tilde{F}(v|I,x) = G\big\{\xi^{-1}_{(\delta,G)}(v,I,x)|I,x\big\}$$

is well-defined, where the (conditional) support of $\tilde{F}$ is given by $\big[\underline{v}_{\tilde{F}}(I,x),\ \bar{v}_{\tilde{F}}(I,x)\big]$ with

$$\underline{v}_{\tilde{F}}(I,x) = \underline{b}_G(I,x) \quad \text{and} \quad \bar{v}_{\tilde{F}}(I,x) = \xi_{(\delta,G)}\big\{\bar{b}_G(I,x),I,x\big\}.$$

Further, we have $\tilde{F} \in \mathcal{F}$, $s_{(\delta,\tilde{F})} = \xi^{-1}_{(\delta,G)}$, and $G_{(\delta,\tilde{F})} = G$.

Proof. For the first statement, recall from Guerre, Perrigne, and Vuong (2009, Lemma 1.(ii)) that $G \in \mathcal{G}_0$ if and only if $\partial_b \xi_{(\delta_0,G)}(b,I,x) > 0$ for all $(b,x) \in \mathcal{S}_G(I)$ and for all $I \in \mathcal{I}$. Therefore, the desired characterization of $\mathcal{G}_0$ follows from the fact that $\lambda^{-1}_{\delta_0}$ is just the identity function.

For the second statement, note that $\xi^{-1}_{(\delta,G)}$ is well-defined because $G \in \mathcal{G}_0$ and hence $\xi_{(\delta,G)}(\cdot,I,x)$ is monotonic by part (1). Therefore, $\tilde{F}$ is well-defined as well. Then, $\tilde{F} \in \mathcal{F}$ follows from Properties 1 and the smoothness properties of the equilibrium bidding function (Guerre, Perrigne, and Vuong, 2009, Theorem 1). Now, we consider proving $G_{(\delta,\tilde{F})} = G$. By the definition of $\tilde{F}$, we have $G(b|I,x) = \tilde{F}\big\{\xi_{(\delta,G)}(b,I,x)|I,x\big\}$. Therefore, $G_{(\delta,\tilde{F})} = G$ will follow if we show that $s_{(\delta,\tilde{F})} = \xi^{-1}_{(\delta,G)}$, which we will prove below.

First, note that $\xi^{-1}_{(\delta,G)}$ satisfies the boundary condition $\xi^{-1}_{(\delta,G)}\big\{\underline{v}_{\tilde{F}}(I,x),I,x\big\} = \underline{v}_{\tilde{F}}(I,x)$ because $\underline{v}_{\tilde{F}}(I,x) = \underline{b}_G(I,x)$. Second, by construction of $\tilde{F}$, we have

$$\frac{\tilde{f}(v|I,x)}{\tilde{F}(v|I,x)} = \partial_v \xi^{-1}_{(\delta,G)}(v,I,x) \frac{g\big\{\xi^{-1}_{(\delta,G)}(v,I,x)|I,x\big\}}{G\big\{\xi^{-1}_{(\delta,G)}(v,I,x)|I,x\big\}}. \tag{S.1}$$

Further, by (A.3), we know that

$$\frac{G\big\{\xi^{-1}_{(\delta,G)}(v,I,x)|I,x\big\}}{g\big\{\xi^{-1}_{(\delta,G)}(v,I,x)|I,x\big\}} = (I-1)\lambda_\delta\big\{v - \xi^{-1}_{(\delta,G)}(v,I,x)\big\}. \tag{S.2}$$

Combining (S.1) and (S.2) shows that

$$\partial_v \xi^{-1}_{(\delta,G)}(v,I,x) = (I-1)\frac{\tilde{f}(v|I,x)}{\tilde{F}(v|I,x)}\lambda_\delta\{v - \xi^{-1}_{(\delta,G)}(v,I,x)\}, \tag{S.3}$$

which is a differential equation corresponding to (5). Therefore, the equality $\xi^{-1}_{(\delta,G)} = s_{(\delta,\tilde{F})}$ follows from the uniqueness of the solution to (5). □

Lemma S.2. *Let $\phi : [0,1] \to \mathbb{R}$ be a function whose first derivative is continuous on $[0,1]$. Further, suppose that it satisfies the differential equation*

$$\forall \alpha \in (0,1],\ \partial_\alpha \phi(\alpha) = \frac{A(\alpha)}{\alpha} - \frac{B(\alpha)}{\alpha}\phi(\alpha),$$

where A and B are continuous functions on $[0,1]$, $B(\alpha) = B(0) + O(\alpha)$, and $B(0) > 0$. Then, we can write

$$\phi(\alpha) = \int_0^1 \Omega(t,\alpha)dt \quad \textit{with} \quad \Omega(t,\alpha) = \frac{A(t\alpha)}{t}\exp\left\{-\int_t^1 \frac{B(u\alpha)}{u}du\right\}.$$

Proof. Note that for any $\alpha_0 \in (0,1]$, the function

$$\begin{aligned}\phi(\alpha) = \left[\phi(\alpha_0) + \int_{\alpha_0}^{\alpha}\frac{A(t)}{t}\exp\Big\{\int_{\alpha_0}^{t}\frac{B(u)}{u}du\Big\}dt\right]\exp\Big\{-\int_{\alpha_0}^{\alpha}\frac{B(u)}{u}du\Big\}\\ = \phi(\alpha_0)\exp\Big\{-\int_{\alpha_0}^{\alpha}\frac{B(u)}{u}du\Big\} + \int_{\alpha_0}^{\alpha}\frac{A(t)}{t}\exp\Big\{-\int_{t}^{\alpha}\frac{B(u)}{u}du\Big\}dt\end{aligned}$$

is clearly a solution to the differential equation for $\alpha \in [\alpha_0, 1]$. Taking $\alpha_0 \downarrow 0$ and noting that ϕ is continuous at 0 and

$$\exp\Big\{-\int_{\alpha_0}^{\alpha}\frac{B(u)}{u}du\Big\} \to 0,$$

we obtain

$$\phi(\alpha) = \int_0^{\alpha}\frac{A(t)}{t}\exp\Big\{-\int_t^{\alpha}\frac{B(u)}{u}du\Big\}dt.$$

To complete the proof, apply the change of variables. □

Let ι denote the identity function. Hereafter, we consider the following spaces of functions:

$$\tilde{\mathcal{D}}(c) = \left\{\lambda : [0,c] \to \mathbb{R} :\ \exists\tilde{\lambda} : \mathbb{R} \to \mathbb{R} \text{ such that } \forall y \in [0,c], \tilde{\lambda}(y) = \lambda(y) \text{ and } \iota - \tilde{\lambda}^{-1} \in \mathcal{D}_{\mathrm{b}}(c,\varphi_c)\right\},$$

$$\tilde{\mathcal{F}}(c) = \left\{\vartheta : [0,1] \times \mathcal{I} \times \mathcal{X} \to \mathbb{R} :\ \exists F \in \mathcal{F}_{\mathrm{b}}(c) \text{ such that } \forall (I,x) \in \mathcal{I} \times \mathcal{X},\ \vartheta(\cdot|I,x) = F^{-1}(\cdot|I,x)\right\},$$

and we let $\tilde{\Theta}(c) = \tilde{\mathcal{D}}(c) \times \tilde{\mathcal{F}}(c)$. We also define the function

$$\breve{s}_{(\lambda,\vartheta)}(\alpha,I,x) := s_{(\iota-\tilde{\lambda}^{-1},\vartheta^{-1})}\{\vartheta(\alpha|I,x),I,x\}$$

for $\alpha \in [0,1]$, where $\tilde{\lambda}$ represents the extension associated with λ from the definition of $\tilde{\mathcal{D}}(c)$. We remark that $\breve{s}_{(\lambda,\vartheta)}$ is well-defined because the values of $\{\lambda(u) : u > c\}$ do not affect the equilibrium bidding function: recall that $\bar{v}(F) \leq c$ by the construction of $\mathcal{F}_{\mathrm{b}}(c)$. Also, it satisfies the differential equation

$$\partial_\alpha \breve{s}_{(\lambda,\vartheta)}(\alpha, I, x) = \frac{(I-1) \cdot \lambda\big\{\vartheta(\alpha|I,x) - \breve{s}_{(\lambda,\vartheta)}(\alpha, I, x)\big\}}{\alpha} \tag{S.4}$$

for all $\alpha \in (0,1]$, with the initial condition $\breve{s}_{(\lambda,\vartheta)}(0, I, x) = \vartheta(0|I,x)$.

Lemma S.3. *For any given $c > 1$, there exists a constant $\underline{c}_\alpha > 0$ such that*

$$\inf_{(\lambda,\vartheta)\in\tilde{\Theta}(c)} \ \inf_{(\alpha,I,x)\in[0,1]\times\mathcal{I}\times\mathcal{X}} \partial_\alpha \breve{s}_{(\lambda,\vartheta)}(\alpha, I, x) \geq \underline{c}_\alpha. \tag{S.5}$$

Further, there exists a finite constant $\bar{c}_\alpha > 0$ such that, for any nonnegative integers $t_0, t_1, \ldots, t_D$ satisfying $\sum_{j=0}^D t_j \leq S + 1$, we have

$$\sup_{(\lambda,\vartheta)\in\tilde{\Theta}(c)} \ \sup_{(\alpha,I,x)\in[0,1]\times\mathcal{I}\times\mathcal{X}} \left|\partial_{\alpha^{t_0}} \partial_{x^{t_1}} \ldots \partial_{x^{t_D}} \breve{s}_{(\lambda,\vartheta)}(\alpha, I, x)\right| \leq \bar{c}_\alpha. \tag{S.6}$$

Proof. Without loss of generality, we assume that $D = 1$ to simplify the exposition. Since $I \in \mathcal{I}$ will be fixed throughout the proof, we will omit I from the function argument notation.

From (S.4), it follows that

$$\begin{cases} \partial_\alpha \partial_x \breve{s}_{(\lambda,\vartheta)}(\alpha, x) = \dfrac{\breve{K}_{(\lambda,\vartheta)}(\alpha, x)\partial_x \vartheta(\alpha|x)}{\alpha} - \dfrac{\breve{K}_{(\lambda,\vartheta)}(\alpha, x)}{\alpha} \partial_x \breve{s}_{(\lambda,\vartheta)}(\alpha, x), \\ \partial_{\alpha^2} \breve{s}_{(\lambda,\vartheta)}(\alpha, x) = \dfrac{\breve{K}_{(\lambda,\vartheta)}(\alpha, x)\partial_\alpha \vartheta(\alpha|x)}{\alpha} - \dfrac{\breve{K}_{(\lambda,\vartheta)}(\alpha, x) + 1}{\alpha} \partial_\alpha \breve{s}_{(\lambda,\vartheta)}(\alpha, x), \end{cases}$$

where

$$\breve{K}_{(\lambda,\vartheta)}(\alpha, x) := (I-1)\lambda'\{\vartheta(\alpha|x) - \breve{s}_{(\lambda,\vartheta)}(\alpha, x)\}$$

satisfies $\breve{K}_{(\lambda,\vartheta)}(\alpha, x) = \breve{K}_{(\lambda,\vartheta)}(0, x) + O(\alpha)$ with $\breve{K}_{(\lambda,\vartheta)}(0, x) \geq I - 1$. We emphasize that $I \geq 2$. Then, after applying Lemma S.2, we obtain the following representations:

$$\partial_\alpha \breve{s}_{(\lambda,\vartheta)}(\alpha, x) = \int_0^1 \Omega^{(1)}_{(\lambda,\vartheta)}(t, \alpha, x) dt \quad \text{and} \quad \partial_x \breve{s}_{(\lambda,\vartheta)}(\alpha, x) = \int_0^1 \Omega^{(2)}_{(\lambda,\vartheta)}(t, \alpha, x) dt, \tag{S.7}$$

where we write

$$\Omega^{(j)}_{(\lambda,\vartheta)}(t, \alpha, x) = \frac{A^{(j)}_{(\lambda,\vartheta)}(t\alpha, x)}{t} \exp\left\{-\int_t^1 \frac{B^{(j)}_{(\lambda,\vartheta)}(u\alpha, x)}{u} du\right\}$$

for $j = 1, 2$, $A^{(1)}_{(\lambda,\vartheta)} = \breve{K}_{(\lambda,\vartheta)} \cdot \partial_\alpha \vartheta$, $A^{(2)}_{(\lambda,\vartheta)} = \breve{K}_{(\lambda,\vartheta)} \cdot \partial_x \vartheta$, $B^{(1)}_{(\lambda,\vartheta)} = \breve{K}_{(\lambda,\vartheta)} + 1$, and $B^{(2)}_{(\lambda,\vartheta)} = \breve{K}_{(\lambda,\vartheta)}$. Since

$$I - 1 \ \leq \ \breve{K}_{(\lambda,\vartheta)}(\alpha, x) \ \leq \ (I-1) \sup_{(\lambda,\vartheta)\in\tilde{\Theta}(c)} \ \sup_{y\in[0,c]} |\lambda'(y)| \ \leq \ (I-1)\varphi_c$$

for any $(\alpha, x) \in [0,1] \times \mathcal{X}$, we know that $I \leq B^{(1)}_{(\lambda,\vartheta)}(u\alpha, x) \leq (I-1)\varphi_c + 1$ and $I - 1 \leq B^{(2)}_{(\lambda,\vartheta)}(u\alpha, x) \leq (I-1)\varphi_c$. As a result, we can bound

$$t^{(I-1)\varphi_c+1} \leq C^{(j)}_{(\lambda,\vartheta)}(t,\alpha,x) := \exp\left\{-\int_t^1 \frac{B^{(j)}_{(\lambda,\vartheta)}(u\alpha,x)}{u}du\right\} \leq t^{I-1} \tag{S.8}$$

for $j = 1,2$. Further, since $c^{-1} \leq \partial_\alpha \vartheta(\alpha|x) \leq c$, we have $c^{-1} \leq A^{(1)}_{(\lambda,\vartheta)}(t\alpha, x) \leq (I-1)c\varphi_c$ and

$$c^{-1}t^{(I-1)\cdot\varphi_c} \leq \Omega^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x) \leq (I-1)c\varphi_c t^{I-2}. \tag{S.9}$$

Combining the first expression in (S.7) with (S.9), we can prove the first claim of the lemma by taking $\underline{c}_\alpha = c^{-1}\int_0^1 t^{(I-1)\varphi_c}dt$.

For the second claim of the lemma, we consider $\left|\partial_{\alpha^{t_0}}\partial_{x^{t_1}}\breve{s}_{(\lambda,\vartheta)}(\alpha,x)\right|$. We start with the simplest case $S = 1$ as an illustration. As $t_0 + t_1 \leq 2$, there are six cases to consider. The case of $(t_0,t_1) = (0,0)$ is trivial because $\left|\bar{v}_F(x)\right| \leq c$. The case $(t_0,t_1) = (1,0)$ follows from the first expression in (S.7) and the inequality (S.9), which yields

$$\left|\partial_\alpha \breve{s}_{(\lambda,\vartheta)}(\alpha,x)\right| \leq \breve{c}_1 := (I-1)c\varphi_c\int_0^1 t^{I-2}dt < \infty.$$

To deal with the case $(t_0,t_1) = (0,1)$, note that $\left|\partial_x\vartheta(\alpha|x)\right| \leq c^2$ by the implicit function theorem, which implies

$$\left|\Omega^{(2)}_{(\lambda,\vartheta)}(t,\alpha)\right| \leq (I-1)c^2\varphi_c t^{I-2}.$$

As a result, from the second expression in (S.7), we obtain the uniform bound

$$\left|\partial_x \breve{s}_{(\lambda,\vartheta)}(\alpha,x)\right| \leq \breve{c}_2 := (I-1)c^2\varphi_c\int_0^1 t^{I-2}dt < \infty.$$

Now consider the cases $(t_0,t_1) = (2,0)$ and $(t_0,t_1) = (0,2)$. By the dominated convergence theorem and expression (S.7), we have

$$\partial_{\alpha^2}\breve{s}_{(\lambda,\vartheta)}(\alpha,x) = \int_0^1 \partial_\alpha\Omega^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x)dt \quad \text{and} \quad \partial_{x^2}\breve{s}_{(\lambda,\vartheta)}(\alpha,x) = \int_0^1 \partial_x\Omega^{(2)}_{(\lambda,\vartheta)}(t,\alpha,x)dt.$$

With a slight abuse of notation, letting ∂_1 and ∂_2 denote partial derivatives of a given function with respect to its first and second argument, respectively, we can write

$$\partial_\alpha\Omega^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x) = \left\{\partial_1 A^{(1)}_{(\lambda,\vartheta)}(t\alpha,x) - \frac{A^{(1)}_{(\lambda,\vartheta)}(t\alpha,x)}{t}\int_t^1 \partial_1 B^{(1)}_{(\lambda,\vartheta)}(tu,x)du\right\}C^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x),$$

$$\partial_x\Omega^{(2)}_{(\lambda,\vartheta)}(t,\alpha,x) = \left\{\frac{\partial_2 A^{(2)}_{(\lambda,\vartheta)}(t\alpha,x)}{t} - \frac{A^{(2)}_{(\lambda,\vartheta)}(t\alpha,x)}{t}\int_t^1 \frac{\partial_2 B^{(2)}_{(\lambda,\vartheta)}(tu,x)}{u}du\right\}C^{(2)}_{(\lambda,\vartheta)}(t,\alpha,x).$$

We observe that

$$\begin{aligned} \left|\partial_1 B^{(1)}_{(\lambda,\vartheta)}(tu,x)\right| &= \left|\partial_1 B^{(2)}_{(\lambda,\vartheta)}(tu,x)\right| \le \breve{c}_3 := (I-1)\varphi_c\left(c+\breve{c}_1\right), \\ \left|\partial_2 B^{(1)}_{(\lambda,\vartheta)}(tu,x)\right| &= \left|\partial_2 B^{(2)}_{(\lambda,\vartheta)}(tu,x)\right| \le \breve{c}_4 := (I-1)\varphi_c(c^2+\breve{c}_2). \end{aligned} \tag{S.10}$$

In addition, we have

$$\begin{aligned} \left|\partial_1 A^{(1)}_{(\lambda,\vartheta)}(tu,x)\right| &\le \breve{c}_5 := \breve{c}_3 c + (I-1)\varphi_c c^3, \\ \left|\partial_2 A^{(2)}_{(\lambda,\vartheta)}(tu,x)\right| &\le \breve{c}_6 := \breve{c}_4 c^2 + 4(I-1)\varphi_c c^6. \end{aligned} \tag{S.11}$$

Therefore, in view of (S.8), we can bound

$$\begin{aligned} \left|\partial_{\alpha^2}\breve{s}_{(\lambda,\vartheta)}(\alpha,x)\right| &\le \int_0^1 \left|\partial_\alpha \Omega^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x)\right| dt \le \int_0^1 \breve{c}_5\left\{t+\breve{c}_4(1-t)\right\} t^{I-2} dt < \infty, \\ \left|\partial_{x^2}\breve{s}_{(\lambda,\vartheta)}(\alpha,x)\right| &\le \int_0^1 \left|\partial_x \Omega^{(2)}_{(\lambda,\vartheta)}(t,\alpha,x)\right| dt \le \int_0^1 \breve{c}_6(1-\breve{c}_5\log t) t^{I-2} dt < \infty. \end{aligned}$$

Thus, the only remaining case is $(t_0,t_1)=(1,1)$. But this case can be shown in the same way as $\partial_{\alpha^2}\breve{s}_{(\lambda,\vartheta)}$ and $\partial_{x^2}\breve{s}_{(\lambda,\vartheta)}$ because we have that

$$\partial_\alpha\partial_x\breve{s}_{(\lambda,\vartheta)}(\alpha,x) = \int_0^1 \partial_x\Omega^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x)dt = \int_0^1 \partial_\alpha\Omega^{(2)}_{(\lambda,\vartheta)}(t,\alpha,x)dt$$

by the dominated convergence theorem and we can also use (S.10). So the proof for the case of $S=1$ is complete.

Now consider the general case $S\in\mathbb{N}$ and suppose that $t_0+t_1=S$. From eqs. (S.7), we can write

$$\begin{aligned} \partial_{\alpha^{t_0+1}}\partial_{x^{t_1}}\breve{s}_{(\lambda,\vartheta)}(t,\alpha,x) &= \int_0^1 \partial_{\alpha^{t_0}}\partial_{x^{t_1}}\Omega^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x)dt, \\ \partial_{\alpha^{t_0}}\partial_{x^{t_1+1}}\breve{s}_{(\lambda,\vartheta)}(t,\alpha,x) &= \int_0^1 \partial_{\alpha^{t_0}}\partial_{x^{t_1}}\Omega^{(2)}_{(\lambda,\vartheta)}(t,\alpha,x)dt. \end{aligned} \tag{S.12}$$

Next, we elaborate the integrands of the right-hand side expressions to show that the integrals are uniformly bounded. Since the two cases can be dealt with similarly, we focus on the first one, i.e. (S.12). Combining Leibniz's rule with Faà Di Bruno's formula (see e.g. Roman, 1980), we obtain

$$\partial_{\alpha^{t_0}}\Omega^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x) = C^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x)\ \times$$
$$\sum_{r=0}^{t_0}\sum \frac{t_0!}{r!(t_0-r)!}\frac{r!}{m_1!m_2!\dots m_r!}\left\{\frac{A^{(1)}_{(\lambda,\vartheta)}(t\alpha,x)}{t}\right\}^{[t_0-r]}\prod_{j=1}^{r}\left\{\left[-\int_t^1\left\{\frac{B^{(1)}_{(\lambda,\vartheta)}(u\alpha,x)}{u}\right\}^{[j]}du\right]\Big/ j!\right\}^{m_j},$$

where here the superscript $^{[\ell]}$ denotes the ℓ^{th} order partial derivatives with respect to α and the second summation is over all nonnegative integers $m_1,\dots,m_r$ such that $m_1+2m_2+\dots+r\cdot m_r=r$.

This is clearly a weighted sum of

$$C^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x)\left\{\frac{A^{(1)}_{(\lambda,\vartheta)}(t\alpha,x)}{t}\right\}^{[t_0-r]}\prod_{j=1}^{r}\left\{\left[-\int_t^1\left\{\frac{B^{(1)}_{(\lambda,\vartheta)}(u\alpha,x)}{u}\right\}^{[j]}du\right]/j!\right\}^{m_j}.$$

Applying again Leibniz's rule, we obtain that $\partial_{\alpha^{t_0}}\partial_{x^{t_1}}\Omega^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x)$ is a weighted sum of

$$\underbrace{\left\{C^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x)\right\}^{<a>}}_{F_1}\underbrace{\left[\left\{\frac{A^{(1)}_{(\lambda,\vartheta)}(t\alpha,x)}{t}\right\}^{[t_0-r]}\right]^{<b>}}_{F_2}\underbrace{\prod_{j=1}^{r}\left[\left\{\left[-\int_t^1\left\{\frac{B^{(1)}_{(\lambda,\vartheta)}(u\alpha,x)}{u}\right\}^{[j]}du\right]/j!\right\}^{m_j}\right]^{<\ell_j>}}_{F_3},$$

where $\{a,b,\ell_1,\dots,\ell_q\}$ are nonnegative integers such that $a+b+\sum_j^r\ell_j=t_1$ and here the superscript $^{<\ell>}$ denotes the ℓ^{th} order derivative with respect to x. Below, we consider the factors F_1, F_2, and F_3 separately. Consider F_1 first: by Faà Di Bruno's formula, such a factor is a weighted sum of

$$C^{(1)}_{(\lambda,\vartheta)}(t,\alpha,x)\prod_{i=1}^{a}\left[\left\{-\int_t^1\frac{\partial_{x^i}B^{(1)}_{(\lambda,\vartheta)}(u\alpha,x)}{u}du\right\}/i!\right]^{\tilde m_i}, \tag{S.13}$$

where $\tilde m_1,\dots,\tilde m_a$ are nonnegative integers such that $\tilde m_1+2\tilde m_2+\dots+a\tilde m_a=a$. Proceeding as in eqs. (S.10) and (S.11), by the construction of $\tilde\Theta(c)$, we can find a finite constant $\check c_7>0$ such that

$$\max_{i=1,\dots,S}\sup_{(\lambda,\vartheta)}\sup_{(\alpha,x)}\max\left\{\left|\partial_{\alpha^{S-i}}\partial_{x^i}A^{(1)}_{(\lambda,\vartheta)}(\alpha,x)\right|,\left|\partial_{\alpha^{S-i}}\partial_{x^i}B^{(1)}_{(\lambda,\vartheta)}(\alpha,x)\right|\right\}\le\check c_7.$$

Hence, the absolute value of (S.13) can be bounded above by

$$\check c_7^S t^{I-1}\prod_{i=1}^{a}(-\log t)^{\tilde m_i}, \tag{S.14}$$

where we used the upper inequality in (S.8). Regarding F_2, the worst case arises when $r=t_0$ because otherwise the $t\in(0,1)$ in the denominator disappears. Focusing on such a case, we have that

$$\left|\left[\left\{\frac{A^{(1)}_{(\lambda,\vartheta)}(t\alpha,x)}{t}\right\}\right]^{<b>}\right|=\left|\frac{\partial_{x^b}A^{(1)}_{(\lambda,\vartheta)}(t\alpha,x)}{t}\right|\le\frac{\check c_7}{t}. \tag{S.15}$$

Finally, regarding F_3, note that

$$\left|\int_t^1\left\{\frac{B^{(1)}_{(\lambda,\vartheta)}(u\alpha,x)}{u}\right\}^{[j]}du\right|\le\check c_7\int_t^1u^{j-1}du. \tag{S.16}$$

Combining expressions (S.13)-(S.16) shows that the product of the factors F_1, F_2, and F_3 is bounded by the expression

$$\breve{c}_7^{S+2} t^{I-2} (-\log t)^{\tilde{m}_1+\cdots+\tilde{m}_a} \prod_{j=1}^{r} (1-t^j), \tag{S.17}$$

which is integrable over $t \in (0,1)$. This proves that (S.12) is uniformly bounded. □

Lemma S.4. *Let $\{\phi_L : L \in \mathbb{N}\}$ be an equicontinuous family of real functions defined on $[a,b]$. Assume that $\phi_L(t) \to \phi(t)$ pointwise at every $t \in (a,b)$ for some function ϕ that is continuous on $[a,b]$. Then, the following statements hold.*

(1) $\phi_L \to \phi$ uniformly on $[a,b]$.

(2) Suppose further that the first derivatives of $\{\phi_L : L \in \mathbb{N}\}$ exist on $[a,b]$. Then, if $\{\phi'_L : L \in \mathbb{N}\}$ is a uniformly bounded and equicontinuous family, we have that

$$\sup_{t \in [a,b]} \left| \phi'_L(t) - \phi'(t) \right| \to 0.$$

Proof. To prove the first statement, by compactness and equicontinuity, it suffices to show that $\phi_L(t) \to \phi(t)$ for $t = a$ and $t = b$: see e.g. Rudin (1976, p. 168). Since the two cases are exactly symmetric, we focus on $t = a$. Pick an arbitrary $\varepsilon > 0$. Since $\{\phi_L : L \in \mathbb{N}\}$ is equicontinuous and ϕ is continuous, we can choose some $\tilde{a} > a$ such that for all $L \in \mathbb{N}$, $|\phi_L(\tilde{a}) - \phi_L(a)| \le \varepsilon/3$, and $|\phi(\tilde{a}) - \phi(a)| \le \varepsilon/3$. Since we know that $\phi_L(\tilde{a}) \to \phi(\tilde{a})$, we can also choose $L_\epsilon \in \mathbb{N}$ such that $L \ge L_\varepsilon$ implies $|\phi_L(\tilde{a}) - \phi(\tilde{a})| \le \varepsilon/3$. Therefore, by using the triangular inequality, $L \ge L_\varepsilon$ implies that

$$|\phi_L(a) - \phi(a)| \le |\phi_L(a) - \phi_L(\tilde{a})| + |\phi_L(\tilde{a}) - \phi(\tilde{a})| + |\phi(\tilde{a}) - \phi(a)| \le \varepsilon.$$

For the second statement, suppose that ϕ'_L does not converge uniformly. Then, there exists some $\epsilon > 0$ and a subsequence $\{\phi'_{L_m} : m \in \mathbb{N}\}$ such that

$$\sup_{t \in [a,b]} \left| \phi'_{L_m}(t) - \phi'(t) \right| > \epsilon \tag{S.18}$$

for all $m \in \mathbb{N}$. But since $\{\phi'_{L_m} : m \in \mathbb{N}\}$ is a uniformly bounded and equicontinuous family, by the Ascoli–Arzelá theorem (see e.g. Rudin, 1987, Theorem 11.28), there is a further subsequence $\{\phi'_{L_{m_j}} : j \in \mathbb{N}\}$ that uniformly converges to some (continuous) limiting function, say $\tilde{\phi}'$:

$$\sup_{t \in [a,b]} \left| \phi'_{L_{m_j}}(t) - \tilde{\phi}'(t) \right| \to 0.$$

Now, note that $\phi_{L_{m_j}}(t) = \phi_{L_{m_j}}(a) + \int_a^t \phi'_{L_{m_j}}(u)du$ for all $t \in [a,b]$ and define $\tilde{\phi}(t) = \phi(a) + \int_a^t \tilde{\phi}'(u)du$. Then, by construction, we must have $\sup_{t\in[a,b]}\left|\phi_{L_{m_j}}(t) - \tilde{\phi}(t)\right| \to 0$, which means that $\tilde{\phi} = \phi$ and therefore $\tilde{\phi}' = \phi'$. So we have $\sup_{t\in[a,b]}\left|\phi'_{L_{m_j}}(t) - \phi'(t)\right| \to 0$, but this contradicts (S.18). □

Now consider an arbitrary $(\lambda_*, \vartheta_*) \in \tilde{\Theta}(c)$ together with a sequence $\left\{(\lambda_L, \vartheta_L) \in \tilde{\Theta}(c) : L \in \mathbb{N}\right\}$ that satisfies the following conditions:

(A.C1) $\lambda_L(u) \to \lambda_*(u)$ at every $u \in [0,c]$;

(A.C2) $\vartheta_L(\alpha|I,x) \to \vartheta_*(\alpha|I,x)$ at every $(\alpha, I, x) \in [0,1] \times \mathcal{I} \times \mathcal{X}$.

Before proceeding to the next lemma, we remark that for conditions (A.C1) and (A.C2), uniform and pointwise convergence are equivalent by compactness of the domains and equicontinuity of the sequences. We note that the set $\mathcal{I}$ is irrelevant because it is finite.

Lemma S.5. *If $\left\{(\lambda_L, \vartheta_L) \in \tilde{\Theta}(c) : L \in \mathbb{N}\right\}$ satisfies conditions (A.C1) and (A.C2), then*

$$\sup_{(\alpha,I,x)\in[0,1]\times\mathcal{I}\times\mathcal{X}} \left|\partial_{\alpha^t}\breve{s}_{(\lambda_L,\vartheta_L)}(\alpha,I,x) - \partial_{\alpha^t}\breve{s}_{(\lambda_*,\vartheta_*)}(\alpha,I,x)\right| \to 0$$

for any $t = 0, 1, \ldots, S$.

Proof. First, note that for every $I \in \mathcal{I}$ and for every $t = 0, \ldots, S$, $\{\partial_{\alpha^t}\breve{s}_{(\lambda_L,\vartheta_L)}(\cdot, I, \cdot) : L \in \mathbb{N}\}$ is an equicontinuous family by Lemma S.3, where $\mathcal{I}$ is finite and $[0,1]\times\mathcal{X}$ is compact. Therefore, it suffices to establish pointwise convergence. So we fix $(I,x) \in \mathcal{I}\times\mathcal{X}$ and abbreviate $\breve{s}_L(\alpha) = \breve{s}_{(\lambda_L,\vartheta_L)}(\alpha, I, x)$ as well as $\breve{s}_*(\alpha) = \breve{s}_{(\lambda_*,\vartheta_*)}(\alpha, I, x)$.

Consider the case $t = 0$. Our arguments for this proof are similar to the ones employed in the proof of Theorem 4.1 in Coddington and Levinson (1955, ch. 2). Suppose that there exists $\tilde{\alpha} \in [0,1]$ such that $\breve{s}_L(\tilde{\alpha})$ does not converge to $\breve{s}_*(\tilde{\alpha})$. Then, there exists some $\epsilon > 0$ and a subsequence $\left\{\breve{s}_{L_m}(\tilde{\alpha}) : m \in \mathbb{N}\right\}$ such that $\left|\breve{s}_{L_m}(\tilde{\alpha}) - \breve{s}_*(\tilde{\alpha})\right| \geq \epsilon$ for all m. Since $\{\breve{s}_{L_m}\}$ is uniformly bounded and equicontinuous by Lemma S.3, the Ascoli-Arzela theorem shows that there exists a further subsequence $\left\{\breve{s}_{L_{m_j}} : j \in \mathbb{N}\right\}$ that connverges uniformly to some function $\tilde{s}$. Such a subsequence satisfies (S.4), and hence

$$\breve{s}_{L_{m_j}}(\tilde{\alpha}) = \vartheta_{L_{m_j}}(0|I,x) + (I-1)\int_0^{\tilde{\alpha}} \frac{\lambda_{L_{m_j}}\left\{\vartheta_{L_{m_j}}(t|I,x) - \breve{s}_{L_{m_j}}(t)\right\}}{t} dt. \tag{S.19}$$

Note that $\lambda_{L_{m_j}}\left\{\vartheta_{L_{m_j}}(t|I,x) - \breve{s}_{L_{m_j}}(t)\right\} \to \lambda_*\left\{\vartheta_*(t|I,x) - \tilde{s}(t)\right\}$ pointwise at $t \in [0,1]$ because of conditions (A.C1)-(A.C2) and by the construction of $\left\{\breve{s}_{L_{m_j}} : j \in \mathbb{N}\right\}$. Therefore, taking $j \to \infty$ on both sides of (S.19) and applying the dominated convergence theorem yields

$$\tilde{s}(\tilde{\alpha}) = \vartheta_*(0|I,x) + (I-1)\int_0^{\tilde{\alpha}} \frac{\lambda_*\{\vartheta_*(t|I,x) - \tilde{s}(t)\}}{t} dt, \tag{S.20}$$

which implies that $\tilde{s}(\tilde{\alpha}) = \breve{s}_*(\tilde{\alpha})$ because $\breve{s}_*$ uniquely solves the differential equation implied by (S.4) given the initial condition. But this contradicts $\left|\breve{s}_{L_m}(\tilde{\alpha}) - \breve{s}_*(\tilde{\alpha})\right| \geq \epsilon$ for every $m \in \mathbb{N}$.

The cases of $t = 1, \ldots, S$ follow from the case of $t = 0$ and Lemmas S.3 and S.4. □

Before proceeding to the next lemma, we introduce some additional notation. Define

$$\breve{R}_{(\lambda,\vartheta)}(\alpha|I,x) = \lambda\left\{\vartheta(\alpha|I,x) - \breve{s}_{(\lambda,\vartheta)}(\alpha,I,x)\right\}.$$

Note that $G^{-1}_{(\delta,F)}(\alpha|I,x) = \breve{s}_{(\lambda_\delta,F^{-1})}(\alpha,I,x)$ and $R_{(\delta,F)}(\alpha|I,x) = \breve{R}_{(\lambda_\delta,F^{-1})}(\alpha|I,x)$. So by expression (7), we have

$$\breve{R}_{(\lambda_\delta,F^{-1})}(\alpha|I,x) = R_{(\delta,F)}(\alpha|I,x) = \mathcal{R}_{G_{(\delta,F)}}\left\{\breve{s}_{(\lambda_\delta,F^{-1})}(\alpha,I,x) \;\middle|\; I,x\right\},$$

which implies that $\partial_\alpha \breve{R}_{(\lambda_\delta,F^{-1})}(\alpha|I,x) = \partial_b \mathcal{R}_{G_{(\delta,F)}}\left\{\breve{s}_{(\lambda_\delta,F^{-1})}(\alpha,I,x) \;\middle|\; I,x\right\} \cdot \partial_\alpha \breve{s}_{(\lambda_\delta,F^{-1})}(\alpha,I,x)$. Now, we define

$$\breve{\mathcal{R}}^*_I(\lambda,\vartheta,\alpha,x) = \frac{\partial_\alpha \breve{R}_{(\lambda,\vartheta)}(\alpha|I,x)}{\partial_\alpha \breve{s}_{(\lambda,\vartheta)}(\alpha,I,x)}, \tag{S.21}$$

and we have

$$\breve{\mathcal{R}}^*_I(\lambda_\delta,F^{-1},\alpha,x) = \partial_b \mathcal{R}_{G_{(\delta,F)}}\left\{\breve{s}_{(\lambda_\delta,F^{-1})}(\alpha,I,x)|I,x\right\}$$

for every $(\delta,F) \in \Theta_{\rm b}(c)$. Also, define

$$\underline{\mathcal{R}}(\lambda,\vartheta) = \min_{(\alpha,I,x)\in[0,1]\times\mathcal{I}\times\mathcal{X}} \breve{\mathcal{R}}^*_I(\lambda,\vartheta,\alpha,x),$$

and we obtain

$$\underline{\mathcal{R}}(\lambda_\delta,F^{-1}) = \min_{I\in\mathcal{I}} \min_{(b,x)\in\mathcal{S}_{G_{(\delta,F)}}(I)} \partial_b \mathcal{R}_{G_{(\delta,F)}}(b|I,x). \tag{S.22}$$

In addition, we equip $\tilde{\Theta}(c)$ and $\tilde{\Theta}(c) \times [0,1] \times \mathcal{X}$ with the metrics defined by

$$d_\Theta\left\{(\lambda,\vartheta),(\tilde{\lambda},\tilde{\vartheta})\right\} = \max\left\{\sup_{y\in[0,c]} |\lambda(y) - \tilde{\lambda}(y)|, \sup_{(\alpha,I,x)\in[0,1]\times\mathcal{I}\times\mathcal{X}} \left|\vartheta(\alpha|I,x) - \tilde{\vartheta}(\alpha|I,x)\right|\right\}$$

and

$$\breve{d}\left\{(\lambda,\vartheta,\alpha,x),(\tilde{\lambda},\tilde{\vartheta},\tilde{\alpha},\tilde{x})\right\} = \max\left\{d_\Theta\left((\lambda,\vartheta),(\tilde{\lambda},\tilde{\vartheta})\right), \|(\alpha,x) - (\tilde{\alpha},\tilde{x})\|\right\},$$

respectively. We are now ready to state the next lemma.

Lemma S.6. *$\underline{\mathcal{R}}$ is continuous on $\tilde{\Theta}(c)$.*

Proof. For the first statement, by the Theorem of the Maximum (see e.g., Ok, 2007), it suffices to show that $\breve{\mathcal{R}}_I^*$ is continuous on $\tilde{\Theta}(c) \times [0,1] \times \mathcal{X}$ for each $I \in \mathcal{I}$. Fix $I \in \mathcal{I}$, pick any $(\lambda, \vartheta, \alpha, x) \in \tilde{\Theta}(c) \times [0,1] \times \mathcal{X}$, and consider an arbitrary sequence $\big\{(\lambda_L, \vartheta_L, \alpha_L, x_L) \in \tilde{\Theta}(c) \times [0,1] \times \mathcal{X} : L \in \mathbb{N}\big\}$ that satisfies

$$\breve{d}\big\{(\lambda_L, \vartheta_L, \alpha_L, x_L), (\lambda, \vartheta, \alpha, x)\big\} \to 0. \tag{S.23}$$

In view of the definition in (S.21), we treat the numerator and the denominator separately. First, the denominator $\partial_\alpha \breve{s}$ is continuous at $(\lambda, \vartheta, \alpha, x)$ because

$$\begin{aligned}
&\left|\partial_\alpha \breve{s}_{(\lambda_L,\vartheta_L)}(\alpha_L, I, x_L) - \partial_\alpha \breve{s}_{(\lambda,\vartheta)}(\alpha, I, x)\right| \\
&\quad \le \left|\partial_\alpha \breve{s}_{(\lambda_L,\vartheta_L)}(\alpha_L, I, x_L) - \partial_\alpha \breve{s}_{(\lambda,\vartheta)}(\alpha_L, I, x_L)\right| + \left|\partial_\alpha \breve{s}_{(\lambda,\vartheta)}(\alpha_L, I, x_L) - \partial_\alpha \breve{s}_{(\lambda,\vartheta)}(\alpha, I, x)\right| \\
&\qquad \le \sup_{(\alpha,I,x)} \left|\partial_\alpha \breve{s}_{(\lambda_L,\vartheta_L)}(\alpha, I, x) - \partial_\alpha \breve{s}_{(\lambda,\vartheta)}(\alpha, I, x)\right| + \ \bar{c}_\alpha \cdot \|(\alpha_L, x_L) - (\alpha, x)\|,
\end{aligned} \tag{S.24}$$

where the constant $\bar{c}_\alpha > 0$ comes from Lemma S.3; here, by Lemma S.5 and eq. (S.23), the two terms on the utmost right–hand side of (S.24) converge to zero. Now, considering the numerator $\partial_\alpha R$, we write

$$\begin{aligned}
&\partial_\alpha R_{(\lambda_L,\vartheta_L)}(\alpha_L | I, x_L) \\
&\quad = \lambda_L' \big\{\vartheta_L(\alpha_L | I, x_L) - \breve{s}_{(\lambda_L,\vartheta_L)}(\alpha_L, I, x_L)\big\}\big\{\partial_\alpha \vartheta_L(\alpha_L | I, x_L) - \partial_\alpha \breve{s}_{(\lambda_L,\vartheta_L)}(\alpha_L, I, x_L)\big\}.
\end{aligned} \tag{S.25}$$

By using the triangular inequality similarly to (S.24), we have

$$\left|\vartheta_L(\alpha_L | I, x_L) - \vartheta(\alpha | I, x)\right| \le \sup_{(\alpha,I,x)} \left|\vartheta_L(\alpha | I, x) - \vartheta(\alpha | I, x)\right| + (c + c^2)\|(\alpha_L, x_L) - (\alpha, x)\|, \tag{S.26}$$

so the right-hand side of (S.26) converges to zero by (S.23). Also, a similar argument shows that $\breve{s}_{(\lambda_L,\vartheta_L)}(\alpha_L, I, x_L) \to \breve{s}_{(\lambda,\vartheta)}(\alpha, I, x)$. Since (S.23) and Lemma S.4 imply that λ_L' converges to λ' uniformly on $[0, c]$, a similar use of the triangular inequality shows that

$$\lambda_L' \big\{\vartheta_L(\alpha_L | I, x_L) - \breve{s}_{(\lambda_L,\vartheta_L)}(\alpha_L, I, x_L)\big\} \to \lambda' \big\{\vartheta(\alpha | I, x) - \breve{s}_{(\lambda,\vartheta)}(\alpha, I, x)\big\}. \tag{S.27}$$

Similarly, condition (S.23) and Lemma S.4 imply that $\partial_\alpha \vartheta_L(\cdot, I, \cdot)$ converges uniformly to $\partial_\alpha \vartheta(\cdot, I, \cdot)$ on $[0,1] \times \mathcal{X}$. Hence, by the same trick of the triangular inequality, we know that

$$\partial_\alpha \vartheta_L(\alpha_L, I, x_L) \to \partial_\alpha \vartheta(\alpha, I, x). \tag{S.28}$$

Finally, combining together (S.24), (S.27), (S.28), and (S.25) shows the desired result:

$$\partial_\alpha R_{(\lambda_L,\vartheta_L)}(\alpha_L|I,x_L)$$
$$\to \lambda'\big\{\vartheta(\alpha|I,x) - \breve{s}_{(\lambda,\vartheta)}(\alpha,I,x)\big\}\big\{\partial_\alpha\vartheta(\alpha|I,x) - \partial_\alpha\breve{s}_{(\lambda,\vartheta)}(\alpha,I,x)\big\} = \partial_\alpha R_{(\lambda,\vartheta)}(\alpha|I,x). \quad \square$$

Lemma S.7. *If $\{(\delta_L, F_L) \in \Theta_{\rm b}(c) : L \in \mathbb{N}\}$ satisfies conditions (C1) and (C2) of Section 3, then*

(1) $\{(\lambda_{\delta_L}, F_L^{-1}) \in \tilde{\Theta}(c) : L \in \mathbb{N}\}$ satisfies (A.C1) and (A.C2) with $(\lambda_*, \vartheta_*) = (\iota, F_*^{-1})$;

(2) there exist $\tilde{L} \in \mathbb{N}$ and $\underline{c}_\xi > 0$ such that $L \geq \tilde{L}$ implies $G_L := G_{(\delta_L,F_L)} \in \mathcal{G}_0$ and

$$\min_{I\in\mathcal{I}} \min_{(b,x)\in\mathscr{S}_{G_L}(I)} \partial_b\xi_{(\delta_0,G_L)}(b,I,x) \;\geq\; \underline{c}_\xi.$$

Proof. Pick any $y \in [0,c]$ and note that $\lambda_{\delta_L}^{-1}(y) \leq y \leq c$. Then,

$$\big|\lambda_{\delta_L}(y) - y\big| = \Big|\lambda_{\delta_L}(y) - \lambda_{\delta_L}\big\{\lambda_{\delta_L}^{-1}(y)\big\}\Big| \leq \varphi_c\big|y - \lambda_{\delta_L}^{-1}(y)\big| = \varphi_c\big|\delta_L(y)\big|,$$

which proves that (C1) implies that (A.C1) is satisfied as we set $\lambda_* = \iota$. In order to show that (C2) implies (A.C2), fix $(I,x) \in \mathcal{I}\times\mathcal{X}$ and note that $F_L^{-1}(\alpha|I,x) \to F_*^{-1}(\alpha|I,x)$ at every $\alpha \in (0,1)$ (see e.g., Lehmann and Romano, 2005, Lemma 11.2.1). So, the desired results follows immediately from Lemma S.4.(1).

Regarding the second statement, combine Lemma S.6 and part (1) of this lemma with (S.22) and part (1) of Lemma S.1 to obtain

$$\underline{\mathcal{R}}(\lambda_{\delta_L}, F_L^{-1}) \to \underline{\mathcal{R}}(\iota, F_*^{-1}) > -1.$$

Consequently, there are $\varepsilon > 0$ and $\tilde{L} \in \mathbb{N}$ such that $\underline{\mathcal{R}}(\lambda_{\delta_L}, F_L^{-1}) \geq -1 + \varepsilon$ for all $L \geq \tilde{L}$. To complete the proof, choose $\underline{c}_\xi = \varepsilon$ and note that $\partial_b\xi_{(\delta_0,G_L)}(b,I,x) = 1 + \partial_b\mathcal{R}_{G_L}(b|I,x) \geq 1 + \underline{\mathcal{R}}(\lambda_{\delta_L}, F_L^{-1}) \geq \underline{c}_\xi$ for all $L \geq \tilde{L}$. $\square$

Lemma S.8. *There exists $\underline{c}_g > 0$ such that*

$$\inf_{(\delta,F)\in\Theta_{\rm b}(c)} \min_{I\in\mathcal{I}} \min_{(b,x)\in\mathcal{S}_{G_{(\delta,F)}}(I)} g_{(\delta,F)}(b|I,x) \geq \underline{c}_g.$$

Further, there exist finite constants $\bar{c}_G > 0$ and $\bar{c}_\xi > 0$ such that for any nonnegative integers $t_0, t_1, \ldots, t_D$ satisfying $\sum_{j=0}^D t_j \leq S+1$, we have

$$\sup_{(\delta,F)\in\Theta_{\rm b}(c)} \max_{I\in\mathcal{I}} \max_{(b,x)\in\mathcal{S}_{G_{(\delta,F)}}(I)} \big|\partial_{b^{t_0}}\partial_{x^{t_1}}\ldots\partial_{x^{t_D}}\; G_{(\delta,F)}(b|I,x)\big| \leq \bar{c}_G,$$
$$\sup_{(\delta,F)\in\Theta_{\rm b}(c)} \max_{I\in\mathcal{I}} \max_{(b,x)\in\mathcal{S}_{G_{(\delta,F)}}(I)} \big|\partial_{b^{t_0}}\partial_{x^{t_1}}\ldots\partial_{x^{t_D}}\; \xi_{(\delta_0,G_{(\delta,F)})}(b,I,x)\big| \leq \bar{c}_\xi.$$

Proof. Since $G^{-1}_{(\delta,F)}(\alpha|I,x) = \breve{s}_{(\lambda_\delta,F^{-1})}(\alpha,I,x)$, we know

$$\bar{c}_\alpha^{-1} \le \left\{\partial_\alpha \breve{s}_{(\lambda_\delta,F^{-1})}(\alpha,I,x)\right\}^{-1} = g\left\{G^{-1}_{(\delta,F)}(\alpha|I,x)|I,x\right\} \le \underline{c}_\alpha^{-1},$$

where $\underline{c}_\alpha > 0$ and $\bar{c}_\alpha > 0$ are from Lemma S.3. Therefore, we can set $\underline{c}_g = \bar{c}_\alpha^{-1}$. Also, the constants $\bar{c}_G$ and $\bar{c}_\xi$ can be similarly obtained by using Lemma S.3 together with the implicit function theorem and the chain rule. Specifically, $\bar{c}_\xi$ can be obtained from (6) and (A.3) and the fact that

$$\mathcal{R}_{G_{(\delta,F)}}(b|I,x) = R_{(\delta,F)}\left\{G_{(\delta,F)}(b|I,x) \;\middle|\; I,x\right\}. \qquad \square$$

For the next lemma, we will abbreviate $G_L = G_{(\delta_L,F_L)}$, $s_L = s_{(\delta_L,F_L)}$, and $\mathcal{R}_L = \mathcal{R}_{G_L}$. Further, we will use $G_* = G_{(\delta_0,F_*)}$. The same convention applies to other objects such as the boundaries of the conditional support, $\breve{s}$, and ξ. For instance, $\underline{b}_*(I,x)$ and $\bar{b}_*(I,x)$ will denote the boundaries of the support of $G_*(\cdot|I,x)$.

Lemma S.9. *Consider a sequence $\{(\delta_L,F_L) \in \Theta_{\mathrm{b}}(c) : L \in \mathbb{N}\}$ satisfying conditions (C1) and (C2). For any $(I,x) \in \mathcal{I}\times\mathcal{X}$ and $b \in \left(\underline{b}_*(I,x),\ \bar{b}_*(I,x)\right)$, we have that $G_L(b|I,x) \to G_*(b|I,x)$, $g_L(b|I,x) \to g_*(b|I,x)$, and $\xi_L(b,I,x) \to \xi_*(b,I,x)$.*

Proof. Fix $(I,x) \in \mathcal{I}\times\mathcal{X}$ and $b \in \left(\underline{b}_*(I,x),\ \bar{b}_*(I,x)\right)$. Note that $\underline{b}_L(I,x) = \breve{s}_L(0,I,x)$ and $\bar{b}_L(I,x) = \breve{s}_L(1,I,x)$. Therefore, it follows from Lemma S.5 that there exist $\epsilon > 0$ and $L_* \in \mathbb{N}$ such that $L \ge L_*$ implies that $[b-\epsilon, b+\epsilon] \subset \left(\underline{b}_*(I,x),\ \bar{b}_*(I,x)\right) \cap \left(\underline{b}_L(I,x),\ \bar{b}_L(I,x)\right)$. So, in the rest of the proof, consider only $L \ge L_*$ without loss of generality. Pick $\tilde{\alpha} \in (0,1)$ such that $b = \breve{s}_*(\tilde{\alpha},I,x)$ and use the mean value theorem to obtain

$$\begin{aligned}\left|G_L(b|I,x) - G_*(b|I,x)\right| &= \left|\breve{s}_L^{-1}\left\{\breve{s}_*(\tilde{\alpha},I,x),I,x\right\} - \tilde{\alpha}\right| \\ &= \left|\breve{s}_L^{-1}\left\{\breve{s}_*(\tilde{\alpha},I,x),I,x\right\} - \breve{s}_L^{-1}\left\{\breve{s}_L(\tilde{\alpha},I,x),I,x\right\}\right| \le \bar{c}_\alpha^{-1}\cdot\left|\breve{s}_*(\tilde{\alpha},I,x) - \breve{s}_L(\tilde{\alpha},I,x)\right| \to 0,\end{aligned}$$

where the constant $\bar{c}_\alpha > 0$ is from Lemma S.3 and the last convergence is by Lemmas S.5 and S.7; here, the second equality uses the fact that $\breve{s}_*(\tilde{\alpha},I,x) \in \left(s_L(0,I,x),\ s_L(1,I,x)\right)$ for all $L \ge L_*$. Now, for the convergence of the PDFs, note that the first derivatives of $\{G_L(\cdot|I,x) : L \ge L_*\}$ and $G_*(\cdot|I,x)$ exist on $[b-\epsilon, b+\epsilon]$, so Lemma S.4 can be applied. Finally, the convergence of $\xi_L(b,I,x)$ is automatic because $\xi_L(b,I,x) = b + \mathcal{R}_L(b|I,x) - \delta_L\left[\mathcal{R}_L(b|I,x)\right]$. $\square$

For the next lemma, let $\tilde{\Psi}(\delta_1,\delta_2,F) = \left(\tilde{\Psi}_1(\delta_1,\delta_2,F)\ \ldots\ \tilde{\Psi}_Q(\delta_1,\delta_2,F)\right)^\top$ with

$$\tilde{\Psi}_q(\delta_1,\delta_2,F) = \int_{\mathcal{X}_T}\int_0^1 \delta_1\left\{R_{(\delta_2,F)}(\alpha|I_q,x)\right\}d\alpha dx \quad \text{for } q = 1,\ldots,Q, \tag{S.29}$$

where δ_1 is a continuous real function and $(\delta_2, F) \in \Theta$.

Lemma S.10. *Suppose that $(\delta_2, F) \in \Theta^{\mathrm{cp}}$ and let δ_1 be a continuous function that satisfies: (i) $\delta_1(0) = 0$, (ii) it is nondecreasing on $\big[0,\ \tilde{R}_{(\delta_2,F)}(T)\big]$, and (iii) $\delta_1(u) > 0$ for some $u \in \big[0,\ \tilde{R}_{(\delta_2,F)}(T)\big]$. Then, for any $\omega \in \mathcal{W}$, we have $\omega^\intercal \tilde{\Psi}(\delta_1, \delta_2, F) > 0$.*

Proof. By condition (iii), we know that there exists $\epsilon > 0$ such that $\delta_1(u) = \epsilon$ for some $u \in [0, \tilde{R}_{(\delta_2,F)}(T)]$. Let $\tilde{u} = \min\big\{u \in [0, \tilde{R}_{(\delta_2,F)}(T)] :\ \delta_1(u) = \epsilon\big\}$. Further, pick $(\tilde{\alpha}, \tilde{x}) \in [0,1] \times \mathcal{X}$ such that $R_{(\delta_2,F)}(\tilde{\alpha}|I_1, \tilde{x}) = \tilde{u}$: this is possible because $R_{(\delta_2,F)}(0|I_1, x) = 0$ for all $x \in \mathcal{X}$, $(\delta_2, F) \in \Theta^{\mathrm{cp}}$, $[0,1] \times \mathcal{X}$ is compact, and $R_{(\delta_2,F)}(\cdot|I_1, \cdot)$ is continuous. Then, by construction,

$$\tilde{u} = R_{(\delta_2,F)}(\tilde{\alpha}|I_1, \tilde{x}) > R_{(\delta_2,F)}(\tilde{\alpha}|I_q, \tilde{x})$$

for $q = 2, \ldots, Q$. Therefore,

$$\delta_1(\tilde{u}) - \delta_1\big\{R_{(\delta_2,F)}(\tilde{\alpha}|I_q, \tilde{x})\big\} > 0,$$

where the inequality is strict because of the definition of $\tilde{u}$ and condition (ii): if $\delta_1\big\{R_{(\delta_2,F)}(\tilde{\alpha}|I_q, \tilde{x})\big\} = \epsilon$, then $\tilde{u}$ would not be the minimum. By continuity, we can choose some $\tilde{\epsilon} > 0$ and open balls $\mathcal{B}(\tilde{\alpha}) \subseteq (0,1)$ and $\mathcal{B}(\tilde{x}) \subseteq \mathcal{X}$ containing $\tilde{\alpha}$ and $\tilde{x}$, respectively, such that for all $(\alpha, x) \in \mathcal{B}(\tilde{\alpha}) \times \mathcal{B}(\tilde{x})$ and $q = 2, \ldots, Q$,

$$\delta_1\big\{R_{(\delta_2,F)}(\alpha|I_1, x)\big\} - \delta_1\big\{R_{(\delta_2,F)}(\alpha|I_q, x)\big\} \geq \tilde{\epsilon}. \tag{S.30}$$

Moreover, since δ_1 is nondecreasing and $(\delta_2, F) \in \Theta^{\mathrm{cp}}$, we must have

$$\delta_1\big\{R_{(\delta_2,F)}(\alpha|I_1, x)\big\} - \delta_1\big\{R_{(\delta_2,F)}(\alpha|I_q, x)\big\} \geq 0 \tag{S.31}$$

for all $(\alpha, x) \notin \mathcal{B}(\tilde{\alpha}) \times \mathcal{B}(\tilde{x})$ and $q = 2, \ldots, Q$. Therefore, for any $\omega \in \mathcal{W}$,

$$\begin{aligned} \omega^\intercal \tilde{\Psi}(\delta_1, \delta_2, F) &= \sum_{q=2}^{Q} -\omega_q \int_{\mathcal{X}_T} \int_0^1 \Big[\delta_1\big\{R_{(\delta_2,F)}(\alpha|I_1, x)\big\} - \delta_1\big\{R_{(\delta_2,F)}(\alpha|I_q, x)\big\}\Big] d\alpha dx \\ &\geq \sum_{q=2}^{Q} -\omega_q \int_{\mathcal{B}(\tilde{x})} \int_{\mathcal{B}(\tilde{\alpha})} \Big[\delta_1\big\{R_{(\delta_2,F)}(\alpha|I_1, x)\big\} - \delta_1\big\{R_{(\delta_2,F)}(\alpha|I_q, x)\big\}\Big] d\alpha dx \\ &\geq \sum_{q=2}^{Q} -\omega_q \int_{\mathcal{B}(\tilde{x})} \int_{\mathcal{B}(\tilde{\alpha})} \tilde{\epsilon}\ d\alpha dx > 0, \end{aligned}$$

where the first equality follows from $\omega_1 = -\sum_{q=2}^{Q} \omega_q > 0$, the first inequality is from (S.31) and the fact that $\omega_q \leq 0$ for all $q = 2, \ldots, Q$, and the second inequality follows from (S.30). □

Lemma S.11. *If Assumption B.3 is satisfied, then we have $\underline{\kappa}(\omega) = \min_{\delta \in \bar{\mathcal{K}}} \kappa(\omega, \delta)$ for every $\omega \in \mathcal{W}$. Further, $\underline{\kappa}$ is continuous on $\mathcal{W}$.*

Proof. Fix $\omega \in \mathcal{W}$. Observe that $\kappa(\omega, \cdot)$ is continuous on $(\mathscr{C}[0, \bar{U}], d_\infty)$. This implies that the set

$$\mathcal{M}(\omega) := \left\{\delta \in \bar{\mathcal{K}} : \kappa(\omega, \delta) = \inf_{\tilde{\delta} \in \bar{\mathcal{K}}} \kappa(\omega, \tilde{\delta})\right\},$$

is nonempty (Rudin, 1976, Theorem 4.16). So, we can choose $\delta^* \in \mathcal{M}(\omega) \subset \bar{\mathcal{K}}$ so that $\kappa(\omega, \delta^*) = \min_{\delta \in \bar{\mathcal{K}}} \kappa(\omega, \delta) = \inf_{\delta \in \bar{\mathcal{K}}} \kappa(\omega, \delta)$.

Now, we will prove that $\underline{\kappa}(\omega) := \inf_{\delta \in \mathcal{K}} \kappa(\omega, \delta)$ is indeed equal to $\kappa(\omega, \delta^*)$. First, note that

$$\underline{\kappa}(\omega) := \inf_{\delta \in \mathcal{K}} \kappa(\omega, \delta) \geq \inf_{\delta \in \bar{\mathcal{K}}} \kappa(\omega, \delta) = \min_{\delta \in \bar{\mathcal{K}}} \kappa(\omega, \delta) = \kappa(\omega, \delta^*).$$

Suppose that the inequality is strict: i.e. $\inf_{\delta \in \mathcal{K}} \kappa(\omega, \delta) - \kappa(\omega, \delta^*) = \epsilon$ for some $\epsilon > 0$. Then, by continuity, there exists some $\tilde{\epsilon} > 0$ such that $d_\infty(\delta, \delta^*) < \tilde{\epsilon}$ implies $\left|\kappa(\omega, \delta) - \kappa(\omega, \delta^*)\right| < \epsilon/2$. Since $\delta^* \in \bar{\mathcal{K}}$, we can choose $\tilde{\delta}^* \in \mathcal{K}$ such that $d_\infty(\tilde{\delta}^*, \delta^*) < \tilde{\epsilon}$. But then, we have $\kappa(\omega, \tilde{\delta}^*) \leq \kappa(\omega, \delta^*) - \epsilon/2$, which contradicts the fact that $\kappa(\omega, \delta^*)$ is the infimum. Finally, continuity of $\underline{\kappa}$ follows by the Theorem of the Maximum (see e.g., Ok, 2007). □

Lemma S.12. *Consider an arbitrary $\delta \in \mathcal{K}(\underline{c})$. Let $\delta^{(M)}$ be the Bernstein polynomial associated with δ, i.e. $\delta^{(M)}(u) = \sum_{m=1}^{M} \delta\big(m\varphi_c/M\big)\mathrm{P}_m^{(M)}(u)$. Then, there is some general constant $C > 0$ such that*

$$\sup_{\delta \in \mathcal{K}(\underline{c})} \sup_{u \in [0, \varphi_c]} \left|\delta^{(M)}(u) - \delta(u)\right| \leq \frac{C}{\sqrt{M}}.$$

Proof. By Theorem 3.1 in DeVore and Lorentz (1993, Chapter 10), we know that

$$\left|\delta^{(M)}(u) - \delta(u)\right| \leq C_1 \mathcal{B}_2\left\{\delta, \frac{u(1-u)}{\sqrt{M}}\right\},$$

where C_1 is a general constant, which does not depend on δ, and $\mathcal{B}_2(\delta, t)$ is the second modulus of smoothness of the function δ, i.e.

$$\mathcal{B}_2(\delta, t) = \sup_{\tilde{t} \in [0, t]} \sup_{u \in [0, \varphi_c - 2\tilde{t}]} \left|\delta(u) + \delta(u + 2\tilde{t}) - 2\delta(u + \tilde{t})\right|.$$

Here, note that $u(1-u) \leq \varphi_c/2$ and that

$$\sup_{\tilde{t} \in [0, t]} \sup_{u \in [0, \varphi_c - 2\tilde{t}]} \left|\delta(u) + \delta(u + 2\tilde{t}) - 2\delta(u + \tilde{t})\right| \leq 2t$$

for every $t \in [0, \varphi_c]$, where the inequality follows from the fact that δ is Lipschitz continuous with Lipschitz constant smaller than 1. □

S.1.2. **Statistical results.** Throughout this subsection, we assume that Assumption 1 is always satisfied and consider an arbitrary sequence $\{(\delta_L, F_L) \in \Theta_{\mathrm{b}}(c) : L \in \mathbb{N}\}$, where (δ_L, F_L) is associated with the DGP of the sample (8) of size L. In this context, e.g. when we say *under* H^{local}, it means that such a sequence satisfies the corresponding conditions. We abbreviate $G_L := G_{(\delta_L, F_L)}$, $g_L := g_{(\delta_L, F_L)}$, and $\Sigma_L := \Sigma_{G_L}$, where Σ_{G_L} is obtained from Assumption 2.(2).

Lemma S.13. *If Assumption 2 holds and* $\|\boldsymbol{\omega}_L - w\| = o_{P_L}(1)$ *for some* $w \in \mathcal{W}$*, then*

$$\left| \frac{\sqrt{r(\omega)^\top \Sigma_L r(\omega)}}{\hat{\boldsymbol{\sigma}}(\boldsymbol{\omega}_L)} - 1 \right| = o_{P_L}(1).$$

Proof. Due to the mean value theorem and the fact that $\inf_L r(\omega)^\top \Sigma_L r(\omega) > 0$ by Assumption 2, it suffices to show $\left| r(\boldsymbol{\omega}_L)^\top \hat{\boldsymbol{\Sigma}} r(\boldsymbol{\omega}_L) - r(\omega)^\top \Sigma_L r(\omega) \right| = o_{P_L}(1)$: we will establish this below. Let $\boldsymbol{r}_L = r(\boldsymbol{\omega}_L)$ and $r_\omega = r(\omega)$. By the triangular inequality, we have

$$\left| \boldsymbol{r}_L^\top \hat{\boldsymbol{\Sigma}} \boldsymbol{r}_L - r_\omega^\top \Sigma_L r_\omega \right| \le \left| \boldsymbol{r}_L^\top \hat{\boldsymbol{\Sigma}} \boldsymbol{r}_L - \boldsymbol{r}_L^\top \Sigma_L \boldsymbol{r}_L \right| + \left| \boldsymbol{r}_L^\top \Sigma_L \boldsymbol{r}_L - r_\omega^\top \Sigma_L r_\omega \right|, \tag{S.32}$$

where

$$\left| \boldsymbol{r}_L^\top \hat{\boldsymbol{\Sigma}} \boldsymbol{r}_L - \boldsymbol{r}_L^\top \Sigma_L \boldsymbol{r}_L \right| = \left| \boldsymbol{r}_L^\top (\hat{\boldsymbol{\Sigma}} - \Sigma_L) \boldsymbol{r}_L \right| \le Q^2 \|\hat{\boldsymbol{\Sigma}} - \Sigma_L\| \|\boldsymbol{r}_L\|^2 = o_{P_L}(1)$$

because $\|\boldsymbol{\Sigma} - \Sigma_L\| = o_{P_L}(1)$ by Assumption 2.(3) and $\|\boldsymbol{r}_L\| \le \|r_\omega\| + o_{P_L}(1)$ due to $\|\boldsymbol{\omega}_L - w\| = o_{P_L}(1)$. Regarding the second term on the right-hand side of (S.32), we observe that

$$\begin{aligned}\left| \boldsymbol{r}_L^\top \Sigma_L \boldsymbol{r}_L - r_\omega^\top \Sigma_L r_\omega \right| &\le \left| (\boldsymbol{r}_L - r_\omega)^\top \Sigma_L (\boldsymbol{r}_L - r_\omega) \right| + 2 \left| r_\omega^\top \Sigma_L (\boldsymbol{r}_L - r_\omega) \right| \\ &\le Q^2 \left(\sup_L \|\Sigma_L\| \right) \left(\|\boldsymbol{r}_L - r_\omega\|^2 + 2 \|r_\omega\| \|\boldsymbol{r}_L - r_\omega\| \right) = o_{P_L}(1). \quad \square\end{aligned}$$

For the next lemma, denote $\zeta(\omega, F_L) = \inf_{\delta \in \mathcal{K}} \omega^\top \Psi(\delta, F_L)$ and $\hat{\boldsymbol{\zeta}}(\omega) = \min_{\delta \in \mathcal{K}^{(L)}} \omega^\top \hat{\boldsymbol{\Psi}}(\delta)$.

Lemma S.14. *If Assumption B.4 holds, then* $\sup_{\omega \in \mathcal{W}_1} \left| \hat{\boldsymbol{\zeta}}(\omega) - \zeta(\omega, F_L) \right| = o_{P_L}(1)$ *under* H^{local}.

Proof. Choose any $\omega \in \mathcal{W}_1$. For each $L \in \mathbb{N}$, pick any $\tilde{\delta}_{1,\omega}^{(L)} \in \mathcal{K}^{(L)}$ such that

$$\omega^\top \Psi(\tilde{\delta}_{1,\omega}^{(L)}) = \min_{\delta \in \mathcal{K}^{(L)}} \omega^\top \Psi(\delta),$$

which is possible because $\mathcal{K}^{(L)}$ is compact and $\omega^\top \Psi(\cdot)$ is continuous. Then,

$$\begin{aligned}\min_{\delta \in \mathcal{K}^{(L)}} \omega^\top \hat{\boldsymbol{\Psi}}(\delta) &\le \omega^\top \hat{\boldsymbol{\Psi}}\left(\tilde{\delta}_{1,\omega}^{(L)}\right) \le \omega^\top \Psi\left(\tilde{\delta}_{1,\omega}^{(L)}, F_L\right) + Q\|\omega\| \sup_{\delta \in \mathcal{K}} \left\| \hat{\boldsymbol{\Psi}}(\delta) - \Psi(\delta, F_L) \right\| \\ &\le \inf_{\delta \in \mathcal{K}} \omega^\top \Psi(\delta, F_L) + Q \sup_{\delta \in \mathcal{K}} \left\| \hat{\boldsymbol{\Psi}}(\delta) - \Psi(\delta, F_L) \right\| + \sup_{\omega \in \mathcal{W}_1} \left| \min_{\delta \in \mathcal{K}^{(L)}} \omega^\top \Psi(\delta, F_L) - \inf_{\delta \in \mathcal{K}} \omega^\top \Psi(\delta, F_L) \right|,\end{aligned}$$

where the first inequality is by definition, and the second and third inequalities are by the triangular inequality. Therefore, it follows that

$$\hat{\boldsymbol{\zeta}}(\omega) - \zeta(\omega, F_L) \le Q \sup_{\delta\in\mathcal{K}} \left\| \hat{\boldsymbol{\Psi}}(\delta) - \Psi(\delta, F_L) \right\| + \sup_{\omega\in\mathcal{W}_1} \left| \min_{\delta\in\mathcal{K}^{(L)}} \omega^\top \Psi(\delta, F_L) - \inf_{\delta\in\mathcal{K}} \omega^\top \Psi(\delta, F_L) \right|, \tag{S.33}$$

where the right-hand side does not depend on ω. Similarly, we consider bounding $\zeta(\omega, F_L) - \hat{\boldsymbol{\zeta}}(\omega)$. Specifically, pick any $\tilde{\boldsymbol{\delta}}_{2,\omega}^{(L)} \in \mathcal{K}^{(L)}$ such that $\omega^\top \hat{\boldsymbol{\Psi}}(\tilde{\boldsymbol{\delta}}_{2,\omega}^{(L)}) = \min_{\delta\in\mathcal{K}^{(L)}} \omega^\top \hat{\boldsymbol{\Psi}}(\delta)$, and we obtain

$$\begin{aligned} \inf_{\delta\in\mathcal{K}} \omega^\top \Psi(\delta, F_L) &\le \omega^\top \Psi\left(\tilde{\boldsymbol{\delta}}_{2,\omega}^{(L)}, F_L\right) \le \omega^\top \hat{\boldsymbol{\Psi}}\left(\tilde{\boldsymbol{\delta}}_{2,\omega}^{(L)}\right) + Q\|\omega\| \sup_{\delta\in\mathcal{K}} \left\| \hat{\boldsymbol{\Psi}}(\delta) - \Psi(\delta, F_L) \right\| \\ &\le \min_{\delta\in\mathcal{K}^{(L)}} \omega^\top \hat{\boldsymbol{\Psi}}(\delta) + Q \sup_{\delta\in\mathcal{K}} \left\| \hat{\boldsymbol{\Psi}}(\delta) - \Psi(\delta, F_L) \right\|. \end{aligned}$$

So, we have

$$\zeta(\omega, F_L) - \hat{\boldsymbol{\zeta}}(\omega) \le Q \sup_{\delta\in\mathcal{K}} \left\| \hat{\boldsymbol{\Psi}}(\delta) - \Psi(\delta, F_L) \right\|, \tag{S.34}$$

where the right-hand side does not depend on ω. Combining (S.33) and (S.34) leads to

$$\sup_{\omega\in\mathcal{W}_1} \left| \hat{\boldsymbol{\zeta}}(\omega) - \zeta(\omega, F_L) \right| \le Q \sup_{\delta\in\mathcal{K}} \left\| \hat{\boldsymbol{\Psi}}(\delta) - \Psi(\delta, F_L) \right\| + \sup_{\omega\in\mathcal{W}_1} \left| \min_{\delta\in\mathcal{K}^{(L)}} \omega^\top \Psi(\delta, F_L) - \inf_{\delta\in\mathcal{K}} \omega^\top \Psi(\delta, F_L) \right|. \tag{S.35}$$

Therefore, the lemma statement follows from Assumption B.4. □

S.1.2.1. *Statistical results for Subsection 4.3.2.* The objective of this subsection is to show that the estimators $\hat{\mathbb{M}}(I_q)$ and $\hat{\boldsymbol{\Sigma}}$ proposed in Subsection 4.3.2 satisfy Assumption 2. For this purpose, we will assume that Assumptions 1 and 3 are always satisfied. We further assume that $T > 0$ to avoid boundary problems. Without loss of generality, we suppose that the kernel K has support $[-1, 1]^D$ and, since the asymptotic analysis is performed taking $L \to \infty$, we will consider L to be sufficiently large so that $\{x + th : \ x \in \mathcal{X}, \ t \in [-1, 1]^D\} \subset \mathcal{X}$.

As a starting point, we write

$$\hat{\mathbb{M}}(I_q) = \frac{1}{L} \sum_{l=1}^{L} \left\{ \frac{\dot{\boldsymbol{b}}_l \mathbb{1}(\boldsymbol{x}_l \in \mathcal{X}_T) \mathbb{1}(\boldsymbol{I}_l = I_q)}{\tilde{\boldsymbol{f}}(I_q, \boldsymbol{x}_l)} \right\} \tag{S.36}$$

with

$$\tilde{\boldsymbol{f}}(I_q, x) = \frac{1}{L} \sum_{l=1}^{L} K\left(\frac{x - \boldsymbol{x}_l}{h}\right) \frac{\mathbb{1}(\boldsymbol{I}_l = I_q)}{h^D}. \tag{S.37}$$

Since Assumption 1 is satisfied for each $L \in \mathbb{N}$, we observe that $\{(\dot{\boldsymbol{b}}_l, \boldsymbol{I}_l, \boldsymbol{x}_l) : l = 1, \ldots, L\}$ are IID random vectors each having joint density $\dot{g}_L \cdot f_{\boldsymbol{I}\boldsymbol{x}}$, where $f_{\boldsymbol{I}\boldsymbol{x}}$ is the joint density of $(\boldsymbol{I}_l, \boldsymbol{x}_l)$ and $\dot{g}_L(\cdot|I, x) \coloneqq \dot{g}_{(\delta_L, F_L)}(\cdot|I, x)$ denotes the conditional density of $\dot{\boldsymbol{b}}_l = \boldsymbol{I}_l^{-1} \sum_{p=1}^{\boldsymbol{I}_l} \boldsymbol{b}_{pl}$ given $(\boldsymbol{I}_l, \boldsymbol{x}_l) = (I, x)$. We recall that $f_{\boldsymbol{I}\boldsymbol{x}}$ does not vary with L.

Lemma S.15. *For any $q = 1, \ldots, Q$, we have*

$$\sup_{x \in \mathcal{X}_T} \left| \tilde{\boldsymbol{f}}(I_q, x) - f_{\boldsymbol{I}\boldsymbol{x}}(I_q, x) \right| = O_{P_L}\left(\sqrt{\frac{\log L}{Lh^D}} + h^P \right).$$

Proof. Note that $\tilde{\boldsymbol{f}}$ is a function of $\{(\boldsymbol{I}_l, \boldsymbol{x}_l) : l = 1, \ldots, L\}$, so its distribution does not depend on the underlying sequence $\{(\delta_L, F_L) \in \Theta_{\mathrm{b}}(c) : \ L \in \mathbb{N}\}$ introduced at the beginning of this subsection. Therefore, this is a standard result based on a fixed DPG, and it follows from e.g. Sections 1.10 and 1.11 in Li and Racine (2007). □

Now define

$$\tilde{\mathbb{M}}(I_q) = \frac{1}{L} \sum_{l=1}^{L} \left\{ \frac{\dot{\boldsymbol{b}}_l \mathbb{1}(\boldsymbol{x}_l \in \mathcal{X}_T) \mathbb{1}(\boldsymbol{I}_l = I_q)}{f(I_q, \boldsymbol{x}_l)} \right\}, \tag{S.38}$$

which is the infeasible version of $\hat{\mathbb{M}}(I_q)$, and let

$$\tilde{\mathbb{D}}(I_q) = -\frac{1}{L} \sum_{l=1}^{L} \left\{ \frac{\dot{\boldsymbol{b}}_l \mathbb{1}(\boldsymbol{x}_l \in \mathcal{X}_T) \mathbb{1}(\boldsymbol{I}_l = I_q)}{f^2(I_q, \boldsymbol{x}_l)} \right\} \left\{ \tilde{\boldsymbol{f}}(I_q, \boldsymbol{x}_l) - f(I_q, \boldsymbol{x}_l) \right\}. \tag{S.39}$$

Lemma S.16. *For any $q = 1, \ldots, Q$, we have*

$$\sqrt{L} \left\{ \hat{\mathbb{M}}(I_q) - \tilde{\mathbb{M}}(I_q) - \tilde{\mathbb{D}}(I_q) \right\} = o_{P_L}(1).$$

Proof. By the second-order Taylor expansion and the fact that $f_{\boldsymbol{I}\boldsymbol{x}}(I_q, \cdot)$ is bounded away from zero, there is some constant $C > 0$ such that

$$\left| \hat{\mathbb{M}}(I_q) - \tilde{\mathbb{M}}(I_q) - \tilde{\mathbb{D}}(I_q) \right| \le \left\{ C + o_{P_L}(1) \right\} \left| \frac{1}{L} \sum_{l=1}^{L} \dot{\boldsymbol{b}}_l \mathbb{1}(\boldsymbol{x}_l \in \mathcal{X}_T) \mathbb{1}(\boldsymbol{I}_l = I_q) \right| \sup_{x \in \mathcal{X}_T} \left| \tilde{\boldsymbol{f}}(I_q, x) - f(I_q, x) \right|^2.$$

Therefore, the desired result follows from Lemma S.15 because

$$\boldsymbol{b}_{pl} \mathbb{1}(\boldsymbol{x}_l \in \mathcal{X}_T) \mathbb{1}(\boldsymbol{I}_l = I_q) \le \bar{b}_{G_L}(I_q, \boldsymbol{x}_l) \le \sup_L \sup_{x \in \mathcal{X}} \bar{b}_{G_L}(I_q, x) < \infty$$

holds with $\mathbb{P}_{(\delta_L, F_L)}$-probability one by Lemma S.3. □

Before stating the next lemma, we introduce additional notation and make several remarks. We will write $d\varrho_L(b, I, x)$ when we integrate with respect to the measure represented by $\dot{g}_L(\cdot|I, \cdot) f_{\boldsymbol{I}\boldsymbol{x}}(I, \cdot)$. To be specific,

$$\int \phi(b, x) d\varrho_L(b, I, x) = \int_{\mathbb{R}^D} \int_{\mathbb{R}} \phi(b, x) \dot{g}_L(b|I, x) f_{\boldsymbol{I}\boldsymbol{x}}(I, x) db dx$$

for any integrable function ϕ defined on $\mathbb{R}^{1+D}$. Then, letting $z = (b, I, x)$ and $\tilde{z} = (\tilde{b}, \tilde{I}, \tilde{x})$, we define

$$\begin{aligned} m(z, \tilde{z}) &= \frac{b\mathbb{1}(x \in \mathcal{X}_T)\mathbb{1}(I = I_q)}{f^2_{\boldsymbol{I}\boldsymbol{x}}(I_q, x)} K\Big(\frac{x - \tilde{x}}{h}\Big)\frac{\mathbb{1}(\tilde{I} = I_q)}{h^D}, \\ m_1(\tilde{z}) &= \int m(z, \tilde{z}) d\varrho_L(z), \quad \text{and} \\ m_2(z) &= \int m(z, \tilde{z}) d\varrho_L(\tilde{z}). \end{aligned}$$

Note that the functions m, m_1 and m_2 depend on q and L, but they are suppressed from the notation for the sake of simplicity.

Several remarks are noteworthy before stating the lemma. First, since b appears linearly in the m function, there is no difference in integrating with respect to $\dot{g}_L(\cdot|I, x)$ and $g_L(\cdot|I, x)$. Specifically, abbreviating $\mathbb{E}_L = \mathbb{E}_{G_{(\delta_L, F_L)}}$, we note that

$$\mathbb{E}_L(\dot{\boldsymbol{b}}_l | \boldsymbol{x}_l = x, \boldsymbol{I}_l = I) = \mathbb{E}_L(\boldsymbol{b}_{pl} | \boldsymbol{x}_l = x, \boldsymbol{I}_l = I) = \int b g_L(b|I, x) db$$

for any $p = 1, \dots, I$ and $l = 1, \dots, L$. Second, we can write

$$\begin{aligned} m_1(\tilde{z}) &= \int_{\mathcal{X}_T} \frac{\mathbb{E}_L(\boldsymbol{b}|\boldsymbol{x} = x, \boldsymbol{I} = I_q)}{f_{\boldsymbol{I}\boldsymbol{x}}(I_q, x)} K\Big(\frac{x - \tilde{x}}{h}\Big)\frac{\mathbb{1}(\tilde{I} = I_q)}{h^D} dx \quad \text{and} \\ m_2(z) &= \frac{b\mathbb{1}(x \in \mathcal{X}_T)\mathbb{1}(I = I_q)}{f^2_{\boldsymbol{I}\boldsymbol{x}}(I_q, x)} \int K\Big(\frac{x - \tilde{x}}{h}\Big)\frac{1}{h^D} f_{\boldsymbol{I}\boldsymbol{x}}(I_q, \tilde{x}) d\tilde{x}. \end{aligned}$$

Third, by using the standard bias expansion technique, we know that there exists a finite constant $c_{m_2} > 0$ such that for all L,

$$\sup_z \left| m_2(z) - \frac{b\mathbb{1}(x \in \mathcal{X}_T)\mathbb{1}(I = I_q)}{f_{\boldsymbol{I}\boldsymbol{x}}(I_q, x)} \right| \le c_{m_2} h^P. \tag{S.40}$$

Now, we are ready to establish a stochastic equicontinuity condition under the sequence $\{G_L : L \in \mathbb{N}\}$. Define

$$\begin{aligned} \boldsymbol{SE}_{qL} = &\frac{1}{\sqrt{L}} \sum_{l=1}^{L} \left\{ \frac{\dot{\boldsymbol{b}}_l \mathbb{1}(\boldsymbol{x}_l \in \mathcal{X}_T)\mathbb{1}(\boldsymbol{I}_l = I_q)}{f^2_{\boldsymbol{I}\boldsymbol{x}}(I_q, \boldsymbol{x}_l)} \tilde{\boldsymbol{f}}(I_q, \boldsymbol{x}_l) - \int \frac{b\mathbb{1}(x \in \mathcal{X}_T)\mathbb{1}(I = I_q)}{f^2_{\boldsymbol{I}\boldsymbol{x}}(I_q, x)} \tilde{\boldsymbol{f}}(I_q, x) d\varrho_L(b, I, x) \right\} \\ &- \frac{1}{\sqrt{L}} \sum_{l=1}^{L} \left\{ \frac{\dot{\boldsymbol{b}}_l \mathbb{1}(\boldsymbol{x}_l \in \mathcal{X}_T)\mathbb{1}(\boldsymbol{I}_l = I_q)}{f^2_{\boldsymbol{I}\boldsymbol{x}}(I_q, \boldsymbol{x}_l)} f_{\boldsymbol{I}\boldsymbol{x}}(I_q, \boldsymbol{x}_l) - \int \frac{b\mathbb{1}(x \in \mathcal{X}_T)\mathbb{1}(\boldsymbol{I} = I_q)}{f^2_{\boldsymbol{I}\boldsymbol{x}}(I_q, x)} f_{\boldsymbol{I}\boldsymbol{x}}(I_q, x) d\varrho_L(b, I, x) \right\}. \end{aligned}$$

Lemma S.17. *For any $q = 1, \dots, Q$, $\boldsymbol{SE}_{qL} = o_{P_L}(1)$.*

Proof. Chose any $q = 1, \dots, Q$. First, by the above definitions, we have that

$$\int \frac{b\mathbb{1}(x \in \mathcal{X}_T)\mathbb{1}(I = I_q)}{f_{\boldsymbol{Ix}}^2(I_q, x)} \Big\{\tilde{\boldsymbol{f}}(I_q, x) - f_{\boldsymbol{Ix}}(I_q, x)\Big\} d\varrho_L(b, I, x)$$

$$= \frac{1}{L}\sum_{l=1}^{L} m_1(\boldsymbol{z}_l) - \mathbb{E}_L\left\{\frac{\boldsymbol{b}\mathbb{1}(\boldsymbol{x} \in \mathcal{X}_T)\mathbb{1}(\boldsymbol{I} = I_q)}{f_{\boldsymbol{Ix}}(I_q, \boldsymbol{x})}\right\} \quad \text{(S.41)}$$

and

$$\begin{aligned}
\int m_1(\tilde{z}) d\varrho_L(\tilde{z}) &= \iint \frac{\mathbb{1}(x \in \mathcal{X}_T)\mathbb{E}_L(\boldsymbol{b}|\boldsymbol{x} = x, \boldsymbol{I} = I_q)}{f_{\boldsymbol{Ix}}(I_q, x)} K\Big(\frac{x - \tilde{x}}{h}\Big)\frac{1}{h^D} f_{\boldsymbol{Ix}}(I_q, \tilde{x}) dx d\tilde{x} \\
&= \int \frac{\mathbb{1}(x \in \mathcal{X}_T)\mathbb{E}_L(\boldsymbol{b}|\boldsymbol{x} = x, \boldsymbol{I} = I_q)}{f_{\boldsymbol{Ix}}(I_q, x)} \int K\Big(\frac{x - \tilde{x}}{h}\Big)\frac{1}{h^D} f_{\boldsymbol{Ix}}(I_q, \tilde{x}) d\tilde{x} dx \\
&= \int \frac{\mathbb{1}(x \in \mathcal{X}_T)\mathbb{E}_L(\boldsymbol{b}|\boldsymbol{x} = x, \boldsymbol{I} = I_q)}{f_{\boldsymbol{Ix}}(I_q, x)} \int K(t) f_{\boldsymbol{Ix}}(I_q, x - th) dt dx.
\end{aligned}$$

Therefore, by noting that $x \in \mathcal{X}_T$, expanding $f_{\boldsymbol{Ix}}(I_q, x - th)$, and using the fact that

$$\sup_L \sup_x \mathbb{E}_L(\boldsymbol{b}|\boldsymbol{x} = x, \boldsymbol{I} = I_q) \le \sup_L \sup_x \bar{b}_{G_L}(I_q, x) < \infty$$

by Lemma S.3, we obtain

$$\int m_1(\tilde{z}) d\varrho_L(\tilde{z}) = \int_{\mathcal{X}_T} \mathbb{E}_L(\boldsymbol{b}|\boldsymbol{x} = x, \boldsymbol{I} = I_q) dx + O(h^P) = \mathbb{E}_L\left\{\frac{\boldsymbol{b}\mathbb{1}(\boldsymbol{x} \in \mathcal{X}_T)\mathbb{1}(\boldsymbol{I} = I_q)}{f_{\boldsymbol{Ix}}(I_q, \boldsymbol{x})}\right\} + O(h^P).$$

Then, (S.41) can be rewritten as

$$\int \frac{b\mathbb{1}(x \in \mathcal{X}_T)\mathbb{1}(I = I_q)}{f_{\boldsymbol{Ix}}^2(I_q, x)} \Big\{\tilde{\boldsymbol{f}}(I_q, x) - f_{\boldsymbol{Ix}}(I_q, x)\Big\} d\varrho_L(d, I, x)$$

$$= \frac{1}{L}\sum_{l=1}^{L} m_1(\boldsymbol{z}_l) - \int m_1(\tilde{z}) d\varrho_L(\tilde{z}) + O(h^P). \quad \text{(S.42)}$$

Now, we note that

$$\frac{1}{L}\sum_{l=1}^{L} \frac{\dot{\boldsymbol{b}}_l \mathbb{1}(\boldsymbol{x}_l \in \mathcal{X}_T)\mathbb{1}(\boldsymbol{I}_l = I_q)}{f_{\boldsymbol{Ix}}^2(I_q, \boldsymbol{x}_l)} \Big\{\tilde{\boldsymbol{f}}(I_q, \boldsymbol{x}_l) - f_{\boldsymbol{Ix}}(I_q, \boldsymbol{x}_l)\Big\}$$

$$\begin{aligned}
&= \frac{1}{L^2}\sum_{l=1}^{L}\sum_{\tilde{l}=1}^{L} m(\boldsymbol{z}_l, \boldsymbol{z}_{\tilde{l}}) - \frac{1}{L}\sum_{l=1}^{L} \frac{\dot{\boldsymbol{b}}_l \mathbb{1}(\boldsymbol{x}_l \in \mathcal{X}_T)\mathbb{1}(\boldsymbol{I}_l = I_q)}{f_{\boldsymbol{Ix}}(I_q, \boldsymbol{x}_l)} \\
&= \frac{1}{L^2}\sum_{l=1}^{L}\sum_{\tilde{l}=1}^{L} m(\boldsymbol{z}_l, \boldsymbol{z}_{\tilde{l}}) - \frac{1}{L}\sum_{l=1}^{L} m_2(\boldsymbol{z}_l) + O_{P_L}(h^P), \quad \text{(S.43)}
\end{aligned}$$

where the last two terms of the second equality are obtained by (S.40). Combining (S.42) and (S.43), we can write

$$\boldsymbol{SE}_{qL} = \sqrt{L}\left\{ \frac{1}{L^2}\sum_{l=1}^{L}\sum_{\tilde{l}=1}^{L} m(\boldsymbol{z}_l, \boldsymbol{z}_{\tilde{l}}) - \frac{1}{L}\sum_{l=1}^{L}\left\{ m_1(\boldsymbol{z}_l) + m_2(\boldsymbol{z}_l)\right\} + \int m_1(\tilde{z}) d\varrho_L(\tilde{z}) \right\} + O_{P_L}(\sqrt{L}h^P),$$

to which we apply the arguments in the proof of Lemma 8.4 in Newey and McFadden (1994). Then, we obtain

$$\boldsymbol{SE}_{qL} = O_{P_L}\left(\frac{1}{L}\mathbb{E}_L\left\{ |m(\boldsymbol{z}_1, \boldsymbol{z}_1)| \right\} + \frac{1}{L}\left[\mathbb{E}_L\left\{ m^2(\boldsymbol{z}_1, \boldsymbol{z}_2) \right\} \right]^{1/2} \right) + O_{P_L}(\sqrt{L}h^P).$$

To complete the proof, note that

$$\frac{1}{L}\mathbb{E}_L\left\{ |m(\boldsymbol{z}_1, \boldsymbol{z}_1)| \right\} = \frac{1}{Lh^D}|K(0)|\mathbb{E}_L\left\{ \frac{|\dot{\boldsymbol{b}}_1|\mathbb{1}(\boldsymbol{x}_1 \in \mathcal{X}_T)\mathbb{1}(\boldsymbol{I}_1 = I_q)}{f^2_{\boldsymbol{Ix}}(I_q, \boldsymbol{x}_1)} \right\} \leq \frac{|K(0)|}{Lh^D}c^3$$

and

$$\begin{aligned} &\frac{1}{L}\left[\mathbb{E}_L\left\{ m^2(\boldsymbol{z}_1, \boldsymbol{z}_2) \right\} \right]^{1/2} \\ &= \frac{1}{L}\left[\iint \frac{\mathbb{E}_L(\boldsymbol{b}|\boldsymbol{x} = x, \boldsymbol{I} = I_q)}{f^4_{\boldsymbol{Ix}}(I_q, x)} K\Big(\frac{x - \tilde{x}}{h}\Big)\frac{1}{h^{2D}} f_{\boldsymbol{Ix}}(I_q, x) f_{\boldsymbol{Ix}}(I_q, \tilde{x}) dx d\tilde{x} \right]^{1/2} = O\Big(\frac{1}{L\sqrt{h^D}}\Big), \end{aligned}$$

where the last equality is obtained by a standard change of variables. Therefore, the lemma statement follows from the fact that $\sqrt{L}h^P \to 0$ and $Lh^D \to \infty$. □

Lemmas S.16 and S.17 show that

$$\begin{aligned} \sqrt{L}\left\{ \hat{\mathbb{M}}(I_q) - \mathbb{M}_{G_L}(I_q) \right\} &= \sqrt{L}\left\{ \tilde{\mathbb{M}}(I_q) - \mathbb{M}_{G_L}(I_q) \right\} \\ &- \sqrt{L}\left\{ \int \frac{\mathbb{E}_L(\boldsymbol{b} \mid \boldsymbol{x} = x, \boldsymbol{I} = I_q)\mathbb{1}(\boldsymbol{x} \in \mathcal{X}_T)}{f_{\boldsymbol{Ix}}(I_q, x)} \tilde{\boldsymbol{f}}(I_q, x) dx - \mathbb{M}_{G_L}(I_q) \right\} + o_{P_L}(1). \quad \text{(S.44)} \end{aligned}$$

The next lemma shows that the second term on the right-hand side of (S.44) can be approximated by a sample average uniformly over L, also. Let $\mu_L(I, x) = \mathbb{E}_L(\boldsymbol{b}|\boldsymbol{I} = I, \boldsymbol{x} = x)$ and define

$$\nu_L(I_q, x) = \frac{\mathbb{1}(x \in \mathcal{X}_T)\mu_L(I_q, x)}{f_{\boldsymbol{Ix}}(I_q, x)},$$

where we note that

$$\mathbb{E}_L\Big\{\mathbb{1}(\boldsymbol{I}=I_q)\nu_L(I_q,\boldsymbol{x})\Big\} = \sum_{I\in\mathcal{I}}\int \mu_L(I_q,x)\frac{\mathbb{1}(x\in\mathcal{X}_T)\mathbb{1}(I=I_q)}{f_{\boldsymbol{I}\boldsymbol{x}}(I_q,x)}f_{\boldsymbol{I}\boldsymbol{x}}(I,x)dx$$
$$= \int_{\mathcal{X}_T}\mu_L(I_q,x)dx = \mathbb{M}_{G_L}(I_q).$$

Lemma S.18. *For any $q=1,\dots,Q$,*

$$\sqrt{L}\int \nu_L(I_q,x)\tilde{\boldsymbol{f}}(I_q,x)dx = \frac{1}{\sqrt{L}}\sum_{l=1}^{L}\mathbb{1}(\boldsymbol{I}_l=I_q)\nu_L(I_q,\boldsymbol{x}_l)+o_{P_L}(1).$$

Proof. Note that

$$\int \nu_L(I_q,x)\tilde{\boldsymbol{f}}(I_q,x)dx = \frac{1}{L}\sum_{l=1}^{L}\mathbb{1}(\boldsymbol{I}_l=I_q)\tilde{\nu}_L(I_q,\boldsymbol{x}_l),$$

where we define

$$\tilde{\nu}_L(I_q,\boldsymbol{x}_l) = \int \nu_L(I_q,x)K\Big(\frac{x-\boldsymbol{x}_l}{h}\Big)\frac{1}{h^D}dx.$$

Therefore, it suffices to show that

$$\frac{1}{\sqrt{L}}\sum_{l=1}^{L}\mathbb{1}(\boldsymbol{I}_l=I_q)\Big\{\tilde{\nu}_L(I_q,\boldsymbol{x}_l)-\nu_L(I_q,\boldsymbol{x}_l)\Big\} = o_{P_L}(1).$$

We will show this by using the Markov inequality. Squaring and taking expectations on the right-hand side yields an upper bound of

$$\mathbb{E}_L\Big[\mathbb{1}(\boldsymbol{I}=I_q)\Big\{\int \nu_L(I_q,\boldsymbol{x}+th)K(t)dt-\nu_L(I_q,\boldsymbol{x})\Big\}^2\Big]$$
$$+L\bigg(\mathbb{E}_L\Big[\mathbb{1}(\boldsymbol{I}=I_q)\Big\{\tilde{\nu}_L(I_q,\boldsymbol{x})-\nu_L(I_q,\boldsymbol{x})\Big\}\Big]\bigg)^2. \quad \text{(S.45)}$$

In view of the first term, observe that there exists a finite constant $c_\nu>0$ such that $|\nu_L(I_q,\tilde{x})-\nu_L(I_q,x)|<c_\nu\|\tilde{x}-x\|$ for all $(\tilde{x},x)\in(\mathcal{X}\times\mathcal{X})\cup\{(\mathbb{R}\backslash\mathcal{X})\times(\mathbb{R}\backslash\mathcal{X})\}$ and all $L\in\mathbb{N}$: such a constant can be obtained from Lemma S.8. Therefore, $\nu_L(I_q,x+th)-\nu_L(I_q,x)\to 0$ pointwise at each fixed $(x,t)\in\{\mathbb{R}^D\backslash\text{boundary}(\mathcal{X})\}\times[-1,1]^D$.[1] Then, since boundary($\mathcal{X}$) has Lebesgue measure zero, we have

$$\mathbb{E}_L\Big[\mathbb{1}(\boldsymbol{I}=I_q)\Big\{\int \nu_L(I_q,\boldsymbol{x}+th)K(t)dt-\nu_L(I_q,\boldsymbol{x})\Big\}^2\Big]$$
$$=\int\Big[\int\Big\{\nu_L(I_q,x+th)-\nu_L(I_q,x)\Big\}K(t)dt\Big]^2 f_{\boldsymbol{I}\boldsymbol{x}}(I_q,x)dx\to 0 \quad \text{(S.46)}$$

by the dominated convergence theorem.

[1]We remark that $\nu_L(I_q,x)$ does not necessarily converge, but $\nu_L(I_q,x+th)-\nu_L(I_q,x)$ does on the specified set.

For the second term of (S.45), note that

$$\sqrt{L}\mathbb{E}\Big[\mathbb{1}(\boldsymbol{I}=I_q)\big\{\tilde{\nu}_L(I_q,\boldsymbol{x})-\nu_L(I_q,\boldsymbol{x})\big\}\Big]$$

$$=\sqrt{L}\left\{\iint \nu_L(I_q,x)K\Big(\frac{x-\tilde{x}}{h}\Big)\frac{1}{h^D}dx f_{\boldsymbol{I}\boldsymbol{x}}(I_q,\tilde{x})d\tilde{x}-\int \nu_L(I_q,x)f_{\boldsymbol{I}\boldsymbol{x}}(I_q,x)dx\right\}$$

$$=\sqrt{L}\left\{\int \nu_L(I_q,x)\int K(t)f_{\boldsymbol{I}\boldsymbol{x}}(I_q,x-th)dtdx-\int \nu_L(I_q,x)f_{\boldsymbol{I}\boldsymbol{x}}(I_q,x)dx\right\}=O\big(\sqrt{L}h^P\big),$$

where the last equality follows from expanding $f_{\boldsymbol{I}\boldsymbol{x}}(I_q,x-th)$. Finally, note that $\sqrt{L}h^P\to 0$ by Assumption 3. □

Before stating the next lemmas, we define Σ_{G_L} to be a $Q\times Q$ diagonal matrix whose (q,q)-element is given by

$$\Sigma_{G_L}^{(q,q)}=\mathbb{E}_L\left[\frac{\big\{\dot{\boldsymbol{b}}-\mu_L(I_q,\boldsymbol{x})\big\}^2\mathbb{1}(\boldsymbol{x}\in\mathcal{X}_T)\mathbb{1}(\boldsymbol{I}=I_q)}{f_{\boldsymbol{I}\boldsymbol{x}}^2(I_q,\boldsymbol{x})}\right].$$

Lemma S.19. *For any $q=1,\dots,Q$, we have $0<\inf_L\Sigma_{G_L}^{(q,q)}\le\sup_L\Sigma_{G_L}^{(q,q)}<\infty$; in particular, Σ_{G_L} is positive definite for all $L\in\mathbb{N}$. Further, $\{\Sigma_{G_L}:L\in\mathbb{N}\}$ also satisfies Assumption 2.2.(b).*

Proof. Note that

$$\Sigma_{G_L}^{(q,q)}=\mathbb{E}\left(\mathbb{E}_{G_L}\left[\frac{\big\{\dot{\boldsymbol{b}}-\mu_L(I_q,\boldsymbol{x})\big\}^2\mathbb{1}(\boldsymbol{x}\in\mathcal{X}_T)}{f_{\boldsymbol{x}|\boldsymbol{I}}^2(\boldsymbol{x}|I_q)}\middle|\boldsymbol{x},\boldsymbol{I}=I_q\right]\middle|\ \boldsymbol{I}=I_q\right)\frac{1}{f_{\boldsymbol{I}}(I_q)}$$

$$=\mathbb{E}\left\{\frac{\mathbb{V}_{G_L}(\boldsymbol{b}|\boldsymbol{x},\boldsymbol{I}=I_q)\mathbb{1}(\boldsymbol{x}\in\mathcal{X}_T)}{I_q f_{\boldsymbol{x}|\boldsymbol{x}}^2(\boldsymbol{x}|I_q)}\ \middle|\ \boldsymbol{I}=I_q\right\}\frac{1}{f_{\boldsymbol{I}}(I_q)}. \quad \text{(S.47)}$$

Therefore, using the fact that $\bar{b}_{G_L}(I_q,\cdot)$ is uniformly bounded and $f_{\boldsymbol{x}|\boldsymbol{I}}(\cdot|I_q)$ is bounded away from zero shows that $\sup_L\Sigma_{G_L}^{(qq)}<\infty$. For the infimum part, we note that $\inf_L\mathbb{V}_{G_L}(\boldsymbol{b}|\boldsymbol{x}=x,\boldsymbol{I}=I_q)>0$ for all $x\in\mathcal{X}$ because $\inf_{(L,x)}\big\{\bar{b}_{G_L}(I_q,x)-\underline{b}_{G_L}(I_q,x)\big\}>0$ by Lemma S.8. Finally, we have that condition 2.(b) of Assumption 2 is satisfied by the dominated convergence theorem. □

Lemma S.20. *For any $r\in\mathbb{R}^Q\backslash\{0\}$, we have*

$$\frac{\sqrt{L}r^\top\big(\hat{\mathbb{M}}-\mathbb{M}_{G_L}\big)}{\sqrt{r^\top\Sigma_{G_L}r}}\xrightarrow{d}N(0,1)$$

under the sequence $\{G_L:L\in\mathbb{N}\}$.

Proof. By combining (S.44) with Lemma S.18, we obtain

$$\frac{\sqrt{L} r^\top \{\hat{\mathbb{M}} - \mathbb{M}_{G_L}\}}{\sqrt{r^\top \Sigma_{G_L} r}} = \frac{1}{\sqrt{L}} \sum_{l=1}^{L} \left\{ \frac{\sum_{q=1}^{Q} r_q \varsigma_{lL}(I_q)}{\sqrt{r^\top \Sigma_{G_L} r}} \right\} + o_{P_L}(1)$$

with

$$\varsigma_{lL}(I_q) := \frac{\{\dot{\boldsymbol{b}}_l - \mu_L(I_q, \boldsymbol{x}_l)\} \mathbb{1}(\boldsymbol{x}_l \in \mathcal{X}_T) \mathbb{1}(\boldsymbol{I}_l = I_q)}{f_{\boldsymbol{Ix}}(I_q, \boldsymbol{x}_l)}.$$

We recall that

$$\inf_L \mathbb{V}_{G_L} \Big\{ \sum_{q=1}^{Q} r_q \varsigma_{lL}(I_q) \Big\} = \inf_L r^\top \Sigma_{G_L} r > 0$$

by Lemma S.19, and we note that $\sup_L |\varsigma_{lL}(I_q)| < \infty$ with $\mathbb{P}_{G_L}$-probability one. Therefore, the Lindeberg condition is satisfied, so the lemma statement follows from the central limit theorem for triangular arrays. □

Now write $m_L(I_q, x) = \mu_L(I_q, x) f_{\boldsymbol{Ix}}(I_q, x)$ and

$$\hat{\boldsymbol{m}}(I_q, x) = \frac{1}{L} \sum_{l=1}^{L} \dot{\boldsymbol{b}}_l K\Big(\frac{x - \boldsymbol{x}_l}{h}\Big) \frac{1}{h^D} \mathbb{1}(\boldsymbol{I}_l = I_q).$$

Lemma S.21. *For any $q = 1, \ldots, Q$,*

$$\sup_{x \in \mathcal{X}_T} \left| \hat{\boldsymbol{m}}(I_q, x) - m_L(I_q, x) \right| = O_{P_L}\left(\sqrt{\frac{\log L}{L h^D}} + h^P \right).$$

Proof. Although we have $\{\mathbb{P}_{G_L} : L \in \mathbb{N}\}$ instead of having a fixed underlying probability, the standard proof goes through without many modifications. We spell it out here for the sake of clarity. Fix $q = 1, \ldots, Q$ and note that

$$\begin{aligned} &\sup_{x \in \mathcal{X}_T} \left| \hat{\boldsymbol{m}}(I_q, x) - m_L(I_q, x) \right| \\ &\qquad \le \sup_{x \in \mathcal{X}_T} \left| \hat{\boldsymbol{m}}(I_q, x) - \mathbb{E}_L\{\hat{\boldsymbol{m}}(I_q, x)\} \right| + \sup_{x \in \mathcal{X}_T} \left| \mathbb{E}_L\{\hat{\boldsymbol{m}}(I_q, x)\} - m_L(I_q, x) \right|. \qquad \text{(S.48)} \end{aligned}$$

Considering the second term, we can bound

$$\begin{aligned} \sup_{x \in \mathcal{X}_T} \left| \mathbb{E}_L\{\hat{\boldsymbol{m}}(I_q, x)\} - m_L(I_q, x) \right| &= \sup_{x \in \mathcal{X}_T} \left| \int m_L(I_q, s) K\Big(\frac{x - s}{h}\Big) \frac{1}{h^D} ds - m_L(I_q, x) \right| \\ &\le C h^P \sup_{x \in \mathcal{X}_T} \max_{P_1, \ldots, P_D} \left| \partial_{x_1^{P_1}} \partial_{x_2^{P_2}} \ldots \partial_{x_D^{P_D}} m_L(I_q, x) \right| \end{aligned}$$

for some constant $C < \infty$ that depends only on the kernel, where the maximum is taken over all integers $P_1, \ldots, P_D$ such that $P_1 + \cdots + P_D = P$, and we have that

$$\sup_L \sup_{x \in \mathcal{X}_T} \max \left| \partial_{x_1^{P_1}} \partial_{x_2^{P_2}} \ldots \partial_{x_D^{P_D}} m_L(I_q, x) \right| < \infty$$

by Lemma S.8. Thus, we only need to focus on the first term on the right-hand side of (S.48). Define

$$\tilde{r}_L = \left(\frac{L}{h^{D+2} \log L} \right)^{D/2}$$

and partition $\mathcal{X}_T = \prod_{d=1}^D [\underline{x}_d + T, \bar{x}_d - T]$ into $\lceil \tilde{r}_L \rceil$ cubes $\{\mathcal{X}_{T,(j,L)} : j = 1, \ldots, \lceil \tilde{r}_L \rceil\}$ such that

$$\sup_{x \in \mathcal{X}_{T,(j,L)}} \| x - x_{(j,L)} \| \ \leq \ \bar{c}_{\mathcal{X}_T} \tilde{r}_L^{-1/D}, \tag{S.49}$$

where we define $\lceil x \rceil = \inf\{n \in \mathbb{N} : n \geq x\}$, $x_{(j,L)}$ is the center of each cube, and

$$\bar{c}_{\mathcal{X}_T} = \max \{ \bar{x}_d - \underline{x}_d - 2T : \ d = 1, \ldots, D \}.$$

Then, we can bound

$$\sup_{x \in \mathcal{X}_T} \left| \hat{\boldsymbol{m}}(I_q, x) - \mathbb{E}_L \{ \hat{\boldsymbol{m}}(I_q, x) \} \right| \leq \boldsymbol{F}_{1L} + \boldsymbol{F}_{2L} + F_{3L},$$

where

$$\begin{cases} \boldsymbol{F}_{1L} := \max_j \sup_{x \in \mathcal{X}_{T,(j,L)}} \left| \hat{\boldsymbol{m}}(I_q, x) - \hat{\boldsymbol{m}}(I_q, x_{(j,L)}) \right|, \\ \boldsymbol{F}_{2L} := \max_j \left| \hat{\boldsymbol{m}}(I_q, x_{(j,L)}) - \mathbb{E}_L \{ \hat{\boldsymbol{m}}(I_q, x_{(j,L)}) \} \right|, \\ F_{3L} := \max_j \sup_{x \in \mathcal{X}_{T,(j,L)}} \left| \mathbb{E}_L \left\{ \hat{\boldsymbol{m}}(I_q, x_{(j,L)}) - \hat{\boldsymbol{m}}(I_q, x) \right\} \right|. \end{cases} \tag{S.50}$$

Considering $\boldsymbol{F}_{1L}$ and F_{3L}, from inequality (S.49) and by Lipschitz-continuity of K, we obtain

$$\left| \hat{\boldsymbol{m}}(I_q, x) - \hat{\boldsymbol{m}}(I_q, x_{(j,L)}) \right| \leq \frac{\tilde{c}_K \cdot \bar{c}_{\mathcal{X}_T}}{h^{D+1} \cdot \tilde{r}_L^{1/D}} = \tilde{c}_K \bar{c}_{\mathcal{X}_T} \sqrt{\frac{\log L}{L h^D}} \quad \text{with } \mathbb{P}_{G_L}\text{-probability one,}$$

where $\tilde{c}_K$ is the Lipschitz constant of the kernel. We remark that neither $\bar{c}_K$ nor $\bar{c}_{\mathcal{X}_T}$ depends on L, and $\log L/(Lh^D) \to 0$. Therefore, $\boldsymbol{F}_{2L}$ is the only remaining term that needs to be considered.

In the rest of this proof, we will show that $\boldsymbol{F}_{2L} = O_{P_L} \left\{ \sqrt{\log L/(Lh^D)} \right\}$. Define

$$\boldsymbol{Z}_{lL}(I_q, x) = \frac{1}{Lh^D} \left[\dot{\boldsymbol{b}}_l K\left(\frac{x - \boldsymbol{x}_l}{h} \right) \mathbb{1}(\boldsymbol{I}_l = I_q) - \mathbb{E}_L \left\{ \dot{\boldsymbol{b}}_l K\left(\frac{x - \boldsymbol{x}_l}{h} \right) \mathbb{1}(\boldsymbol{I}_l = I_q) \right\} \right]$$

so we can write

$$\boldsymbol{W}_L(I_q, x) = \hat{\boldsymbol{m}}(I, x) - \mathbb{E}_L\{\hat{\boldsymbol{m}}(I, x)\} = \sum_{l=1}^{L} \boldsymbol{Z}_{lL}(I_q, x).$$

Then, note that for any $C_L > 0$ that may depend on L, we have

$$\begin{aligned}\mathbb{P}_{G_L}(\boldsymbol{F}_{2L} > C_L) &\leq \lceil \tilde{r}_L \rceil \sup_{x \in \mathcal{X}_T} \mathbb{P}_{G_L}\{|\boldsymbol{W}_L(I_q, x)| > C_L\} \\ &\leq \lceil \tilde{r}_L \rceil \sup_{x \in \mathcal{X}_T} \Big[\mathbb{P}_{G_L}\Big\{\sum_{l=1}^{L} \boldsymbol{Z}_{lL}(I_q, x) > C_L\Big\} + \mathbb{P}_{G_L}\Big\{-\sum_{l=1}^{L} \boldsymbol{Z}_{lL}(I_q, x) > C_L\Big\}\Big]. \quad \text{(S.51)}\end{aligned}$$

Now define $A_L = \sqrt{Lh^D \log(L)}$ and observe that

$$\sup_{x \in \mathcal{X}_T} A_L |\boldsymbol{Z}_{lL}(I_q, x)| \leq 2c_K \sup_L \sup_{x \in \mathcal{X}_T} \bar{b}_{G_L}(I_q, x) \sqrt{\frac{\log L}{Lh^D}} \quad \text{with } \mathbb{P}_{G_L}\text{-probability one,}$$

where $c_K = \sup_t |K(t)|$. Further, since the right-hand side goes to zero, note that

$$\sup_{x \in \mathcal{X}_T} A_L |\boldsymbol{Z}_{lL}(I_q, x)| \leq 1/2$$

with $\mathbb{P}_{G_L}$-probability one when L is sufficiently large, and consequently

$$\mathbb{E}_L\Big[\exp\{\pm A_L \cdot \boldsymbol{Z}_{lL}(I_q, x)\}\Big] \leq 1 + \mathbb{E}_L\{A_L^2 \cdot \boldsymbol{Z}_{lL}^2(I_q, x)\} \leq \exp\Big[\mathbb{E}_L\{A_L^2 \boldsymbol{Z}_{lL}^2(I_q, x)\}\Big],$$

where we use the inequalities $\exp(y) \leq 1 + y + y^2$ for all $y \in [-1/2, 1/2]$ and $1 + \tilde{y} \leq \exp(\tilde{y})$ for all $\tilde{y} \in \mathbb{R}$. Therefore, by the Markov inequality and the fact that $\boldsymbol{Z}_{lL}(I_q, x)$ are independent across $l = 1, 2, \ldots, L$, we have

$$\begin{aligned}\mathbb{P}_{G_L}\Big\{\pm \sum_{l=1}^{L} \boldsymbol{Z}_{lL}(I_q, x) > C_L\Big\} &\leq \exp(-A_L C_L) \prod_{l=1}^{L} \mathbb{E}_L\big[\exp\{\pm A_L \boldsymbol{Z}_{lL}(I_q, x)\}\big] \\ &\leq \exp(-A_L C_L) \exp\Big[\sum_{l=1}^{L} \mathbb{E}_L\{A_L^2 \boldsymbol{Z}_{lL}^2(I_q, x)\}\Big]. \quad \text{(S.52)}\end{aligned}$$

Hence, combining (S.51) and (S.52) shows that

$$\mathbb{P}_{G_L}(\boldsymbol{F}_{2L} > C_L) \leq 2\lceil \tilde{r}_L \rceil \exp(-A_L C_L) \sup_{x \in \mathcal{X}_L} \exp\Big[\sum_{l=1}^{L} A_L^2 \mathbb{E}_L\{\boldsymbol{Z}_{lL}^2(I_q, x)\}\Big], \quad \text{(S.53)}$$

where

$$\begin{aligned}\mathbb{E}_L\{\boldsymbol{Z}_{lL}^2(I_q, x)\} &\leq \frac{1}{L^2 h^{2D}} \mathbb{E}_L\Big\{\dot{\boldsymbol{b}}_l^2 K^2\Big(\frac{x - \boldsymbol{x}_l}{h}\Big) \mathbb{1}(\boldsymbol{I}_l = I_q)\Big\} \\ &= \frac{1}{L^2 h^D} \int_{\mathcal{X}} \mathbb{E}_L(\dot{\boldsymbol{b}}_l^2 | \boldsymbol{I} = I_q, \boldsymbol{x} = x - sh) K^2(s) f_{\boldsymbol{I}\boldsymbol{x}}(I_q, x - sh) ds \leq \frac{c_z}{L^2 h^D}\end{aligned}$$

and $c_z = \sup_L \sup_{x\in\mathcal{X}}\{\bar{b}_{G_L}(I_q,x)\}^2 \sup_{x\in\mathcal{X}} f_{\boldsymbol{I}\boldsymbol{x}}(I_q,x)\int K^2(t)dt < \infty$. So, we obtain from (S.53) that

$$\mathbb{P}_{G_L}(\boldsymbol{F}_{2L} > C_L) \le 2\lceil \tilde{r}_L\rceil \exp\Big(-A_L C_L + \frac{c_z A_L^2}{Lh^D}\Big).$$

Taking $C_L = \tilde{c}\sqrt{\log L/(Lh^D)}$ with $\tilde{c} > c_z$ yields

$$\mathbb{P}_{G_L}\left(\boldsymbol{F}_{2L} > \tilde{c}\sqrt{\frac{\log L}{Lh^D}}\right) \le \frac{2\lceil \tilde{r}_L\rceil}{L^{\tilde{c}-c_z}} = \frac{2\tilde{r}_L}{L^{\tilde{c}-c_z}} + o(1) = \frac{2}{(\log L)^{D/2}h^{D(D+2)/2}L^{\tilde{c}-c_z-D/2}} + o(1). \tag{S.54}$$

Finally, setting $\tilde{c} = D(D+2)/2 + D/2 + c_z$ leads to the desired result:

$$\mathbb{P}_{G_L}\left(\boldsymbol{F}_{2L} > \tilde{c}\sqrt{\frac{\log L}{Lh^D}}\right) \le \frac{2}{(\log L)^{D/2}(Lh)^{D(D+2)/2}} + o(1) = o(1). \tag{S.55}$$

□

Combining together Lemmas S.15 and S.21 yields

$$\sup_{x\in\mathcal{X}_T} \left|\hat{\boldsymbol{\mu}}(I_q,x) - \mu_L(I_q,x)\right| = o_{P_L}(1) \tag{S.56}$$

for any $q = 1,\dots,Q$. Finally, the next lemma establishes the consistency of $\hat{\boldsymbol{\Sigma}}$.

Lemma S.22. $\|\hat{\boldsymbol{\Sigma}} - \Sigma_{G_L}\| = o_{P_L}(1)$.

Proof. Since $\hat{\boldsymbol{\Sigma}}$ and Σ_{G_L} are diagonal matrices, it suffices to consider the (q,q)th element for an arbitrary $q = 1,\dots,Q$. By the Taylor expansion,

$$\begin{aligned}&\left|\frac{\{b-\hat{\boldsymbol{\mu}}(I_q,x)\}^2}{\hat{\boldsymbol{f}}^2(I_q,x)} - \frac{\{b-\mu_L(I_q,x)\}^2}{f^2_{\boldsymbol{I}\boldsymbol{x}}(I_q,x)}\right| \\ &\qquad \le \frac{2|b-\breve{\boldsymbol{\mu}}(I_q,x)|}{\breve{\boldsymbol{f}}^2(I_q,x)}\left|\hat{\boldsymbol{\mu}}(I_q,x)-\mu_L(I_q,x)\right| + \frac{2\{b-\breve{\boldsymbol{\mu}}(I_q,x)\}^2}{\breve{\boldsymbol{f}}^3(I_q,x)}\left|\hat{\boldsymbol{f}}(I_q,x) - f_{\boldsymbol{I}\boldsymbol{x}}(I_q,x)\right|,\end{aligned} \tag{S.57}$$

where $\breve{\boldsymbol{\mu}}(I_q,x) = \mu_L(I_q,x) + t_\mu\{\hat{\boldsymbol{\mu}}(I_q,x) - \mu_L(I_q,x)\}$ and $\breve{\boldsymbol{f}}(I_q,x) = f_{\boldsymbol{I}\boldsymbol{x}}(I_q,x) + t_f\{\hat{\boldsymbol{f}}(I_q,x) - f_{\boldsymbol{I}\boldsymbol{x}}(I_q,x)\}$ for some $t_\mu, t_f \in [0,1]$. Therefore, by Lemma S.15, expression (S.56), and by the facts that μ_L is uniformly bounded and $f_{\boldsymbol{I}\boldsymbol{x}}$ is bounded away from zero, we know that there are some finite constants $C_1, C_2 > 0$ such that the right-hand side of (S.57) is bounded by

$$\{C_1 + o_{P_L}(1)\}\left|\hat{\boldsymbol{\mu}}(I_q,x) - \mu_L(I_q,x)\right| + \{C_2 + o_{P_L}(1)\}\left|\hat{\boldsymbol{f}}(I_q,x) - f_{\boldsymbol{I}\boldsymbol{x}}(I_q,x)\right| = o_{P_L}(1).$$

Hence, all that remains to be shown is that

$$\left|\frac{1}{L}\sum_{l=1}^{L}\frac{\{\dot{\boldsymbol{b}}_l - \mu_L(I_q,\boldsymbol{x}_l)\}^2 \mathbb{1}(\boldsymbol{x}_l\in\mathcal{X}_T)\mathbb{1}(\boldsymbol{I}_l = I_q)}{f^2_{\boldsymbol{I}\boldsymbol{x}}(I_q,\boldsymbol{x}_l)} - \Sigma_{G_L}^{(q,q)}\right| = o_{P_L}(1).$$

However, this follows from Lemma 11.4.3 in Lehmann and Romano (2005) because of Lemma S.19 and the facts that $\{\dot{\boldsymbol{b}}_l - \mu_L(I_q, \boldsymbol{x}_l)\}^2$ is uniformly bounded with probability one and that $f_{\boldsymbol{I}\boldsymbol{x}}(I_q, \cdot)$ is bounded away from zero. □

S.2. Extension: binding reserve price

So far we have considered the case where there is no binding reserve price. In practice, however, the seller may announce a binding reserve price before the auction to increase the expected revenue. This situation raises a problem because the bidders who have valuations smaller than the reserve price do not participate in the auction, so the number of actual bidders is different from that of potential bidders: we refer to Hu, Matthews, and Zou (2010) for a detailed description of this model. In this subsection, we argue that our results can be extended to the case of having a binding reserve price, as long as we can group the observed auctions into clusters such that they share the same number of potential bidders within the cluster. Below we elaborate the idea.

Let $\boldsymbol{I}_l$ and $\boldsymbol{I}_l^*$ denote the number of potential and actual bidders, respectively, in auction l. We suppose that $\boldsymbol{I}_l$ is unobserved and that its support, $\mathcal{I} = \{I_1, \ldots, I_Q\}$, is unknown except for Q. We further assume that the observed auctions can be put into Q clusters, where each cluster consists of auctions with the same number of potential bidders. Formally, we observe $\boldsymbol{C}_l := \big(\mathbb{1}(\boldsymbol{I}_l = I_1) \ \ldots \ \mathbb{1}(\boldsymbol{I}_l = I_Q)\big)^\top$ for each $l = 1, \ldots, L$ without knowing $I_1, \ldots, I_Q$. So, we consider the sample

$$\Big\{(\boldsymbol{b}_{pl}^*, \boldsymbol{I}_l^*, \boldsymbol{x}_l, \boldsymbol{C}_l) : \ p = 1, \ldots, \boldsymbol{I}_l^*, \ l = 1, \ldots, L\Big\}, \tag{S.58}$$

where $\boldsymbol{b}_{pl}^*$ denotes the truncated equilibrium bid.

Regarding the reserve price, we assume that it is determined as a function of $\boldsymbol{I}_l$ and $\boldsymbol{x}_l$ when the value distribution is given. Precisely, for $F \in \mathcal{F}$, let $\phi_F^{\mathrm{rp}}(I, x)$ be the reserve price given $\boldsymbol{I}_l = I$ and $\boldsymbol{x}_l = x$, where $\phi_F^{\mathrm{rp}}(I, \cdot)$ admits $S + 1$ continuous derivatives on $\mathcal{X}$ and $\underline{v}_F(I, x) \leq \phi_F^{\mathrm{rp}}(I, x) < \bar{v}_F(I, x)$ for all $(I, x) \in \mathcal{I} \times \mathcal{X}$. This setup is in line with Guerre, Perrigne, and Vuong (2009). Also, a similar environment has been used in Guerre, Perrigne, and Vuong (2000) and Marmer and Shneyerov (2012), albeit that both papers assume $Q = 1$.

The analysis of Sections 2 and 3 can be extended as follows. First, we introduce the truncated value distribution defined by

$$F^*(v|I, x) = \frac{F(v|I, x) - F\left\{\phi_F^{\mathrm{rp}}(I, x)|I, x\right\}}{1 - F\left\{\phi_F^{\mathrm{rp}}(I, x)|I, x\right\}}$$

for $v \geq \phi_F^*(I,x)$, while $F^*(v|I,x) = 0$ if $v \leq \phi_F^{\mathrm{rp}}(I,x)$. Then, we have $F^* \in \mathcal{F}$ by construction, and therefore, for any $\delta \in \mathcal{D}$, $s_{(\delta,F^*)}$ satisfies the differential equation in (5) with the boundary condition $s_{(\delta,F^*)}\{\underline{v}_{F^*}(I,x),I,x\} = \underline{v}_{F^*}(I,x) = \phi_F^{\mathrm{rp}}(I,x)$. Since we do not focus on the identification of F, we can simply view F^* as the new value distribution that generates the bids. One problem with this approach though is that $\boldsymbol{I}_l$ is now unobserved, so first we need to identify $\mathcal{I}$. Without loss of generality, consider the identification of I_1 based on the distribution of $\boldsymbol{I}_l^*$ given $\mathbb{1}(\boldsymbol{I}_l = I_1) = 1$ and $\boldsymbol{x}_l = x$, i.e. given $\boldsymbol{x}_l = x$ within the first cluster. It follows from the auction model with binding reserve price that $\boldsymbol{I}_l^*$ has a binomial distribution with parameters I_1 and $1 - F\{\phi_F^{\mathrm{rp}}(I_1,x)\big|I_1,x\}$ conditional on the first cluster and $\boldsymbol{x}_l = x$. These binomial parameters can be identified because $0 \leq F\{\phi_F^{\mathrm{rp}}(I_1,x)|I_1,x\} < 1$ in the current setup. Therefore, the analysis of Sections 2 and 3 can be performed by using $G_{(\delta,F^*)}$ and the sample in (S.58) in lieu of $G_{(\delta,F)}$ and the sample in (8), respectively. We remark that $F^* \in \mathcal{F}_{\mathrm{b}}(c)$ for some $c > 1$ provided that all the derivatives of $\phi_F^{\mathrm{rp}}(I,x)$ are uniformly bounded across (F,I,x).

The results of Section 4 can also be extended in the same way except that the estimator of $\beta(\delta,F^*)$ now needs to be modified. In this regard, we suggest using

$$\hat{\boldsymbol{\beta}}_q^* = \frac{1}{\hat{\boldsymbol{I}}_q - 1}\left\{\hat{\mathbb{B}}_q^* + (\hat{\boldsymbol{I}}_q - 2)\hat{\mathbb{M}}_q^*\right\},$$

where $q = 1,\ldots,Q$ and

$$\hat{\boldsymbol{I}}_q = \max_{l=1,\ldots,L} \boldsymbol{I}_l^* \mathbb{1}(\boldsymbol{I}_l = I_q).$$

We remark that using $\hat{\boldsymbol{I}}_q$ does not affect the asymptotic analysis as its convergence rate is arbitrarily fast. When $D = 0$, we propose

$$\hat{\mathbb{B}}_q^* = \max_{l=1,\ldots,L} \max_{p=1,\ldots,\boldsymbol{I}_l^*} \boldsymbol{b}_{pl}^* \mathbb{1}(\boldsymbol{I}_l = I_q) \quad \text{and} \quad \hat{\mathbb{M}}_q^* = \frac{\sum_{l=1}^L \boldsymbol{b}_{1l}^* \mathbb{1}(\boldsymbol{I}_l = I_q)}{\sum_{l=1}^L \mathbb{1}(\boldsymbol{I}_l = I_q)}.$$

Note that $\dot{\boldsymbol{b}}_l$ cannot be computed from the new sample, so it has been replaced by $\boldsymbol{b}_{1l}^*$ in the estimator of $\mathbb{M}_G(I_q)$. When $D > 0$, the estimators of $\mathbb{B}_G(I_q)$ can be superconsistently estimated as in Subsection 4.3.1 using the sample (S.58). To compute the estimators suggested in Subsection 4.3.2, we also need to replace $\dot{\boldsymbol{b}}_l$ with $\boldsymbol{b}_{1l}$. Finally, the estimator of Ψ proposed in Subsection B.4.1 can be modified in the same manner.

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