SUPPLEMENT TO TIGHTER BOUNDS IN TRIANGULAR SYSTEMS (2009)

SUNG JAE JUN, JORIS PINKSE, HAIQING XU

CENTER FOR AUCTIONS, PROCUREMENTS AND COMPETITION POLICY
DEPARTMENT OF ECONOMICS
THE PENNSYLVANIA STATE UNIVERSITY

In the main paper we construct bounds for the same object of interest as Chesher (2005) under an alternative, weaker, rank condition. Our analysis was based on global exclusion and independence conditions, whereas Chesher's (2005) analysis is local. The global conditions enable us to obtain identified bounds that depend on multiple instrumental values, which requires a less restrictive rank condition. The purpose of this supplement is to obtain a better understanding of the trade–off between local conditions with a more restrictive rank condition versus global conditions with a less restrictive rank condition.

S1. Re–visiting Chesher (2005)

Consider the model

$$\begin{cases} \boldsymbol{y} = \tilde{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}), \\ \boldsymbol{x} = h(\boldsymbol{z}, \boldsymbol{v}), \end{cases}$$

where $\boldsymbol{x}$ is a discrete endogenous regressor, $\tilde{g}$ and h are weakly monotonic in $\boldsymbol{u}, \boldsymbol{v}$, respectively, and $\boldsymbol{u}$ and $\boldsymbol{v}$ are normalized to have uniform distributions on $\mathscr{U} \equiv (0,1]$. In particular, $h(z,\cdot)$ is the conditional quantile function of $\boldsymbol{x}$ given $\boldsymbol{z} = z$: i.e. $x = h(z,v)$ when $v \in V(x,z) \equiv \big(\mathbb{P}[\boldsymbol{x} < x|\boldsymbol{z} = z], \mathbb{P}[\boldsymbol{x} \leq x|\boldsymbol{z} = z]\big]$, and hence

$$\boldsymbol{x} = x, \boldsymbol{z} = z \Longleftrightarrow \boldsymbol{z} = z, \boldsymbol{v} \in V(x,z).$$

Let x^*, τ^*, v^* be fixed values, where x^* is in the support of $\boldsymbol{x}$ and $(\tau^*, v^*) \in \mathscr{U}^2$. The following four assumptions are from Chesher (2005).

B1: There exist z_1 and z_2 such that $V(x^*, z_1) \leq v^* \leq V(x^*, z_2)$.

B2: For all $\tau \in \mathscr{U}$, $Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau|r, z_j)$ is weakly increasing in $r \in \big(\inf V(x^*, z_1), \sup V(x^*, z_2)\big]$ for $j = 1, 2$.

B3: $Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^*|v^*, z_1) = Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^*|v^*, z_2) = r^*$.

B4: $\tilde{g}(x, z_1, r^*) = \tilde{g}(x, z_2, r^*) = \tilde{g}^*$.

The object of interest is $\tilde{g}^*$. Assumption B1 is a rank condition. The direction of the monotonicity in B2 need not be assumed, but we assume it here for presentational convenience. Assumptions B3 and B4 are quantile invariance and local exclusion conditions, respectively. Chesher's main result is the following.

Lemma 1. $Q_{\boldsymbol{y}|\boldsymbol{x},\boldsymbol{z}}(\tau^*|x^*, z_1) \leq \tilde{g}^* \leq Q_{\boldsymbol{y}|\boldsymbol{x},\boldsymbol{z}}(\tau^*|x^*, z_2)$.

Proof. Since

$$Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^*|V(x^*, z_1), z_1) \leq Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^*|v^*, z_1) = Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^*|v^*, z_2) \leq Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^*|V(x^*, z_2), z_2),$$

the result follows from the monotonicity of $\tilde{g}$. □

In our main paper, we argue that B1 can be too strong. If stronger invariance and exclusion restrictions are satisfied then bounds on $\tilde{g}^*$ that depend on multiple values of $\boldsymbol{z}$ can be constructed.

S2. TOWARD A NEW RANK CONDITION

In the following discussion we focus on obtaining an upper bound.

W1: There exist z_3, z_4 such that $v^* \leq V(x^*, z_4) - V(x^*, z_3) \neq \emptyset$, where $V(x^*, z_3) \subset V(x^*, z_4)$.

W2: For all $\tau \in \mathscr{U}$, $Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau|r, z_j)$ is weakly increasing in $r \in \big(\inf V(x^*, z_4),\ \sup V(x^*, z_4)\big]$ for $j = 3, 4$.

W3: $r^* = Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^*|v^*, z_4)$.

W4: $\tilde{g}^* = \tilde{g}(x^*, z_4, r^*)$.

W5: For all $\tau \in \mathscr{U}$, $Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau|V(x^*, z_3), z_4) = Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau|V(x^*, z_3), z_3) \equiv r^{**}\left(\tau, V(x^*, z_3)\right)$.

W6: For all $\tau \in \mathscr{U}$, $\tilde{g}\left(x, z_3, r^{**}\left(\tau, V(x^*, z_3)\right)\right) = \tilde{g}\left(x, z_4, r^{**}\left(\tau, V(x^*, z_3)\right)\right)$.

Assumption W1 can hold even when there is no value z for which $v^* \leq V(x^*, z)$. Assumptions W2–W4 are an insubstantial departure from B2–B4. W5 and W6 are additions to the local invariance (B3/W3) and local exclusion (B4/W4) restrictions, respectively; once we condition on $\boldsymbol{v} \in V(x^*, z_3)$, the conditional distribution of $\boldsymbol{u}$ given $\boldsymbol{z} = z_3$ is the same as the conditional distribution of $\boldsymbol{u}$ given $\boldsymbol{z} = z_4$.

W5 and W6 are similar to (but weaker than) the independence of $\boldsymbol{z}$ and $\boldsymbol{u}, \boldsymbol{v}$ and the exclusion of $\boldsymbol{z}$ from $\tilde{g}$. Although W5 and W6 can be further localized, we start with them to highlight the main idea; see subsection S3 for further information. Assumptions W5 and W6 are useful for the following reason. Note that

$$\begin{aligned}\mathbb{P}[\boldsymbol{y} \leq y|\boldsymbol{x} = x^*, \boldsymbol{z} = z_4]\mu\left(V(x^*, z_4)\right) &= \mathbb{P}[\tilde{g}\left(x^*, z_4, \boldsymbol{u}\right) \leq y|\boldsymbol{v} \in V(x^*, z_4), \boldsymbol{z} = z_4]\mu\left(V(x^*, z_4)\right) \\ &= \mathbb{P}[\tilde{g}\left(x^*, z_4, \boldsymbol{u}\right) \leq y|\boldsymbol{v} \in V(x^*, z_4) - V(x^*, z_3), \boldsymbol{z} = z_4]\mu\left(V(x^*, z_4) - V(x^*, z_3)\right) \\ &\quad + \mathbb{P}[\tilde{g}\left(x^*, z_4, \boldsymbol{u}\right) \leq y|\boldsymbol{v} \in V(x^*, z_3), \boldsymbol{z} = z_4]\mu\left(V(x^*, z_3)\right), \end{aligned} \tag{1}$$

where RHS1 is what we are trying to learn from (1). The following lemma shows how to identify RHS2 in (1) by using W5–W6.

Lemma 2. $\mathbb{P}[\tilde{g}\left(x^*, z_4, \boldsymbol{u}\right) \leq y|\boldsymbol{v} \in V(x^*, z_3), \boldsymbol{z} = z_4] = \mathbb{P}[\boldsymbol{y} \leq y|\boldsymbol{x} = x^*, \boldsymbol{z} = z_3]$.

Proof. Since

$$\begin{aligned}Q_{\boldsymbol{y}|\boldsymbol{x},\boldsymbol{z}}(\tau|x^*, z_3) &= \tilde{g}\left(x^*, z_3, Q_{\boldsymbol{u}|\boldsymbol{x},\boldsymbol{z}}(\tau|x^*, z_3)\right) = \tilde{g}\left(x^*, z_3, Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau|V(x^*, z_3), z_3)\right) \\ &\overset{W5,W6}{=} \tilde{g}\left(x^*, z_4, Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau|V(x^*, z_3), z_4)\right) = Q_{\tilde{g}(x^*, z_4, \boldsymbol{u})|\boldsymbol{v},\boldsymbol{z}}(\tau|V(x^*, z_3), z_4),\end{aligned}$$

the conditional distribution of $\tilde{g}(x^*, z_4, \boldsymbol{u})$ given $\boldsymbol{v} \in V(x^*, z_3)$ and $\boldsymbol{z} = z_4$ is identified by the conditional distribution of $\boldsymbol{y}$ given $\boldsymbol{x} = x^*$ and $\boldsymbol{z} = z_3$. □

Theorem 1. *$\tilde{g}^* \leq \tilde{g}\left(x^*, z_4, Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^*|V(x^*, z_4) - V(x^*, z_3), z_4)\right)$ is identified.*

Proof. Combining lemma 2 with equation (1) shows that

$$\begin{aligned}&\mathbb{P}[\tilde{g}(x^*, z_4, \boldsymbol{u}) \leq y|\boldsymbol{v} \in V(x^*, z_4) - V(x^*, z_3), \boldsymbol{z} = z_4] \\ &\qquad = \frac{\mathbb{P}[\boldsymbol{y} \leq y|\boldsymbol{x} = x^*, \boldsymbol{z} = z_4]\mu\left(V(x^*, z_4)\right) - \mathbb{P}[\boldsymbol{y} \leq y|\boldsymbol{x} = x^*, \boldsymbol{z} = z_3]\mu\left(V(x^*, z_3)\right)}{\mu\left(V(x^*, z_4) - V(x^*, z_3)\right)}.\end{aligned}$$

Since the conditional distribution of $\tilde{g}(x^*, z_4, \boldsymbol{u})$ is identified, its conditional quantile is also identified. □

Iterating a condition like W1 with more instrumental values leads to the rank condition of our main paper. However, this iteration procedure requires that W5 and W6 be strengthened and in the limit are equivalent to the global conditions of the main paper.

S3. Further Localization

Assumptions W5 and W6 are weaker than but similar to independence/exclusion. If W5 and W6 are not satisfied for all τ but they are satisfied at a particular τ^{**}, then we do not identify the

RHS2 of (1) as we did before, but we can still bound it. To be more concrete, we maintain W1–W4, but we replace W5–W6 with the following assumptions.

W1-L: There exists some τ^{**} such that

$$Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^{**}|V(x^*,z_3),z_3) = Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^{**}|V(x^*,z_3),z_4) = r^{**}\left(\tau^{**},V(x^*,z_3)\right).$$

W2-L: $\tilde{g}\left(x,z_3,r^{**}\left(\tau^{**},V(x^*,z_3)\right)\right) = \tilde{g}\left(x,z_4,r^{**}\left(\tau^{**},V(x^*,z_3)\right)\right)$.

Below we show how to obtain an upper bound of $\tilde{g}^*$ that depends on $V(x^*,z_4) - V(x^*,z_3)$. Let $y^* = Q_{\boldsymbol{y}|\boldsymbol{x},\boldsymbol{z}}(\tau^{**}|x^*,z_3)$. Then,

$$y^* = \tilde{g}\left(x^*,z_3,Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^{**}|V(x^*,z_3),z_3)\right) = \tilde{g}\left(x^*,z_4,Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^{**}|V(x^*,z_3),z_4)\right),$$

which implies that

$$\mathbb{P}\left[\tilde{g}(x^*,z_4,\boldsymbol{u}) < y^*|\boldsymbol{v} \in V(x^*,z_3),z_4\right] < \tau^{**} \leq \mathbb{P}\left[\tilde{g}(x^*,z_4,\boldsymbol{u}) \leq y^*|\boldsymbol{v} \in V(x^*,z_3),z_4\right]. \tag{2}$$

This inquality provides a bound for the RHS2 in (1). To be more concrete, define ϕ_U by

$$\begin{aligned}&\left\{\mu\left(V(x^*,z_4)\right) - \mu\left(V(x^*,z_3)\right)\right\}\phi_U \\ &\qquad = \mathbb{P}\left[\boldsymbol{y} < y^*|\boldsymbol{x} = x^*,\boldsymbol{z} = z_4\right]\mu\left(V(x^*,z_4)\right) - \mathbb{P}\left[\boldsymbol{y} \leq y^*|\boldsymbol{x} = x^*,\boldsymbol{z} = z_3\right]\mu\left(V(x^*,z_3)\right).\end{aligned}$$

We then have the following theorem.

Theorem 2. *If $\tau^* \leq \phi_U$, then $\tilde{g}^* \leq Q_{\boldsymbol{y}|\boldsymbol{x},\boldsymbol{z}}(\tau^{**}|x^*,z_3)$.*

Proof. Note that

$$\begin{aligned}&\left\{\mu\left(V(x^*,z_4)\right) - \mu\left(V(x^*,z_3)\right)\right\}\phi_U \\ &\quad = \mathbb{P}\left[\boldsymbol{y} < y^*|\boldsymbol{x} = x^*,\boldsymbol{z} = z_4\right]\mu\left(V(x^*,z_4)\right) - \mathbb{P}\left[\boldsymbol{y} \leq y^*|\boldsymbol{x} = x^*,\boldsymbol{z} = z_3\right]\mu\left(V(x^*,z_3)\right) \\ &\quad \leq \mathbb{P}\left[\boldsymbol{y} < y^*|\boldsymbol{x} = x^*,\boldsymbol{z} = z_4\right]\mu\left(V(x^*,z_4)\right) - \tau^{**}\mu\left(V(x^*,z_3)\right),\end{aligned} \tag{3}$$

where the last inequality follows from the definition of y^*. Then, by (2), the RHS of (3) is bounded by

$$\begin{aligned}&\mathbb{P}\left[\boldsymbol{y} < y^*|\boldsymbol{x} = x^*,\boldsymbol{z} = z_4\right]\mu\left(V(x^*,z_4)\right) - \mathbb{P}\left[\tilde{g}(x^*,z_4,\boldsymbol{u}) < y^*|\boldsymbol{v} \in V(x^*,z_3),z_4\right]\mu\left(V(x^*,z_3)\right) \\ &\quad = \mathbb{P}\left[\tilde{g}(x^*,z_4,\boldsymbol{u}) < y^*|\boldsymbol{v} \in V(x^*,z_4) - V(x^*,z_3),\boldsymbol{z} = z_4\right]\left\{\mu\left(V(x^*,z_4)\right) - \mu\left(V(x^*,z_3)\right)\right\}.\end{aligned}$$

Therefore, we know that

$$\phi_U \leq \mathbb{P}\left[\tilde{g}(x^*, z_4, \boldsymbol{u}) \leq y^* | \boldsymbol{v} \in V(x^*, z_4) - V(x^*, z_3), \boldsymbol{z} = z_4\right]$$
$$= \mathbb{P}\left[\boldsymbol{y} \leq y^* | \boldsymbol{v} \in V(x^*, z_4) - V(x^*, z_3), \boldsymbol{z} = z_4\right],$$

where the last equality follows from the fact that

$$\boldsymbol{v} \in V(x^*, z_4) - V(x^*, z_3), \boldsymbol{z} = z_4 \Longrightarrow \boldsymbol{v} \in V(x^*, z_4), \boldsymbol{z} = z_4 \Longleftrightarrow \boldsymbol{x} = x^*, \boldsymbol{z} = z_4. \tag{4}$$

Therefore, if $\tau^* \leq \phi_U$, then

$$y^* \geq Q_{\boldsymbol{y}|\boldsymbol{v},\boldsymbol{z}}(\tau^* | V(x^*, z_4) - V(x^*, z_3), z_4) = \tilde{g}\left(x^*, z_4, Q_{\boldsymbol{u}|\boldsymbol{v},\boldsymbol{z}}(\tau^* | V(x^*, z_4) - V(x^*, z_3), z_4)\right) \geq \tilde{g}^*,$$

where the equality is due to (4), and the last inequality follows from W1 and monotonicity. □

For a lower bound, define ϕ_L by

$$\left\{\mu\left(V(x^*, z_4)\right) - \mu\left(V(x^*, z_3)\right)\right\} \phi_L$$
$$= \mathbb{P}\left[\boldsymbol{y} \leq y^* | \boldsymbol{x} = x^*, \boldsymbol{z} = z_4\right] \mu\left(V(x^*, z_4)\right) - \mathbb{P}\left[\boldsymbol{y} < y^* | \boldsymbol{x} = x^*, \boldsymbol{z} = z_3\right] \mu\left(V(x^*, z_3)\right).$$

Since the analysis is symmetric, it will be omitted here.