Local Structural Quantile Effects in a Model with a Nonseparable Control Variable*

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Abstract

I consider a semiparametric version of the nonseparable triangular model of Chesher (2003). The proposed model is linear in coefficients, where the coefficients are unknown functions of unobserved latent variables. Using a control variable idea and quantile regression methods, I propose a simple two–step estimator for the coefficients evaluated at particular values of the latent variables. Under the condition that the instruments are locally relevant, i.e. they affect a particular conditional quantile of interest of the endogenous variable, I establish consistency and asymptotic normality. Simulation experiments confirm the theoretical results.

Key Words: Triangular Models, Local Instruments, Control Variables, Quantile Regression.

JEL Classification Code: C14, C30

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1 Introduction

Consider the triangular model

$$Y = g(D, X, V, \epsilon) \quad \text{and} \quad D = h(X, Z, V), \tag{1}$$

where D is an endogenous variable, and V and ϵ are unobserved random variables. Since the latent variables are nonseparable, the model (1) allows for a wide variety of heterogeneity and many authors have investigated identification issues in this setup (see e.g. Imbens and Newey (2002) and Chesher (2003, 2007)). The purpose of this paper is to propose a new semiparametric estimator that is easy to implement but keeps the flexibility of the nonseparable triangular system. In particular, I consider the semiparametric version

$$Y = D\alpha(V, \epsilon) + X'\beta(V, \epsilon), \tag{2}$$

where the parameters of interest are $\alpha(\tau_1, \tau_2)$ and $\beta(\tau_1, \tau_2)$ at particular values of $V = \tau_1$ and $\epsilon = \tau_2$. I develop a simple two–step estimator based on quantile regression methods, and establish its asymptotic normal distribution.

Unlike fully nonparametric approaches, including many exogenous covariates does not cause the curse of dimensionality. The marginal effect $\alpha(V, \epsilon)$ of the endogenous variable is allowed to depend on both V and ϵ. It will be shown that the asymptotic normality of the estimator only requires that the instruments be locally relevant in the sense that they only need to affect the τ_1 conditional quantile of the endogenous variable when interest is in $V = \tau_1$. The cost of this degree of flexibility is that the proposed estimator converges at a nonparametric rate.

Identification and estimation of heterogeneous marginal effects have been frequently discussed in the literature. Koenker and Basset (1978) provide the first possibility via quantile regression when there is no endogeneity. Chernozhukov and Hansen (2005, 2006) propose a way of modeling endogeneity in quantile regression models, which can be understood in a generalized method of moments (GMM) framework. Imbens and Newey (2002) and Chesher (2003, 2007) consider nonseparable tri-

angular models to capture heterogeneity in marginal effects. In particular, Chesher (2003) provides a set of conditions under which heterogeneous marginal effects are nonparametrically identified in the general system (1). Although his identification results lead to a straightforward nonparametric estimation method, it will suffer from the dimensionality issue when many exogenous covariates are included. Lee (2007) takes a semiparametric approach, where he uses a partially linear quantile regression method. Ma and Koenker (2006) propose a parametric version of Chesher (2003).

The model proposed in this paper can be compared with Lee (2007) and Ma and Koenker (2006). In contrast to Ma and Koenker (2006), the approach of this paper is semiparametric in the sense that the functional form of the random coefficients of the structural equation need not be specified. Lee (2007) takes a control function approach, where he assumes that the control function is additively separable. This separability assumption prevents $\alpha(\tau_1, \tau_2)$ from being heterogeneous over τ_1, and hence Lee (2007) does not preserve the flexibility of the general triangular system.

Since the dependence between the endogenous variable and the random coefficients is due to the fact that they both depend on V, this common latent variable plays the role of a control variable. It is in fact used as a control variable in the estimation procedure, because the two-step procedure essentially fixes V at τ_1 by kernel smoothing.

Note that the parameter $\alpha(\tau_1, \tau_2)$ is a local feature. The reason why this local parameter is focused on is that identification of the average quantile effect (i.e., $\int_0^1 \alpha(s, \tau_2)ds$) requires that the instruments affect all the distributional points of the endogenous variable, which is a strong assumption in nonseparable models. In fact, I investigate the possibility of *local instruments* by conducting small experiments with the Angrist and Krueger data: the instruments do not affect a particular conditional quantile of the endogenous variable although they do affect others. The Monte Carlo results in section 6 confirm that the proposed estimator only requires local relevance of instruments.

2 Motivation: Returns to Education and Instrumental Variables

Consider the following wage equation

$$Y_d = g(d, X, V, \epsilon), \tag{3}$$

where Y_d is a (counterfactual) wage, d is a level of education, X is a vector of individual characteristics, V is unobserved ability, and ϵ is unobserved market fortune. Suppose that g is increasing in ϵ. Individuals choose their level of education to maximize the difference between expected wages given their characteristics and ability and costs:

$$D = \arg\max_d E\big(g(d, X, V, \epsilon)|X, V\big) - C(d, Z, X, V), \tag{4}$$

where C is the cost of getting a particular level of education, which generally depends on other characteristics and unobserved ability. Letting $h(d, X, V) = E\big(g(d, X, V, \epsilon)|X, V\big)$ (i.e. the expected wage with d level of education given individual characteristics and ability), the optimal level of education satisfies the first order condition

$$h_d(D, X, V) = C_d(D, Z, X, V), \tag{5}$$

which shows that D is a function of Z, X, and V.[1] Assume that $h_{dd}(D, X, V) - C_{dd}(D, Z, X, V) < 0$ so that the second order condition is also satisfied. It then follows from the implicit function theorem that

$$\frac{\partial D}{\partial V} = \frac{C_{dV}(D, Z, X, V) - h_{dV}(D, X, V)}{h_{dd}(D, X, V) - C_{dd}(D, Z, X, V)} > 0 \tag{6}$$

as long as $h_{dV}(D, X, V) - C_{dV}(D, Z, X, V) > 0$ (i.e. the net marginal benefit of taking extra education is increasing in V). Therefore, this setup suggests the following model for returns to

[1] Partial derivatives will be denoted by sub–indices.

education

$$\begin{cases} Y = Y_D = g(D, X, V, \epsilon), \\ D = q(Z, X, V), \end{cases} \tag{7}$$

where g is increasing in ϵ and q is increasing in V. Chesher (2003) pointed out that one of the most important features of this model is that the returns to education $g_d(D, X, V, \epsilon)$ generally depends on V and ϵ. He also showed how useful quantile regression and recursive conditioning is to identify $g_d(D, X, \tau_1, \tau_2)$ at particular values of $V = \tau_1$ and $\epsilon = \tau_2$ (and hence at $D = q(Z, X, \tau_1)$).

Chesher's identification conditions include a rank condition, which can be written as

$$\frac{\partial q(Z, X, \tau_1)}{\partial Z} = \frac{C_{dV}(D, Z, X, \tau_1)}{h_{dd}(D, X, \tau_1) - C_{dd}(D, Z, X, \tau_1)} \neq 0, \tag{8}$$

where the first equality is by the implicit function theorem. Note however that the sensitivity of D to changes in Z generally depends on the level of unobserved ability. In particular, no economic theory guarantees that Z satisfies condition (8) for every value of τ_1 in support of V. In the sense of Chesher (2003), Z is a relevant instrument to identify $g_d(D, X, \tau_1, \tau_2)$ only for those values of τ_1 such that $C_{dV}(D, Z, X, \tau_1) \neq 0$. Of course, when $C_{dV}(D, Z, X, \tau_1) \neq 0$ for every value τ_1 in support of V, then $g_d(D, X, \tau_1, \tau_2)$ is identified for every τ_1 and τ_2 in support of V and ϵ, which would provide more complete information about returns to education than any other average notion of returns to education.

Although Chesher (2003) provided a set of conditions to nonparametrically identify $g_d(D, X, \tau_1, \tau_2)$ for particular values of $V = \tau_1$ and $\epsilon = \tau_2$, implementation of this model unavoidablely requires a certain amount of parametrization. The goal of this paper is to semiparametrically implement model (7) with keeping all the flexibility of this model. The essential flexibility to be preserved includes that (i) the returns to education $g_d(D, X, \tau_1, \tau_2)$ are allowed to be heterogeneous over both τ_1 and τ_2, and (ii) the degree of relevance of Z as an instrument is allowed to be heterogeneous over τ_1.

For the second part of flexibility, the data used by Angrist–Krueger (1991) provide an example.[2]

[2]The Angrist and Krueger's data measure education by the years of schooling, which is discrete. This is an issue that is not covered by this paper. Chesher (2005) showed that discrete variations in endogenous variables only deliver

They used birth–quarters as instrumental variables with the idea that the cost of taking extra education varies over different birth–quarters due to the compulsory schooling system of the US. However, since those with lower ability are more likely to be affected by the compulsory schooling system, this idea naturally suggests that birth–quarters will be more relevant for those with lower level of ability. This possiblity will be further discussed in section 5.

The parametrization taken in this paper uses random coefficients e.g. $g(d, X, V, \epsilon) = d\tilde{\alpha}(V, \epsilon) + X'\tilde{\beta}(V, \epsilon)$.[3] To simplify presentation, (V, ϵ) will be assumed to be two independent uniform random variables; otherwise, we may consider $(V, \epsilon) = \big(F_V^{-1}(V_1), F_{\epsilon|V}^{-1}(\epsilon_1|F_V^{-1}(V_1))\big)$, where F_V and $F_{\epsilon|V}$ are the marginal and the conditional distribution functions, and V_1 and ϵ_1 are two independent uniform random variables. It then leads to a model with random coefficients, $Y = D\alpha(V, \epsilon) + X'\beta(V, \epsilon)$, where V and ϵ are independent uniform random variables.

3 The Model

I propose the following model.

$$\begin{cases} Y = D\alpha(V, \epsilon) + X'\beta(V, \epsilon) \\ D = X'\Pi_1(V) + Z'\Pi_2(V), \end{cases} \tag{9}$$

where Y is a scalar outcome variable, D is a scalar endogenous variable, X is a k_x dimensional vector of exogenous covariates, and Z is a k_z dimensional vector of instruments. V and ϵ are both scalars that represent unobserved components. The model is linear in random coefficients. The endogenous variable is correlated with the random coefficients of the structural equation. Since the correlation is due to the fact that V appears twice, the common latent variable V plays the role of a control variable. Now, the following assumptions are made.

A1 (Independence) V and ϵ are independent, and they are normalized to be uniformly distributed.

partial identification.

[3] Given the linear parametrization of g, $C(d, Z, X, V) = A(V)\big(d^2 - 2q^*(Z, X, V)d + \tilde{q}(Z, V, X)\big)$ leads to the equation for the observed education in (7) with $q(Z, X, V) = q^*(Z, X, V) + \zeta(V)$ for some function ζ, where $A(\cdot)$ and $\tilde{q}(\cdot)$ are arbitrary functions. Note however that the linear model considered in section 3 is in fact more flexible than it seems, because as a matter of specification, $Y = g_1(D)'\tilde{\alpha}^*(V, \epsilon) + g_2(X)'\tilde{\beta}^*(V, \epsilon)$ can be easily analyzed, where $g_1(\cdot)$ and $g_2(\cdot)$ are known functions.

X and Z are also assumed to be independent of ϵ and V.

A2 (Monotonicity) $\tau \mapsto D\alpha(V,\tau) + X'\beta(V,\tau)$ is strictly increasing in $\tau \in (0,1)$.

Similarly, $\tau \mapsto X'\Pi_1(\tau) + Z'\Pi_2(\tau)$ is strictly increasing in $\tau \in (0,1)$.

Although it is more restrictive than the fully nonparametric versions of Imbens and Newey (2002) and Chesher (2003), this model still allows for two different kinds of heterogeneity, one kind via V and the other via ϵ. Since it is a linear model, it has practical advantage, also; a completely nonparametric model suffers from the curse of dimensionality when many exogenous variables are included. Although the proposed model is parametric, the control variable V is incorporated in a nonparametric and nonseparable manner, which enables the model to preserve all flexibility of nonseparable models.

The parameter of interest is $\alpha(\tau_1,\tau_2)$ for some fixed $\tau_1,\tau_2 \in (0,1)$, which are chosen by the researcher.[4] It is the marginal effect of the endogenous variable on the outcome variable evaluated at $V = \tau_1, \epsilon = \tau_2$. In the example of section 2, this parameter corresponds to the returns to education for those who have τ_1–level of ability and have τ_2–level of market fortune. Chesher (2003) provides a set of general conditions for nonparametric identification of this parameter, which includes continuous variation of instruments. Instruments are not, however, so rich in practice. Since this is a linear model, discrete instruments can still deliver identification, as long as the following full rank assumption is satisfied.

A3 (Full Rank) $\Pi_2(\tau_1)$ has full column rank for the chosen $\tau_1 \in (0,1)$.

Proposition 1 *Suppose that **A1**, **A2**, and **A3** are satisfied. Let $W = [X' : Z']'$. If $E(WW')$ has full rank, then $\alpha(\tau_1,\tau_2), \beta(\tau_1,\tau_2)$ are identified.*

Proof: See the appendix.

Although proposition 1 is a special case of Chesher (2003, 2007), it exploits the linear quantile regression specification of the model. Therefore, in contrast to Chesher (2003, 2007), instruments need not be continuous and discrete instruments are also allowed.

The nature of instrument relevance is local in the sense that the full rank of $\Pi_2(\tau_1)$ is required only for a particular $\tau_1 \in (0,1)$. For example, consider $D = Z\Phi^{-1}(V) + F^{-1}(V)$, where Z is

[4] In the following discussion, τ_1 and τ_2 will be regarded as fixed values.

non-negative almost surely, $\Phi^{-1}(s)$ is the inverse of the standard normal distribution, and $F^{-1}(s)$ is the inverse of some distribution function. In this case, the instrument is not informative for the conditional median of D, although it is relevant for the other conditional quantiles. When $\alpha(\tau_1, \tau_2)$ can vary over different τ_1, the identification pitfall at $\tau_1 = 0.5$ can be a problem in estimating the average quantile effect $\int_0^1 \alpha(v, \tau_2)dv$. Instead of estimating the average quantile effect, the local parameter $\alpha(\tau_1, \tau_2)$ will be focused on.

Ma and Koenker (2006) discuss estimation of $\alpha(\tau_1, \tau_2)$ and $\beta(\tau_1, \tau_2)$ under additional parametric assumptions. Alternatively, Lee (2007) considers the case where $\alpha(\tau_1, \tau_2)$ and $\beta(\tau_1, \tau_2)$ do not depend on τ_1 except for the intercept and shows that $\alpha(\cdot, \tau_2)$ can be estimated at the regular $\sqrt{n}$ rate. This approach loses the important feature of the triangular system, because it imposes separability of the control variable. In this paper, the control variable is allowed to be arbitrarily involved in the structural equation, which implies that the marginal effect of the endogenous variable can be arbitrarily heterogeneous over different values of $V = \tau_1$. In contrast to Chesher (2003, 2007), the linear structural equation delivers identification when the instruments are discrete. This model is substantially simpler than fully nonparametric and nonseparable models, but yet flexibility of triangular models is well preserved.

For estimation, a local quantile regression approach is taken by using the kernel smoothing idea. I will write $Q_{A|B=b}(\tau)$ for the conditional τ quantile of A given $B = b$. When I do not specify any particular value of b, I will simply write $Q_{A|B}(\tau)$. Recall that

$$\Pr\big(Y \leq D\alpha(\tau_1, \tau_2) + X'\beta(\tau_1, \tau_2) | D = Q_{D|X,Z}(\tau_1), X, Z\big) = \tau_2,^{5}$$

because $D = Q_{D|X=x,Z=z}(\tau_1), X = x, Z = z$ is equivalent to $V = \tau_1, X = x, Z = z$. The idea is that $D\alpha(\tau_1, \tau_2) + X'\beta(\tau_1, \tau_2)$ is the conditional quantile of Y given X, Z for the people whose D is equal to $Q_{D|X,Z}(\tau_1)$. First, let $\rho_\tau(s) = |s| + (2\tau - 1)s$ be the check function for $\tau \in (0, 1)$.

Step 1. Estimate the τ_1 conditional quantile of D given $W_i = [X_i' : Z_i']'$ by $\hat{Q}_{D|W_i}(\tau_1) = W_i'\hat{\Pi}(\tau_1)$,

[5] For the sake of simplicity, the qualifier of "almost surely" will be omitted throughout the paper when it is clear from the context.

where

$$\hat{\Pi}(\tau_1) = \arg\min_{\Pi} \sum_{i=1}^{n} \rho_{\tau_1}(D_i - W_i'\Pi).$$

Step 2. Estimate the parameters of interest by local smoothing quantile regression. Let $\hat{S}_n(\tau_1) = \{1 \leq i \leq n : |D_i - W_i'\hat{\Pi}(\tau_1)| \leq \frac{h_n}{2}\}$, where h_n is a bandwidth choice that shrinks to 0. Then estimate the parameters of interest by

$$\left[\hat{\alpha}(\tau_1, \tau_2) : \hat{\beta}(\tau_1, \tau_2)'\right]' = \arg\min_{a,b} \sum_{i \in \hat{S}_n(\tau_1)} \rho_{\tau_2}(Y_i - D_i a - X_i' b).$$

The second step is another quantile regression with a subsample of those whose first step residuals are sufficiently small. This is kernel-smoothing with a uniform kernel $1\{|s| \leq \frac{h_n}{2}\}$, where $1\{\cdot\}$ denotes the standard indicator function. I focus on the uniform kernel, because regularity conditions for asymptotics become easier to find. As long as there are enough observations with $|D_i - \hat{Q}_{D|X_i,Z_i}(\tau_1)|$ small, this approach is intuitively appealing. The resulting estimators will be shown to be consistent and asymptotically normal under some additional conditions.

4 Asymptotics

This section discusses the asymptotic properties of the proposed two-step estimators. Let $R_i(\tau_1) = D_i - W_i'\Pi(\tau_1)$ and $\epsilon_i(\tau_1, \tau_2) = Y_i - S_i'\theta(\tau_1, \tau_2)$, where $W_i = [X_i', Z_i']', S_i = [D_i, X_i']', \Pi_0 = \Pi(\tau_1) = [\Pi_1(\tau_1)', \Pi_2(\tau_1)]' \in \Xi$, and $\theta_0 = \theta(\tau_1, \tau_2) = [\alpha(\tau_1, \tau_2), \beta(\tau_1, \tau_2)']' \in \Theta$. The data $\{(Y_i, D_i, X_i', Z_i')\}_{i=1}^n$ are *iid*, and the observation index i will be suppressed when generic random variables are considered. The marginal density of A and the conditional density of A given B will be denoted by f_A and $f_{A|B}$, respectively. I make the following assumptions.

Assumption A *There is a compact neighborhood $\mathcal{N}$ around 0 such that (i) for every $\delta \in \mathcal{N}$, $R(\tau_1) + W'\delta$ has a marginal density bounded away from 0 at 0 which is twice continuously differentiable, and (ii)* $\sup_{\delta \in \mathcal{N}} |f_{R(\tau_1)+W'\delta}(t)| \leq b_0(t)$, $\sup_{\delta \in \mathcal{N}} |f'_{R(\tau_1)+W'\delta}(t)| \leq b_1(t)$ *and* $\sup_{\delta \in \mathcal{N}} |f''_{R(\tau_1)+W'\delta}(t)| \leq b_2(t)$ *for some continuous functions $b_0(t)$, $b_1(t)$ and $b_2(t)$.*

Assumption B *$\epsilon(\tau_1, \tau_2)$ has a conditional density at 0 given $R(\tau_1)$ and S.*

Assumptions A and B require that Y and D are continuous random variables but they do not require continuous instruments. Assumption A says that $R(\tau_1) = D - W'\Pi_0$ has a *marginal* density bounded away from 0 at 0 and so do its small perturbations $R(\tau_1) + W'\delta$. The assumptions that Y and D are continuously distributed are crucial but the other assumptions are for regularity. Note also that assumption A ensures that there are sufficiently many observations in $\{1 \leq i \leq n : |R_i(\tau_1)| \leq \delta\}$ for any $\delta > 0$ as n increases.

Note that two sets of conditional moment restrictions are available:

$$\begin{cases} \Pr(D_i \leq W_i'\Pi_0 | W_i) = \tau_1 \\ \Pr(Y_i \leq S_i'\theta_0 | S_i, R_i(\tau_1) = 0) = \tau_2, \end{cases} \tag{10}$$

which implies

$$\begin{cases} E\big(W_i(1\{D_i \leq W_i'\Pi_0\} - \tau_1)\big) = 0 \\ E\big(S_i(1\{Y_i \leq S_i'\theta_0\} - \tau_2) | R_i(\tau_1) = 0\big) f_{R(\tau_1)}(0) = 0. \end{cases} \tag{11}$$

Since D_i is a function of $X_i,\ Z_i$ conditional on $R_i(\tau_1) = 0$, $E\big(S_i(1\{Y_i \leq S_i'\theta_0\} - \tau_2) | R_i(\tau_1) = 0\big) = 0$ obtains by integrating over the distribution of $X_i,\ Z_i$.[6] Although these are not the only moment conditions implied by (10), I do not address the issues of efficiency in this paper.

The proposed estimators can be understood in a generalized method of moments (GMM) framework based on the moment conditions (11) (see e.g., Pakes and Pollard (1989), Newey and McFadden (1994)). In particular, the (quasi) first order conditions of the proposed estimator can be written as

$$\begin{cases} \frac{1}{n}\sum_{i=1}^n W_i(1\{D_i \leq W_i'\hat{\Pi}(\tau_1)\} - \tau_1) = o_p(\frac{1}{\sqrt{n}}) \\ \frac{1}{nh_n}\sum_{i=1}^n S_i(1\{Y_i \leq S_i'\hat{\theta}(\tau_1, \tau_2)\} - \tau_2) 1\{|D_i - W_i'\hat{\Pi}(\tau_1)| \leq \frac{h_n}{2}\} = o_p(\frac{1}{\sqrt{nh_n}}), \end{cases} \tag{12}$$

where h_n is a bandwidth choice with $h_n \downarrow 0$, $nh_n \to \infty$. It is in fact easy to show that

[6] When $R(\tau_1)$ has a density conditional on Y_i and S_i, this conditional expectation can be written as $E\big(S_i(1\{Y_i \leq S_i'\theta_0\} - \tau_2) f_{R(\tau_1)|Y,S}(0|Y_i, S_i)\big) = 0$.

$$M_n(\Pi,\theta)=\begin{bmatrix} M_{1n}(\Pi) \\ M_{2n}(\Pi,\theta)\end{bmatrix}=\begin{bmatrix} \frac{1}{n}\sum_{i=1}^n W_i(1\{D_i\le W_i'\Pi\}-\tau_1) \\ \frac{1}{nh_n}\sum_{i=1}^n S_i(1\{Y_i\le S_i'\theta\}-\tau_2)1\{|D_i-W_i'\Pi|\le \frac{h_n}{2}\}\end{bmatrix}$$
$$\xrightarrow{p} M(\Pi,\theta)=\begin{bmatrix} M_1(\Pi) \\ M_2(\Pi,\theta)\end{bmatrix}=\begin{bmatrix} E\big(W_i(1\{D_i\le W_i'\Pi\}-\tau_1)\big) \\ E\big(S_i(1\{Y_i\le S_i'\theta\}-\tau_2)|D-W'\Pi=0\big)f_{D-W'\Pi}(0)\end{bmatrix}$$

for each Π and θ. The only complication is nondifferentiability of M_n due to the indicators, and the nonparametric convergence rate of the bottom part. Note also that we only need to assume that $D-W'\Pi$ has a marginal density when Π belongs to a neighborhood of Π_0, because Π_0 is consistently estimated from $M_1(\Pi_0)=0$.

Assumption C *The parameters Π_0 and θ_0 uniquely solve $M(\Pi,\theta)=0$, and they are in the interior of the compact parameter space $\Xi\times\Theta$.*

Assumption D *$E\big(M_n(\Pi,\theta)\big)$ and $M(\Pi,\theta)$ are continuous in $(\Pi,\theta)\in\Xi\times\Theta$. It is also assumed that $M(\Pi,\theta)$ is twice differentiable at (Π_0,θ_0).*

Assumption E *$E\big(W_iW_i'f_{R(\tau_1)|W}(0|W_i)\big)$ and $E\big(S_iS_i'f_{\epsilon(\tau_1,\tau_1)|R(\tau_1),S}(0|R_i(\tau_1),S_i)|R_i(\tau_1)=0\big)$ have full column ranks.*

Assumption F *$E\big(||W_i||\sup_s f_{R(\tau_1)|W}(s|W_i)\big)<\infty$.*

Assumption G *$\varphi(\theta,\delta,r)=E\big(S_i(1\{Y_i\le S_i'\theta\}-\tau_2)|R_i(\tau_1)-W_i'\delta=r\big)$ is twice continuously differentiable with respect to r. Moreover, $\frac{\partial\varphi(\theta,\delta,r)}{\partial r}$ and $\frac{\partial^2\varphi(\theta,\delta,r)}{\partial r^2}$ are continuous in $\theta\in\Theta$ and $\delta\in\mathcal{N}$.*

Assumption H *Let w_{ij} and s_{it} be generic elements of W_i and S_i. For some $\nu>0$, $E(|w_{ij}|^{2+\nu})<\infty$, and $E\big(|s_{it}|^{2+\nu}|R(\tau_1)=0\big)<\infty$.*

Assumption C ensures global identification of (Π_0,θ_0), which is satisfied when the conditional density of $R(\tau_1)$ given W and the conditional density of $\epsilon(\tau_1,\tau_2)$ given W and $R(\tau_1)=0$ are bounded away from 0 at 0; see the appendix.

Assumption D is standard. Differentiability of $M(\Pi,\theta)$ is guaranteed e.g. when $T(D_i,W_i) = E\big(S_i(1\{\epsilon(\tau_1,\tau_2) \leq 0\} - \tau_2)|D_i,W_i\big)$ is differentiable with respect to D_i. Assumption E is the standard rank condition but it is local. In particular, note that the full rank assumption fails to hold when $\Pi_2(\tau_1)$ is equal to 0.[7] The rank condition is therefore that the instruments are relevant for the τ_1 conditional quantile of D given W. Assumption G requires smoothness of a regression function. Assumption H assumes that the regressors have sufficient moments, which is needed for a central limit theorem. Since kernel smoothing is used, bias is a problem which I address by undersmoothing.

Assumption I *$h_n \propto n^{-\kappa}$ with $\frac{1}{5} < \kappa < \frac{1}{2(1+\eta)}$ for some $\eta > 0$.*

Lemma 1 *Let $\Xi_0 = \Pi(\tau_1) + \mathcal{N}$. Under assumptions A-B, $\Upsilon_{n,h_n}\Big(E\big(M_n(\Pi,\theta)\big) - M(\Pi,\theta)\Big) = \begin{bmatrix} 0 \\ O(\sqrt{nh_n}h_n^2) \end{bmatrix}$ uniformly over $\Xi_0 \times \Theta$, where $\Upsilon_{n,h_n} = \begin{bmatrix} \sqrt{n}I_{k_x+k_z} & 0 \\ 0 & \sqrt{nh_n}I_{1+k_x} \end{bmatrix}$. In particular, assumption I ensures $\Upsilon_{n,h_n}\Big(E\big(M_n(\Pi,\theta)\big) - M(\Pi,\theta)\Big) = o(1)$ uniformly over $\Xi_0 \times \Theta$.*

Proof: See the appendix.

Now, I state the main theorem.

Theorem 1 *Under assumptions A-I,*

$$\begin{bmatrix} \sqrt{n}\Big(\Big[\hat{\Pi}_1(\tau_1)',\ \hat{\Pi}_2(\tau_1)'\Big]' - [\Pi_1(\tau_1)',\ \Pi_2(\tau_2)']'\Big) \\ \sqrt{nh_n}\Big(\Big[\hat{\alpha}(\tau_1,\tau_2),\ \hat{\beta}(\tau_1,\tau_2)'\Big]' - [\alpha(\tau_1,\tau_2),\ \beta(\tau_1,\tau_2)']'\Big) \end{bmatrix}$$
$$= -\begin{bmatrix} \Gamma_{11}(\tau_1)^{-1} & 0 \\ 0 & \Gamma_{22}(\tau_1,\tau_2)^{-1} \end{bmatrix} \Upsilon_{n,h_n} M_n(\Pi_0,\theta_0) + o_p(1),$$

where $\Gamma_{11}(\tau_1) = E\big(W_iW_i' f_{R(\tau_1)|W}(0|W_i)\big)$ *and* $\Gamma_{22}(\tau_1,\tau_2) = E\big(S_iS_i' f_{\epsilon(\tau_1,\tau_2)|R(\tau_1),S}(0|R_i(\tau_1),S_i)|R_i(\tau_1) =$

[7] Note that given $R(\tau_1) = 0$, $S' = [D : X'] = [X'\Pi_1(\tau_1) + Z'\Pi_2(\tau_1) : X']$.

$0\big) f_{R(\tau_1)}(0)$. *In particular,*

$$\sqrt{nh_n}\Big(\Big[\hat{\alpha}(\tau_1,\tau_2),\ \hat{\beta}(\tau_1,\tau_2)'\Big]' - \big[\alpha(\tau_1,\tau_2),\ \beta(\tau_1,\tau_2)'\big]'\Big) \xrightarrow{d} N\big(0, \Gamma_{22}(\tau_1,\tau_2)^{-1} V_{22}(\tau_1,\tau_2)\Gamma_{22}(\tau_1,\tau_2)^{-1}\big),$$

as $n \to \infty$, *where* $V_{22}(\tau_1,\tau_2) = \tau_2(1-\tau_2)E(S_iS_i'|R_i(\tau_1)=0)f_{R(\tau_1)}(0)$.

Proof: See the appendix.

The center of zero in the asymptotic distribution comes from undersmoothing as most of non-parametric estimators. Since the first–stage estimator has the faster parametric rate of convergence, it does not affect the asymptotic variance of the second–stage estimator. The linear expansion of $\sqrt{nh_n}\big(\hat{\theta}(\tau_1,\tau_2) - \theta_0(\tau_1,\tau_2)\big)$ also suggests joint normality of the estimators with different sets of quantiles.[8]

Corollary 1 *Suppose that the conditions for theorem 1 are satisfied for* $\{(\tau_1,\tau_2),\ (\tilde{\tau}_1,\tilde{\tau}_2)\}$. *Then,*

$$\begin{bmatrix} \sqrt{nh_n}\Big(\Big[\hat{\alpha}(\tau_1,\tau_2),\ \hat{\beta}(\tau_1,\tau_2)'\Big]' - [\alpha(\tau_1,\tau_2),\ \beta(\tau_1,\tau_2)']'\Big) \\ \sqrt{nh_n}\Big(\Big[\hat{\alpha}(\tilde{\tau}_1,\tilde{\tau}_2),\ \hat{\beta}(\tilde{\tau}_1,\tilde{\tau}_2)'\Big]' - [\alpha(\tilde{\tau}_1,\tilde{\tau}_2),\ \beta(\tilde{\tau}_1,\tilde{\tau}_2)']'\Big) \end{bmatrix} \xrightarrow{d} N\left(0, \begin{bmatrix} \Gamma_{22}(\tau_1,\tau_2)^{-1}V_{22}(\tau_1,\tau_2)\Gamma_{22}(\tau_1,\tau_2)^{-1} & COV(\tau_1,\tilde{\tau}_1,\tau_2,\tilde{\tau}_2) \\ COV(\tau_1,\tilde{\tau}_1,\tau_2,\tilde{\tau}_2)' & \Gamma_{22}(\tilde{\tau}_1,\tilde{\tau}_2)^{-1}V_{22}(\tilde{\tau}_1,\tilde{\tau}_2)\Gamma_{22}(\tilde{\tau}_1,\tilde{\tau}_2)^{-1} \end{bmatrix}\right),$$

where $COV(\tau_1,\tau_1,\tau_2,\tilde{\tau}_2) = (\min(\tau_2,\tilde{\tau}_2) - \tau_2\tilde{\tau}_2)\Gamma_{22}(\tau_1,\tau_2)^{-1}E(S_iS_i'|R_i(\tau_1)=0)\Gamma_{22}(\tau_1,\tilde{\tau}_2)^{-1}$ *and* $COV(\tau_1,\tilde{\tau}_1,\tau_2,\tilde{\tau}_2) = 0$ *if* $\tau_1 \neq \tilde{\tau}_1$.

Proof: When $\tau_1 = \tilde{\tau}_1$, it directly follows from the asymptotic expansion of theorem 1. When $\tau_1 \neq \tilde{\tau}_1$, note that $1\{|D_i - W_i'\Pi(\tau_1)| \leq \frac{h_n}{2}\}1\{|D_i - W_i'\Pi(\tilde{\tau}_1)| \leq \frac{h_n}{2}\} = 0$ for sufficiently large n. □

Estimation of $\Gamma_{22}(\tau_1,\tau_2)$ and $V_{22}(\tau_1,\tau_2)$ can be done by various nonparametric methods (see e.g., Koenker (2005), Powell (1986)), although it could be more difficult to estimate $\Gamma_{22}(\tau_1,\tau_2)$. The following proposition shows one possibility.

[8] Uniform inference over $(\tau_1,\tau_2) \in (0,1)^2$ can also be pursued as in e.g. Koenker and Xiao (2002), Chernozhukov and Fernández–Val (2005), and Angrist, Chernozhukov, and Fernández–Val (2006). I thank an anonymous referee for the references.

Proposition 2 *Let $k(\cdot)$ be a kernel such that*

(i) $sup_v|k(v)| < \infty$, $\int |k(v)|dv < \infty$, $\int k(v)^2 dv < \infty$, $\int k(v)dv = 1$, and $\int |k(v)||v|dv < \infty$.

(ii) $k(\cdot)$ is twice differentiable, and $\mathcal{S} = \{v \in \mathbb{R} : |k'(v)| > 0,\ k''(v) = 0\}$ is finite.

Suppose that $\epsilon_i(\tau_1, \tau_2)$ and $R_i(\tau_1)$ have a joint density conditional on S_i such that $\sup_{t,s} |\mathcal{D}_j f_{\epsilon(\tau_1,\tau_2),R(\tau_1)}(t, s|S_i)| \leq \phi(S_i)$ for some $\phi(S_i)$ with $E\big(||S_i||\phi(S_i)\big) < \infty$, where $\mathcal{D}_j$ denotes the derivative with respect to the j^{th} argument. Let $b_{1n} \downarrow 0$, $b_{2n} \downarrow 0$ such that if W_i has bounded support, $\sqrt{n}b_{1n} \to \infty$ and $\sqrt{nh_n}b_{2n} \to \infty$, otherwise $\sqrt{n}b_{1n}^2 \to \infty$ and $\sqrt{nh_n}b_{2n}^2 \to \infty$. Then,

$$\hat{V}_{22}(\tau_1, \tau_2) = \frac{1}{nb_{1n}} \sum_{i=1}^{n} S_i S_i' k\big(\frac{-\hat{R}_i(\tau_1)}{b_{1n}}\big)\tau_2(1-\tau_2) \xrightarrow{p} V_{22}(\tau_1, \tau_2),$$

$$\hat{\Gamma}_{22}(\tau_1, \tau_2) = \frac{1}{nb_{2n}^2} \sum_{i=1}^{n} S_i S_i' k\big(\frac{-\hat{R}_i(\tau_1)}{b_{2n}}\big) k\big(\frac{-\hat{\epsilon}_i(\tau_1, \tau_2)}{b_{2n}}\big) \xrightarrow{p} \Gamma_{22}(\tau_1, \tau_2).$$

Proof: See the appendix.

Existence of the joint conditional density of $\epsilon(\tau_1, \tau_2)$ and $R(\tau_1)$ given S requires that there is at least one cotinuous instrument with non–zero coefficient. This assumption is made for the convenience of the proof, but it seems possible to be relaxed as long as $R(\tau_1)$ has a marginal density. Condition (i) is standard in kernel estimation. Condition (ii) assumes smooth differentiable kernels, which is satisfied by most of commonly used kernels. When W_i does not have bounded support but the bandwidth choices b_{1n} and b_{2n} satisfy the stronger requirements, we do not need condition (ii). As $k(\cdot)$ has higher order derivatives, the requirements for b_{1n} and b_{2n} with unbounded support of W_i become closer to the requirements with bounded support of W_i. The requirement of $\sqrt{nh_n}b_{2n} \to \infty$ shows that estimation of $\Gamma(\tau_1, \tau_2)$ can be quite difficult in practice. Variance estimation of the proposed estimators need further investigation.

5 Experiments using the Angrist and Krueger Data

In this section, I conduct small experiments with the Angrist-Krueger data (see e.g., Angrist and Krueger (1991), Angrist, Imbens, and Krueger (1999)). Returns to schooling is a leading example that shows how useful triangular models are. One of the most interesting features of triangular

models is that it can provide a more complete picture about (random and heterogeneous) marginal effects by investigating two different quantile points. However, even when instruments are not relevant for all different distributional points, analysing a particular quantile of the endogenous variable is still possible; the method proposed in this paper requires that the instruments be relevant for a particular conditional quantile of the endogenous variable. This is a substantially weaker assumption than the traditional rank condition required in simultaneous equations models, where the instruments must affect the conditional mean of the endogenous variable. The Angrist-Krueger data is well-known to suffer from the issue of weak instruments while the sample size is extremely large; $n = 329,509$. The experiments of this section investigate the possibility of local relevance of those *weak* instruments.

One limitation of triangular models employed in this paper is that the assumption of strong monotonicity is required for point-identification (see e.g., Chesher (2003, 2005, 2007)). Since the education variable of the Angrist-Krueger data is observed as the years of schooling, it is a discrete variable, and it clearly does not satisfy the monotonicity assumption. However, since the purpose of this section is not in a rigorous empirical analysis but in investigating potential possibilities and providing an example, I simply pretend that the education variable is continuous by adding a small amount of random noise.[9]

I considered *contaminated* education variables which were obtained by adding a small amount of random noise generated from $N(0, \sigma^2)$ with $\sigma \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$ to the original education variable of the Angrist-Krueger data. I then ran quantile regression of the simulated education variables on 10 dummy variables indicating the years of birth and 30 instruments of the birth-quarters interacted with the birth-years.

Table I shows the p-values of the coefficients of those 30 instruments for various quantiles. Interestingly, those coefficients are quite significant for most quantiles, but not for e.g., 85% quantile. I tried different values of the standard deviations of the random noises, but the coefficients of those instruments for the 85% conditional quantile were insignificant at 5% levels in most cases. In addition, although it is not reported in table I, the Wald statistic testing all the 30 instruments are

[9] Florens, Heckman, Meghir, and Vytlacil (2008) also used the example of returns to education to motivate their analysis, where they also considered continuous endogenous treatments.

irrelevant was largest for the 20% conditional quantile in all five cases.[10]

The fact that the instruments are most relevant for $\tau_1 = 0.20$ needs some comments. Jun (2008) analyzed the Angrist–Krueger data within the framework of the instrumental quantile model (the IV quantile model hereafter) proposed by Chernozhukov and Hansen (2006). In particular, Jun (2008) cast the IV quantile model in a general GMM framework with a moment condition

$$E\Big([X'\ Z']'\big(1\{Y \leq D\alpha_c(\tau) + X'\beta_c(\tau)\} - \tau\big)\Big) = 0, \tag{13}$$

where Y is the log wage, D is the years of schooling, X is 10 dummies indicating the birth years, and Z contains the 30 instruments.[11] He then constructed 95% confidence intervals for $\alpha_c(\tau)$ with various choices of τ without assuming that they are identified. I here recall that his confidence intervals were extremely wide for small values of τ while those for upper quantiles were relatively tighter.[12] The fact that the instruments are weaker for lower quantiles in model (13) shows how different the IV quantile model is from the triangular model considered in this paper. The triangular model is a mutiple equations model, where different (conditional) quantiles of D and different (conditional) quantiles of Y are separately treated and they can be separately interpreted. However, the IV quantile model of Cherhozhukov and Hansen (2006) is a single GMM model, where quantiles of D are irrelevant but quantiles of (counterfactual) outcomes are compared. To be more precise, let Y_d be the counterfactual wage when the level of education is equal to d; the observed wage is $Y = Y_D$. Then, $\alpha_c(\tau)$ captures the difference between $Q_{Y_d|X=x,Z=z}(\tau)$ and $Q_{Y_{d+1}|X=x,Z=z}(\tau)$.[13] Note also that the IV quantile model (13) does not require that D be a continuous variable, because conditional quantiles of D are irrelevant for anaylses.

Figure I shows the point estimates of $\alpha(\tau_1, \tau_2)$ with $\sigma \in \{0.01,\ 0.02\}$. Although $\hat{\alpha}(\tau_1, \tau_2)$ would provide more complete information about the marginal effects if there were no issue of weak

[10] In fact, the Wald statistic divided by 30 was much larger than 10, which is often used for the rule–of–thumb criterion for weak instruments (see e.g. Stock, Wright, and Yogo (2002)).

[11] Understanding the IV quantile model in the GMM framework is a common view. For example, Chernozhukov and Hong (2003) considered the same moment condition and they studied the Laplace–type estimators that are defined by using the quasi–Bayesian posteriors.

[12] For example, the interval for $\alpha_c(0.20)$ was roughly between 0.075 and 0.625 while the interval for $\alpha_c(0.85)$ was between 0.025 and 0.125.

[13] Since quantile is not a linear operator, it is unclear how $\alpha_c(\tau)$ is translated to the triangular model of this paper.

instruments, it is clear that it is not the case; since the first–stage Wald tests for $\tau_1 = 0.85$ do not even reject the null of irrelevant instruments, there is a clear problem due to weak instruments even in these localized estimates. Note however that some values of τ_1 still strongly reject the null of irrelevant instruments in the first stage. Figure I also illustrates the point estimates of $\alpha(0.20, \tau_2)$ and their 95% confidence intervals, where $\tau_1 = 0.20$ is such that the first–stage Wald statistic is largest among other values of τ_1.

Although these are experimental results based on *contaminated* education, they provide another possibility that is comparable to Chernozhukov and Hansen (2006) and Jun (2008). Although the IV quantile model does allow a discrete endogenous variable, it is a single GMM model, and it is not straightforward to assess instruments. The (triangular) model of Chesher (2003) does not allow a discrete endogenous variable, but it provide further localized inference and assessing instruments is more straightforward; in this paper, I considered a semiparametric implementation of Chesher (2003). Several methods have been proposed in the literature to make triangular models more practical. However, they are either fully parametric or separable in control variables (see e.g. Ma and Koenker (2006) and Lee (2007)). The method proposed in this paper provides a simple yet sufficiently flexible way of utilizing triagular models.

6 Simulations: Local Instruments and Identification Pitfalls

In this section, I consider three different data generating processes for Monte Carlo experiments. In the followings, $N^{-1}(s : \mu, \sigma^2)$ denotes the inverse of the distribution function of $N(\mu, \sigma^2)$. The included exogenous variables, X_1 and X_2 are independently generated from $N(0, 1)$ and $N(2, 4)$, respectively for each design. $(\beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3)$ is set to be $(1, 1, 1, 1, 5)$. The parameter of interest is $\alpha(\tau_1, \tau_2)$, the marginal effect of D on Y for those whose ϵ and V are at τ_2 and τ_1, respectively. The true value for each design is given by $\alpha(\tau_1, \tau_2) = 1 - 3\tau_1 + \tau_1 + \exp(\tau_1)$. In the following experiments, $\tau_2 = 0.5$ is used as the various values of τ_1 are tried.

$$\textbf{DGP1} \begin{cases} Y = D(1 - 3V + V^2 + \exp(V)) + X_1\beta_1 + X_2\beta_2 + N^{-1}(\epsilon : 0, 1) \\ D = Z_1\pi_1 + Z_2\pi_2 + X_1\gamma_1 + X_2\gamma_2 + \gamma_3 + N^{-1}(V : 0, 1), \end{cases}$$

where $Z_1 \sim N(0, 1)$, $Z_2 \sim N(0, 1)$ and $(\pi_1, \pi_2) = (1, 1)$.

$$\textbf{DGP2} \begin{cases} Y = D(1 - 3V + V^2 + \exp(V)) + X_1\beta_1 + X_2\beta_2 + N^{-1}(\epsilon : 0, 1) \\ D = Z_1\pi + Z_2 N^{-1}(V : 1, 4) + X_1\gamma_1 + X_2\gamma_2 + \gamma_3 + N^{-1}(V : 0, 1), \end{cases}$$

where $Z_1 \sim N(0,1)$, $Z_2 \sim |N(0,1)|$ and $\pi = 0.01$.

$$\textbf{DGP3} \begin{cases} Y = D(1 - 3V + V^2 + \exp(V)) + X_1\beta_1 + X_2\beta_2 + N^{-1}(\epsilon : 0, 1) \\ D = Z_3 N^{-1}(V : 1, 4) + X_1\gamma_1 + X_2\gamma_2 + \gamma_3 + N^{-1}(V : 0, 1), \end{cases}$$

where $Z_3 \sim Bernoulli(\frac{1}{2})$.

DGP1 is the case where two instruments are globally valid with homoskedastic independent error. DGP2 has two instruments; one is *weak*, and the other is *local*. Since $N^{-1}(0.3 : 1, 4) \approx 0.048$, DGP2 suffers from *weak* identification when τ_1 is chosen to be around 30%. However, when τ_1 is far from 30%, Z_2 is a relevant instrument, and the parameter of interest is identified. Lastly, DGP3 is the case where there is one binary instrument that can be weak for a particular quantile.

For the sample size, I used $n \in \{3,000,\ 5,000,\ 8,000\}$. For the bandwidth, $h_n = IQR \cdot n^{-\frac{1}{3}}$ was used except for Figures III-1 and III-2, where IQR is the interquantile range of the first–stage quantile regression residuals. For Monte Carlo, $1,000$ replications were made.

Figures II-1 and II-2 show the Monte Carlo results from DGP1 and DGP2, respectively; the reulsts from DGP3 are not reported, because they were similar to those of DGP2. For DGP1, RMSEs are small for all the quantile effects. DGP2 is more interesting. Since the instruments are only local in the sense that they are relevant for some quantiles but not for others, RMSEs remain big when τ_1 is close to 30%. In spite of the identification difficulty around $\tau_1 = 30\%$, the instruments are still informative for other quantile effects and RMSEs are reasonably small.

Figures III-1 and III-2 show the sensitivity of RMSEs to different bandwidth choices. Using $n = 5,000$ and $h_n = \kappa \cdot n^{-1/3}$, $\kappa \in \{0.5, 1.0, 1.5, \cdots, 4.0, 4.5, 5.0, 8.0, 10.0, 15.0\}$ were considered, which led to $h_n \in [0.03,\ 0.88]$. As the figures show, RMSEs are quite stable in all cases within wide ranges of bandwidth.

Figure IV illustrates the rejection rates of the t-statistics testing the truth with the nominal size 5% and $n = 8000$. To estimate the standard errors, the crude kernel estimators suggested in proposition 2 were used using the standard normal density as a kernel. In contrast to the other quantile effects, the rejection rates around $\tau_1 = 30\%$ in DGP2, DGP3 are far from 5%.

The size distortions of DGP2 and DGP3 around $\tau_1 = 30\%$ are not surprising in view of the identification difficulty: the instruments are not informative for the quantile effects around $\tau_1 = 30\%$. This is interesting because it suggests that individual heterogeneity can cause instruments to be only locally relevant and yet the local instruments can still be exploited by focusing on a particular local effect. To be more concrete, the densities of the t-statistics are estimated from Monte Carlo and some of them are shown in Figure V. Not surprisingly, when $\tau_1 = 30\%$ in DGP2 and DGP3, the densities of the t-statistics are far from the normal density.

Lastly, Figure VI shows examples of 95% confidence intervals computed by the normal approximation. I simply generated three sets of artificial data from DGP1, DGP2, and DGP3. Then, 95% condidence intervals were computed for various values of τ_1 with fixing $\tau_2 = 0.5$. Again, the confidence bands from DGP2, DGP3 do not cover the truth when τ_1 is around 30%. However, they do cover the truth when τ_1 is far from 30%.

7 Concluding Remarks

Although I focused on the local parameter $\alpha(\tau_1, \tau_2)$, the integrated quantile effect (or the average quantile effect: $\int_0^1 \alpha(v, \tau_2)dv$) is also interesting to consider as long as the instruments are relevant for all the conditional quantiles of the endogenous variable. It can be used as a summary of all the local parameters, and it seems to be possible to recover the regular $\sqrt{n}$ convergence rate.[14] I leave this possiblity for the future research.

There are several limitations in the approach taken in this paper. First, the monotonicity assumption requires that the endogenous variable be a continuous random variable. This can be restrictive in practice. Chesher (2005) discusses a set identification result with a discrete endogenous variable, which is worth further study. Second, the estimation of the variance can be tough in practice. Comparing several methods of variance estimation such as kernel, k–nearest–neighbors, and the bootstrap is worth studying. Testing the location of the local parameter without assuming its identification is another interesting question. Those questions are also left for the future research.

[14] In standard linear quantile regression models, integrating regression quantiles to obtain a location parameter has been considered by e.g. Portnoy and Koenker (1989).

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A Proof of Proposition 1

Let $\Pi_0 = [\Pi_1(\tau_1)' : \Pi_2(\tau_1)']'$. First, note that $Q_{D|W}(\tau_1) = W'\Pi_0$. Multiplying W to both sides and taking expectations yields $E(WW')\Pi_0 = E\big(WQ_{D|W}(\tau_1)\big)$. Therefore, Π_0 is identified by $E(WW')^{-1}E\big(WQ_{D|W}(\tau_1)\big)$. Now, note that

$$\begin{aligned}
&\Pr\big(Y \le Q_{D|W}(\tau_1)\alpha(\tau_1,\tau_2) + X'\beta(\tau_1,\tau_2)\big|D = Q_{D|W}(\tau_1), W\big) \\
&\qquad = \Pr\big(Y \le D\alpha(\tau_1,\tau_2) + X'\beta(\tau_1,\tau_2)\big|D = Q_{D|W}(\tau_1), W\big) \\
&= \Pr\big(Y \le D\alpha(V,\tau_2) + X'\beta(V,\tau_2)\big|V = \tau_1, W\big) = \Pr(\epsilon \le \tau_2\big|V = \tau_1, W) = \Pr(\epsilon \le \tau_2) = \tau_2,
\end{aligned}$$

because $D = Q_{D|W=w}(\tau_1), W = w$ is equivalent to $V = \tau_1, W = w$ due to the strong monotonicity. Note here that independence between V and ϵ is used only for the last equality, which local independence is in fact sufficient for. Now, note that

$$\begin{aligned}
Q_{Y|D=Q_{D|W}(\tau_1),W}(\tau_2) &= Q_{D|W}(\tau_1)\alpha(\tau_1,\tau_2) + X'\beta(\tau_1,\tau_2) \\
&= X'\big(\Pi_1(\tau_1)\alpha(\tau_1,\tau_2) + \beta(\tau_1,\tau_2)\big) + Z'\Pi_2(\tau_1)\alpha(\tau_1,\tau_2).
\end{aligned}$$

Multiplying W to both sides and taking expections shows that
$[\Pi_1(\tau_1)'\alpha(\tau_1,\tau_2) + \beta(\tau_1,\tau_2)',\ \Pi_2(\tau_1)'\alpha(\tau_1,\tau_2)]'$ is identified by $E(WW')^{-1}E\big(WQ_{Y|D=Q_{D|W}(\tau_1),W}(\tau_2)\big)$. Therefore, **A3** results in identification of $\alpha(\tau_1,\tau_2)$ and $\beta(\tau_1,\tau_2)$. □

B Proof of Lemma 1

Note that

$$\Upsilon_{n,h_n}\Big(E\big(M_n(\Pi,\theta)\big) - M(\Pi,\theta)\Big) = \begin{bmatrix} 0 \\ B_n(\Pi,\theta) \end{bmatrix},$$

where

$$B_n(\Pi,\theta) = \sqrt{nh_n}\Big(\frac{1}{h_n}E\big(S_i(1\{Y_i \le S_i\theta\} - \tau_2)1\{|D_i - W_i\Pi| \le \frac{h_n}{2}\}\big) \\ - E\big(S_i(1\{Y_i \le S_i\theta\} - \tau_2)|R_i(\tau_1) - W_i'\delta = 0\big)f_{R_i(\tau_1)-W'\delta}(0)\Big)$$

for $\delta = \Pi - \Pi_0 \in \mathcal{N}$. Let $\varphi(\theta,\delta,r) = E\big(S_i(1\{Y_i \le S_i\theta\} - \tau_2)|R_i(\tau_1) - W_i'\delta = r\big)$, and it follows from the law of iterated expectations and the change–of–variables that

$$B_n(\Pi,\theta) = \sqrt{nh_n}\int_{-1/2}^{1/2} \varphi(\theta,\delta,th_n)f_{R(\tau_1)-W'\delta}(th_n) - \varphi(\theta,\delta,0)f_{R(\tau_1)-W'\delta}(0)dt \\ = \frac{\sqrt{nh_n}h_n^2}{2}\int_{-1/2}^{1/2} \frac{\partial^2\varphi(\theta,\delta,s)f_{R(\tau_1)-W'\delta}(s)}{\partial s^2}\Big|_{s=\bar{s}} t^2dt,$$

where $\bar{s}$ is between 0 and th_n. Therefore, for every $\theta \in \Theta$ and $\Pi \in \Pi_0 + \mathcal{N}$,

$$||B_n(\Pi,\theta)|| \le \sqrt{nh_n}h_n^2\Big(\sup_{-1/2\le r\le 1/2}||\frac{\partial^2\varphi(\theta,\delta,r)}{\partial r^2}||b_0(r) + 2\sup_{-1/2\le r\le 1/2}||\frac{\partial\varphi(\theta,\delta,r)}{\partial r}||b_1(r) \\ + \sup_{-1/2\le r\le 1/2}||\varphi(\theta,\delta,r)||b_2(r)\Big)\int_{-1/2}^{1/2} t^2dt. \quad (14)$$

It then follows that $\sup_{\Pi\in\Pi_0+\mathcal{N},\ \theta\in\Theta}||B_n(\Pi,\theta)|| = O(\sqrt{nh_n}h_n^2)$ from compactness of $\Theta\times\mathcal{N}$ and continuity of the right–hand side of (14). □

C Proof of Theorem 1

In this section, I provide the proof of theorem 1. For the sake of convenience, I splitted the theorem into the consistency part (lemmas 2 and 8) and the normality part (proposition 3). In the following discussion, I will write $M_1(\Pi)$ and $M_2(\Pi,\theta)$ to denote the partitions of $M(\Pi,\theta)$; their sample analogs will be denoted by $M_{1n}(\Pi)$ and $M_{2n}(\Pi,\theta)$, respectively.

Lemma 2 $\sqrt{n}||\hat{\Pi}(\tau_1) - \Pi(\tau_1)|| = O_p(1)$.

Proof: It follows from the fact that $\hat{\Pi}(\tau_1)$ is the standard linear quantile regression estimator. See e.g. Koenker (2005, p 120). □

Lemma 3

$$\frac{1}{nh_n}\sum_{i=1}^{n}|1\{|D_i - W_i\hat{\Pi}| \leq \frac{h_n}{2}\} - 1\{|D_i - W_i\Pi_0| \leq \frac{h_n}{2}\}|||S_i|| = o_p(1).$$

Proof: Note that

$$\begin{aligned}
&|1\{|R_i(\tau_1) - W_i'(\hat{\Pi} - \Pi_0)| \leq \frac{h_n}{2}\} - 1\{|R_i(\tau_1)| \leq \frac{h_n}{2}\}| \\
&\quad \leq 1\{-\frac{h_n}{2} \leq R_i(\tau_1) \leq ||W_i||||\hat{\Pi} - \Pi_0|| - \frac{h_n}{2}\} + 1\{\frac{h_n}{2} \leq R_i(\tau_1) \leq ||W_i||||\hat{\Pi} - \Pi_0|| + \frac{h_n}{2}\} \\
&\quad + 1\{-\frac{h_n}{2} - ||W_i||||\hat{\Pi} - \Pi_0|| \leq R_i(\tau_1) \leq -\frac{h_n}{2}\} + 1\{\frac{h_n}{2} - ||W_i||||\hat{\Pi} - \Pi_0|| \leq R_i(\tau_1) \leq \frac{h_n}{2}\}
\end{aligned}$$

for sufficiently large n, because $|W_i'(\hat{\Pi} - \Pi_0)| = o_p(h_n)$. Since the four terms are similar, I only consider the last one. Let δ_n be a (positive) sequence such that $||\hat{\Pi} - \Pi_0|| = o_p(\delta_n)$ and $\delta_n = o(h_n)$. It then follows that $|W_i||||\hat{\Pi} - \Pi_0|| \leq ||W_i||\delta_n$ with probability approaching to 1. I will show that

$$E\big(\frac{1}{h_n}1\{\frac{h_n}{2} - ||W_i||\delta_n \leq R_i(\tau_1) \leq \frac{h_n}{2}\}||S_i||\big) = o(1).$$

Note first that

$$\begin{aligned}
&E\big(\frac{1}{h_n}1\{\frac{h_n}{2} - ||W_i||\delta_n \leq R_i(\tau_1) \leq \frac{h_n}{2}\}||S_i||\big) \\
&\leq E\big(\frac{1}{h_n}1\{\frac{h_n}{2} - ||W_i||\delta_n \leq R_i(\tau_1) \leq \frac{h_n}{2}\}|D_i|\big) + E\big(\frac{1}{h_n}1\{\frac{h_n}{2} - ||W_i||\delta_n \leq R_i(\tau_1) \leq \frac{h_n}{2}\}||X_i||\big).
\end{aligned}$$

Since the second term is similar, I only consider the first term. Note that

$$\begin{aligned}
E\big(\frac{1}{h_n}1\{\frac{h_n}{2} - ||W_i||\delta_n \leq R_i(\tau_1) \leq \frac{h_n}{2}\}|D_i||W_i\big) &= \int_{\frac{1}{2}-\frac{\delta_n}{h_n}}^{\frac{1}{2}} |th_n + W_i'\Pi_0| f_{R(\tau_1)|W}(th_n|W_i)dt \\
&\leq h_n \int_{\frac{1}{2}-\frac{\delta_n}{h_n}}^{\frac{1}{2}} |t| f_{R(\tau_1)|W}(th_n|W_i)dt + |W_i'\Pi_0| \int_{\frac{1}{2}-\frac{\delta_n}{h_n}}^{\frac{1}{2}} f_{R(\tau_1)|W}(th_n|W_i)dt \to 0
\end{aligned}$$

with probability one. Applying dominated convergence theorem completes the proof. □

Lemma 4 *Let* $\tilde{m}_n(\Pi,\theta) = c'S(1\{Y \le S'\theta\} - \tau_2)1\{|D - W'\Pi| \le \frac{h_n}{2}\}\frac{1}{\sqrt{h_n}}$, *where* c *is an arbitrary conformable vector and* $(\Pi,\theta) \in (\Pi_0 + \mathcal{N}) \times \Theta$. *Then, for any sequence* $\delta_{1n} = o(h_n)$ *and* $\delta_{2n} = o(1)$, $\sup_{||\Pi-\tilde{\Pi}||<\delta_{1n}, ||\theta-\tilde{\theta}||<\delta_{2n}} E\big((\tilde{m}_n(\Pi,\theta) - \tilde{m}_n(\tilde{\Pi},\tilde{\theta}))^2\big) = o(1)$.

Proof: Note that

$$
\begin{aligned}
&(\tilde{m}_n(\Pi,\theta) - \tilde{m}_n(\tilde{\Pi},\tilde{\theta}))^2 \\
&\qquad \le 2\Big((c'S)^2\big|1\{|D - W'\Pi| \le \frac{h_n}{2}\} - 1\{|D - W'\tilde{\Pi}| \le \frac{h_n}{2}\}\big|\frac{1}{h_n} \\
&\qquad\qquad + (c'S)^2\big|1\{Y \le S'\theta\} - 1\{Y \le S'\tilde{\theta}\}\big|1\{|D - W'\tilde{\Pi}| \le \frac{h_n}{2}\}\frac{1}{h_n}\Big). \quad (15)
\end{aligned}
$$

Since $|W'(\Pi - \tilde{\Pi})| = o_p(h_n)$, the first term in the right–hand side of (15) is bounded by

$$
\begin{aligned}
&(c'S)^2\Big(1\{W'\Pi - \frac{h_n}{2} \le D \le W'\tilde{\Pi} - \frac{h_n}{2}\} + 1\{W'\Pi + \frac{h_n}{2} \le D \le W'\tilde{\Pi} + \frac{h_n}{2}\} \\
&\qquad + 1\{W'\tilde{\Pi} - \frac{h_n}{2} \le D \le W'\Pi - \frac{h_n}{2}\} + 1\{W'\tilde{\Pi} + \frac{h_n}{2} \le D \le W'\Pi + \frac{h_n}{2}\}\Big)\frac{1}{h_n}.
\end{aligned}
$$

Since all four terms are similar, I only consider the last one. Taking the expectation of the last term yields

$$
\begin{aligned}
&E\Big((c_1D + c_2'X)^2 1\{W'\tilde{\Pi} + \frac{h_n}{2} \le D \le W'\Pi + \frac{h_n}{2}\}\frac{1}{h_n}\Big) \\
&\qquad = E\Big(\int_{W'\tilde{\Pi}/h_n + \frac{1}{2}}^{W'\Pi/h_n + \frac{1}{2}} (c_1 t h_n + c_2'X)^2 f_{D|W}(t h_n|W)dt\Big) \le C(\Pi,\tilde{\Pi})\frac{||\Pi - \tilde{\Pi}||}{h_n},
\end{aligned}
$$

where $C(\Pi,\tilde{\Pi}) = E\Big(\sup(c_1 s + c_2'X)^2 f_{D|W}(s|W)||W||\Big)$ with sup taken over s between $W'\tilde{\Pi} + \frac{1}{2}$ and $W'\Pi + \frac{1}{2}$. Since $\sup_{(\Pi,\tilde{\Pi})\in\mathcal{N}\times\mathcal{N}} C(\Pi,\tilde{\Pi}) < \infty$, it follows that

$$
\sup_{||\Pi-\tilde{\Pi}||<\delta_{1n}} C(\Pi,\tilde{\Pi})\frac{||\Pi - \tilde{\Pi}||}{h_n} = O(\frac{\delta_{1n}}{h_n}) = o(1).
$$

The second term of the right–hand side of (15) is similar and omitted. □

Lemma 5 *For any sequence $h_n > 0$ with $h_n \downarrow 0, nh_n \to \infty$,*

(i) $\sup_{\theta\in\Theta} ||M_{2n}(\hat{\Pi}, \theta) - M_2(\Pi_0, \theta)|| = o_p(1)$ *as* $n \to \infty$.

Moreover, if we let let $r_n = \sqrt{n}h_n^{\delta}$ *such that* $\delta > 0$ *and* $\sqrt{n}h_n^{1+\delta} \to \infty$, *then for any sequences* $\delta_{1n} \downarrow 0$ *and* $\delta_{2n} \downarrow 0$ *such that* $r_n\delta_{1n} \downarrow 0$, *we have*

(ii) $\sup ||\sqrt{nh_n}\Big(M_{2n}(\Pi, \theta) - E\big(M_{2n}(\Pi, \theta)\big)\Big) - \sqrt{nh_n}\Big(M_{2n}(\Pi_0, \theta_0) - E\big(M_{2n}(\Pi_0, \theta_0)\big)\Big)|| = o_p(1)$

as $n \to \infty$, *where* sup *is taken over* $||\Pi - \Pi_0|| \leq \delta_{1n}$ *and* $||\theta - \theta_0|| \leq \delta_{2n}$.

Proof: Note first that

$$\sup_{\theta} ||M_{2n}(\hat{\Pi}, \theta) - M_{2n}(\Pi_0, \theta)|| \leq \frac{1}{nh_n}\sum_{i=1}^{n} ||S_i|||1\{|D_i - W_i'\hat{\Pi}| \leq \frac{h_n}{2}\} - 1\{|D_i - W_i'\Pi_0| \leq \frac{h_n}{2}\}| = o_p(1)$$

by lemma 3. For part (i), I note that

$$\sup_{\theta} ||M_{2n}(\Pi_0, \theta) - M_2(\Pi_0, \theta)|| \leq \sup_{(\pi,\theta)\in\mathcal{N}\times\Theta} ||M_{2n}(\Pi_0 + \frac{\pi}{r_n}, \theta) - M_2(\Pi_0 + \frac{\pi}{r_n}, \theta)||$$

and I will show that the right–hand side is $o_p(1)$ in the following discussion.

Consider the following class of functions

$$\mathcal{F}_n = \{m_n(\pi, \theta) = c'S(1\{Y \leq S'\theta\} - \tau_2)1\{|R(\tau_1) - W'\frac{\pi}{r_n}| \leq \frac{h_n}{2}\}\frac{1}{\sqrt{h_n}} : (Y, D, X, Z) \times (\pi, \theta) \mapsto \mathbb{R}\},$$

where c is an arbitrary conformable vector and $(\pi, \theta) \in \mathcal{N} \times \Theta$. Note that working with the local parameter π is sufficient, because $\hat{\Pi}$ is already $\sqrt{n}$–consistent for Π_0.

Let $\mu_n(\pi, \theta) = \frac{1}{nh_n}\sum_i^n c'S_i(1\{Y_i \leq S_i'\theta\} - \tau_2)1\{|R_i(\tau_1) - W_i'\frac{\pi}{r_n}| \leq \frac{h_n}{2}\}$ and consider

$$\mathbb{G}_n m_n = \sqrt{n}(\mathbb{P}_n - P)m_n = \sqrt{nh_n}\Big(\mu_n(\pi, \theta) - E\big(\mu_n(\pi, \theta)\big)\Big),$$

which is an empirical process with a sequential class of functions (see e.g., van der Vaart and Wellner (1996, p220-221), van der Vaart (1998, p282)). In particular, corollary 19.35 and theorem

19.28 of van der Vaart (1998) provide sufficient conditions for uniform convergence and stochastic equicontinuity of $\mathcal{F}_n$. Define $\pi_n^*(D,W) = \arg\min_{\pi\in\mathcal{N}} |R(\tau_1) - W'\frac{\pi}{r_n}|$, and $\mathcal{F}_n$ has an envelope function $F_n = 2c'S1\{|R(\tau_1) - W'\frac{\pi_n^*(D,W)}{r_n}| \leq \frac{h_n}{2}\}\frac{1}{\sqrt{h_n}}$. Since $R(\tau_1) - W'\frac{\pi_n^*(D,W)}{r_n}$ has a density, the standard change-of-variable technique shows

$$E(F_n^2) = O(1)$$
$$E(F_n^2 1\{F_n > \epsilon\sqrt{n}\}) = o(1) \text{ for every } \epsilon > 0,$$

where the second equality is because $nh_n \to \infty$. Therefore, in view of lemma 4[15], the uniform entropy conditions of theorem 19.28 of van der Vaart (1998) will imply stochastic equicontinuity of $\mathbb{G}_n m_n$ and the uniform convergence of part (i) will follow from the maximal inequality.

To be more specific, let $J(\delta, \mathcal{F}_n, L_2)$ be the uniform entropy integral of $\mathcal{F}_n$. Since corollary 19.35 of van der Vaart (1998)[16] shows that

$$E(\sup_{\pi,\theta} |\mathbb{G}_n m_n|) \leq CJ(||F_n||_2, \mathcal{F}_n, L_2)$$

for some constant C, $J(||F_n||_2, \mathcal{F}_n, L_2) = O(1)$ will imply that $\sup_{(\pi,\theta)} |\mathbb{G}_n m_n| = o_p(\sqrt{nh_n})$, which will prove part (i) by lemma 1. Also, by theorem 19.28 of van der Vaart (1998), $J(\delta_n, \mathcal{F}_n, L_2) = o(1)$ for every $\delta_n \downarrow 0$ will show stochastic equicontinuity of $\mathbb{G}_n m_n(\pi,\theta)$ so that

$$\sup_{||(\pi,\theta)-(0,\theta_0)||\leq\delta_n} |\mathbb{G}_n m_n(\pi,\theta) - \mathbb{G}_n m_n(0,\theta_0)| = o_p(1)$$

for any $\delta_n \downarrow 0$. It will then prove part (ii), because setting $r_n(\Pi - \Pi_0) = \pi$ shows that the supremum over $||\Pi - \Pi_0|| \leq \delta_{1n}$ is bounded by the supremum over $||\pi|| \leq r_n\delta_{1n}$.

Therefore, showing $J(||F_n||_2, \mathcal{F}_n, L_2) = O(1)$ and $J(\delta_n, \mathcal{F}_n, L_2) = o(1)$ for every $\delta_n \downarrow 0$ will complete the proof of the lemma. By definition of the uniform entropy integral, these two conditions are satisfied when the ϵ–uniform covering number $\sup_Q N(\epsilon||F_n||_2, \mathcal{F}_n, L_2(Q))$ is bounded by a

[15] Note that $r_n h_n \to \infty$.

[16] See also the comments on p289 of van der Vaart (1998). It is worth noting that the maximal inequality is not an asymptotic one.

polynomial of $\frac{1}{\epsilon}$ that does not depend on n.[17] A sufficient condition for this is that $\mathcal{F}_n$ is a Vapnik–Cervonenkis (VC) class with its VC index not depending on n (see e.g. theorem 2.6.7 of van der Vaart and Wellner (1996, hereafter VW)).

In the following four steps, I will show that the VC index of $\mathcal{F}_n$ is finite and that it does not depend on n. VC indices will be denoted by $\mathcal{V}(\cdot)$. First, I state a few useful properties of VC indices, which are used in the following steps.

VC Properties: Let $\mathcal{F}$ and $\mathcal{G}$ be VC classes of sets. Let $\mathcal{H}$ be a VC class of functions, and let g be an arbitraray fixed function. Then,

(i) $\mathcal{I} = \{1_A(x) : A \in \mathcal{F}\}$ is a VC class of functions with $\mathcal{V}(\mathcal{I}) = \mathcal{V}(\mathcal{F})$.

(ii) $\mathcal{F}^c = \{A^c : A \in \mathcal{F}\}$ is a VC class of sets with $\mathcal{V}(\mathcal{F}^c) = \mathcal{V}(\mathcal{F})$.

(iii) $\mathcal{F} \sqcap \mathcal{G} = \{A \cap B : A \in \mathcal{F}, B \in \mathcal{G}\}$ is a VC class of sets with $\mathcal{V}(\mathcal{F} \sqcap \mathcal{G}) \leq \mathcal{V}(\mathcal{F}) + \mathcal{V}(\mathcal{G}) - 1$.

(iv) The collection of all half–spaces in $\mathbb{R}^d$ is a VC class of sets of index $d + 2$. In particular, $\{1\{x'b \leq c\} : b \in \mathbb{R}^d,\ c \in \mathbb{R}\}$ is a VC class of functions of index $d + 2$.

(v) $\mathcal{H} \cdot g = \{f \cdot g : f \in \mathcal{H}\}$ is a VC class of functions with $\mathcal{V}(\mathcal{H} \cdot g) \leq 2\mathcal{V}(\mathcal{H}) - 1$.

These VC properties are now standard in empirical process theory, and their proofs can be found in many places. See e.g. lemmas 2.6.17 and 2.6.18 of VW, and exercise 14 on page 152 of VW. Using these properties, the following four steps prove that the VC index of $\mathcal{F}_n$ is finite and that it does not depend on n.

Step 1: Let $\mathcal{C} = \{f_{A,B}(x) = 1_{A\cap B}(x) - \tau_1 1_B(x) : A \in \mathcal{F}, B \in \mathcal{G}\}$, where $\mathcal{F}$ and $\mathcal{G}$ are VC classes of sets. Then, $\mathcal{V}(\mathcal{C}) \leq 2\big(\mathcal{V}(\mathcal{F}) + 2\mathcal{V}(\mathcal{G}) - 2\big)$.

To prove this, suppose that a collection $\{(x_1, t_1), (x_2, t_2), ..., (x_k, t_k)\}$ is shattered by the collection of subgraphs of $\mathcal{C}$. Then, there should not be any t_j less than $-\tau_1$ or larger than $1 - \tau_1$. Moreover, this collection can be partitioned into two groups. Those points whose t_j is between $1-\tau_1$ and 0 will be shattered by the collection of subgraphs of $\mathcal{C}_1 = \{(1 - \tau_1)1_{A\cap B}(x) : A \in \mathcal{F}, B \in \mathcal{G}\}$. Similarly, those points whose t_j is between 0 and $-\tau_1$ will be shattered by the collection of subgraphs of $\mathcal{C}_2 = \{-\tau_1 1_{B\cap(A\cap B)^c}(x) : A \in \mathcal{F}, B \in \mathcal{G}\}$. It then follows that $\mathcal{V}(\mathcal{C}) \leq \mathcal{V}(\mathcal{C}_1) + \mathcal{V}(\mathcal{C}_2)$. Now, the conclusion of step 1 follows from the VC properties (i), (iii), and (v) above.

[17] Since the ϵ–uniform covering number cannot increase as ϵ becomes larger, it is in fact sufficient to consider $0 < \epsilon < 1$.

Step 2: Let $\mathcal{D}_n = \{1\{|R(\tau_1) - W'\frac{\pi}{r_n}| \leq \frac{h_n}{2}\} : \pi \in \mathbb{R}^{k_x+k_z}\}$, $\mathcal{D}^- = \{1\{R(\tau_1) \leq W'\Pi + h\} : \Pi \in \mathbb{R}^{k_x+k_z},\ h \in \mathbb{R}\}$, and $\mathcal{D}^+ = \{1\{R(\tau_1) \geq W'\Pi - h\} : \Pi \in \mathbb{R}^{k_x+k_z},\ h \in \mathbb{R}\}$. Then, for every n, $\mathcal{V}(\mathcal{D}_n) \leq \mathcal{V}(\mathcal{D}^-) + \mathcal{V}(\mathcal{D}^+) - 1$.

Note first that $\mathcal{D}_n = \{1\{|R(\tau_1) - W'\frac{\pi}{r_n}| \leq \frac{h_n}{2}\} : \pi \in \mathbb{R}^{k_x+k_z}\}$ is a subclass of $\mathcal{D}^* = \{1\{|R(\tau_1) - W'\Pi| \leq h\} : \Pi \in \mathbb{R}^{k_x+k_z},\ h \in \mathbb{R}\}$ for every n. Note also that $\mathcal{D}^-$ and $\mathcal{D}^+$ are VC classes by the VC property (iv) above. It then follows that $\mathcal{V}(\mathcal{D}_n) \leq \mathcal{V}(\mathcal{D}^*) \leq \mathcal{V}(\mathcal{D}^-) + \mathcal{V}(\mathcal{D}^+) - 1$, where the second inequality is due to the VC property (iii) above.

Step 3: Let $\mathcal{F}_n^* = \{(1\{Y \leq S'\theta\} - \tau_1)1\{|R(\tau_1) - W'\frac{\pi}{r_n}| \leq \frac{h_n}{2}\} : \pi \in \mathbb{R}^{k_z+k_x}, \theta \in \mathbb{R}^{1+k_x}\}$, and $\mathcal{E} = \{(1\{Y \leq S'\theta\} - \tau_1) : \theta \in \mathbb{R}^{1+k_x}\}$. Then, $\mathcal{V}(\mathcal{F}_n^*) \leq 2\Big(\mathcal{V}(\mathcal{E}) + 2\mathcal{V}(\mathcal{D}^-) + 2\mathcal{V}(\mathcal{D}^+) - 4\Big)$ for every n.

It follows from steps 1 and 2, because $(1_A - \tau_1)1_B = 1_{A\cap B} - \tau_1 1_B$.

Step 4: $\mathcal{F}_n$ is a VC class, and its VC index does not depend on n.

By the VC property (v) above, $\mathcal{V}(\mathcal{F}_n) \leq 2\mathcal{V}(\mathcal{F}_n^*) - 1$. Therefore, it follows from step 3 that $\mathcal{V}(\mathcal{F}_n) \leq 4\Big(\mathcal{V}(\mathcal{E}) + 2\mathcal{V}(\mathcal{D}^-) + 2\mathcal{V}(\mathcal{D}^+) - 4\Big) - 1$ for all n. Lastly, note that $\mathcal{E}, \mathcal{D}^-, \mathcal{D}^+$ are all VC classes and they do not depend on n. □

Lemma 6 *As $n \to \infty$, $h_n \downarrow 0$, $nh_n \to \infty$,*

$$\Upsilon_{n,h_n}\Big(M_n(\Pi_0, \theta_0) - E\big(M_n(\Pi_0, \theta_0)\big)\Big) \xrightarrow{d} N\Big(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}\Big),$$

where $V_{11} = E(W_iW_i'|R_i(\tau_1) = 0)f_{R(\tau_1)}(0)$ and $V_{22} = \tau_2(1-\tau_2)E(S_iS_i'|R_i(\tau_1) = 0)f_{R(\tau_1)}(0)$.

Remark: Since $R(\tau_1)$ has a conditional density given W, $V_{11} = E\big(W_iW_i'f_{R(\tau_1)|W}(0|W_i)\big)$, which is a more common expression in the quantile literature (e.g. Koenker (2005)). If $R(\tau_1)$ has a density conditional on S, which requires that there is at least one continuous instrument, then $V_{22} = \tau_2(1-\tau_2)E\big(S_iS_i'f_{R(\tau_1)|S}(0|S_i)\big)$.

Proof: For any (conformable) vectors c, d, consider

$$S_n = \frac{[c' : d']\,\Upsilon_{n,h_n}\Big(M_n(\Pi_0,\theta_0) - E\big(M_n(\Pi_0,\theta_0)\big)\Big)}{\sqrt{Var\bigg([c' : d']\,\Upsilon_{n,h_n}\Big(M_n(\Pi_0,\theta_0) - E\big(M_n(\Pi_0,\theta_0)\big)\Big)\bigg)}}.$$

By the Cramer-Wold device, it suffices to show that

$$S_n \quad \xrightarrow{d} N(0,1) \tag{16}$$

$$Var\bigg(\quad \Upsilon_{n,h_n}\Big(M_n(\Pi_0,\theta_0) - E\big(M_n(\Pi_0,\theta_0)\big)\Big)\bigg) \to \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}. \tag{17}$$

Part (17) follows from direct computation, and will be omitted here.

For part (16), write

$$N_n = \big[c' : d'\big]\,\Upsilon_{n,h_n}\Big(M_n(\Pi_0,\theta_0) - E\big(M_n(\Pi_0,\theta_0)\big)\Big) = \sum_{i=1}^{n} \frac{c'W_i m_{1i}}{\sqrt{n}} + \frac{d'S_i m_{2i} k_{in}}{\sqrt{nh_n}},$$

where $m_{1i} = 1\{D_i \le W_i'\Pi_0\} - \tau_1$, $m_{2i} = 1\{Y_i \le S_i'\theta_0\} - \tau_2$, and $k_{in} = 1\{|D_i - W_i\Pi_0| \le \frac{h_n}{2}\}$. By the Liapounov central limit theorem (see e.g., Pagan and Ullah (1999, p 358)), it suffices to shows that

$$\sum_{i}^{n} E\big(\big|\frac{c'W_i m_{1i}}{s_n\sqrt{n}} + \frac{d'S_i m_{2i} k_{in}}{s_n\sqrt{nh_n}}\big|^{2+\nu}\big) = o(1),$$

for some $\nu > 0$, where $s_n^2 = Var(N_n)$. By the C_r inequality,[18]

$$\begin{aligned}\sum_{i}^{n} E\big(\big|\frac{c'W_i m_{1i}}{s_n\sqrt{n}} + \frac{d'S_i m_{2i} k_{in}}{s_n\sqrt{nh_n}}\big|^{2+\nu}\big) &\le C\sum_{i}^{n} E\big(\big|\frac{c'W_i m_{1i}}{s_n\sqrt{n}}\big|^{2+\nu} + \big|\frac{d'S_i m_{2i} k_{in}}{s_n\sqrt{nh_n}}\big|^{2+\nu}\big) \\ &\le \frac{C}{(ns_n^2)^{1+\nu/2}}\sum_{i}^{n} E\big(\big|c'W_i m_{1i}\big|^{2+\nu}\big) + \frac{C}{(nh_n s_n^2)^{1+\nu/2}}\sum_{i}^{n} E\big(\big|d'S_i m_{2i} k_{in}\big|^{2+\nu}\big),\end{aligned}$$

[18] See e.g. Davidson (1994, p 140).

where C is a constant. In view of part (17), s_n^2 is convergent to a positive number, and hence it futher follows that

$$\frac{C}{(ns_n^2)^{1+\nu/2}}\sum_i^n E\big(|c'W_i m_{1i}|^{2+\nu}\big) + \frac{C}{(nh_n s_n^2)^{1+\nu/2}}\sum_i^n E\big(|d'S_i m_{2i} k_{in}|^{2+\nu}\big)$$
$$\leq \frac{1}{n^{1+\nu/2}}O(n) + \frac{1}{(nh_n)^{1+\nu/2}}O(nh_n) = o(1),$$

where ν is chosen to satisfy assumption H. □

Lemma 7 *Let the Jacobian of $M(\Pi,\theta)$ evaluated at Π_0 and θ_0 be given by $\Gamma = \begin{bmatrix} \Gamma_{11} & 0 \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$. Then, $\Gamma_{11} = E\big(W_i W_i' f_{R(\tau_1)|W}(0|W_i)\big)$, and $\Gamma_{22} = E\big(S_i S_i' f_{\epsilon(\tau_1,\tau_2)|R(\tau_1),S}(0|R_i(\tau_1),S_i)|R_i(\tau_1) = 0\big) f_{R(\tau_1)}(0)$.*

Proof: Γ_{11} and Γ_{22} follows from direct calculation, and it will be omitted here. □

Remark: Let $T(D_i,W_i) = E\big(S_i(1\{\epsilon(\tau_1,\tau_2)\leq 0\}-\tau_2)|D_i,W_i\big)$, and assume that $T(D_i,W_i)$ is twice continuously differentiable with respect to D_i. Then, differentiability of $M(\Pi,\theta)$ at Π_0 and θ_0 follows. To see this, consider Γ_{21}. For each $\Pi \in \Pi_0 + \mathcal{N}$,

$$E\big(T(D_i,W_i)1\{|D_i - W_i'\Pi| \leq \frac{h_n}{2}\}\frac{1}{h_n}\big) \to E\big(T(D_i,W_i)|D_i - W_i'\Pi = 0\big) f_{D_i - W_i'\Pi}(0)$$
$$= E\big(S_i(1\{\epsilon(\tau_1,\tau_2)\leq 0\}-\tau_2)|D_i - W_i'\Pi = 0\big) f_{D_i - W_i'\Pi}(0) = M_2(\Pi,\theta_0). \quad (18)$$

Therefore, it suffices to show that the derivatives of the left–hand side of (18) uniformly converges over $\Pi_0 + \mathcal{N}$. Note that

$$E\big(T(D_i,W_i)1\{|D_i - W_i'\Pi| \leq \frac{h_n}{2}\}\frac{1}{h_n}|W_i\big) = \int T(t,W_i)1\{|t - W_i'\Pi| \leq \frac{h_n}{2}\}\frac{1}{h_n} f_{D|W}(t|W_i)dt$$
$$= \int_{-1/2}^{1/2} T(th_n + W_i'\Pi, W_i) f_{D|W}(th_n + W_i'\Pi|W_i)dt,$$

which has a derivative

$$\int_{-1/2}^{1/2} W_i T_D(th_n + W_i'\Pi, W_i) f_{D|W}(th_n + W_i'\Pi|W_i) + W_i T(th_n + W_i\Pi, W_i) f'_{D|W}(th_n + W_i'\Pi|W_i)dt, \tag{19}$$

where T_D denotes the derivative of $T(D_i, W_i)$ with respect to D_i. Since $\Pi_0 + \mathcal{N}$ is compact, uniform convergence of (19) follows from cotinuous differentiability of T_D and $f'_{D|W}(s|W_i)$. □

Remark: $\Gamma_{21} = E\Big(W_i T_D(W_i'\Pi_0, W_i) f_{D|W}(W_i'\Pi_0|W_i) + W_i T(W_i\Pi_0, W_i) f'_{D|W}(W_i'\Pi_0|W_i)\Big)$. In fact, if $R(\tau_1)$ has a conditional density given Y_i, S_i, which is the case when there is at least one continuous instrument, it is simply $\Gamma_{21} = E\big(S_i W_i'(1\{Y_i \leq S_i'\theta_0\} - \tau_2) f'_{R(\tau_1)|Y,S}(0|Y_i, S_i)\big)$. Similarly, Γ_{22} can be written as $E\big(S_i S_i' f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}(0, 0|S_i)\big)$ in that case.

Lemma 8 $\sqrt{nh_n}(\hat{\theta} - \theta_0) = O_p(1) + O(\sqrt{nh_n}h_n^2)$.

Proof: Since

$$\sup_{\theta\in\Theta} ||M_{2n}(\hat{\Pi}, \theta) - M_2(\Pi_0, \theta)|| = o_p(1) \tag{20}$$

by lemma 5, consistency of $\hat{\theta}$ follows from the standard arguments in view of assumption C. Choose an arbitrary $\delta > 0$. Then, continuity of $M_2(\Pi_0, \theta)$ and the compact parameter space guarantees that there is some $\theta^* \in \{\theta \in \Theta : ||\theta - \theta_0|| \geq \delta\}$ such that $\inf_{||\theta-\theta_0||\geq\delta} ||M_2(\Pi_0, \theta)|| = ||M_2(\Pi_0, \theta^*)||$. Since $M_2(\Pi_0, \theta)$ is equal to 0 uniquely at θ_0, it follows that $\inf_{||\theta-\theta_0||\geq\delta} ||M_2(\Pi_0, \theta)|| \geq \epsilon$ for some $\epsilon > 0$. It then follows that $\Pr(||\hat{\theta} - \theta_0|| \geq \delta) \leq \Pr(||M_2(\Pi_0, \hat{\theta})|| \geq \epsilon)$. Now, note that

$$||M_2(\Pi_0, \hat{\theta})|| \leq \sup ||M_{2n}(\hat{\Pi}, \theta) - M_2(\Pi_0, \theta)|| + ||M_{2n}(\hat{\Pi}, \hat{\theta})|| = o_p(1),$$

where the last equality is due to equations (12) and (20). Therefore, $||\hat{\theta} - \theta_0|| = o_p(1)$.

To establish the convergence rate, let $\delta_{2n} \downarrow 0$ be an arbitrary sequence and let $\delta_{1n} \downarrow 0$ be a sequence satisfying the conditions of lemma 5. It then follows from lemma 5 that

$$\begin{aligned}
&\sup ||\sqrt{nh_n}\big(M_{2n}(\Pi,\theta) - M_2(\Pi,\theta)\big) - \sqrt{nh_n}\big(M_{2n}(\Pi_0,\theta_0) - M_2(\Pi_0,\theta_0)\big)|| \\
&\quad \leq \sup ||\sqrt{nh_n}\Big(M_{2n}(\Pi,\theta) - E\big(M_{2n}(\Pi,\theta)\big)\Big) - \sqrt{nh_n}\Big(M_{2n}(\Pi_0,\theta_0) - E\big(M_{2n}(\Pi_0,\theta_0)\big)\Big)|| \\
&\qquad + 2\sqrt{nh_n}\sup ||E\big(M_{2n}(\Pi,\theta)\big) - M_2(\Pi,\theta)|| = o_p(1) + O(\sqrt{nh_n}h_n^2), \quad (21)
\end{aligned}$$

where sup is taken over $||\Pi - \Pi_0|| < \delta_{1n}$ and $||\theta - \theta_0|| < \delta_{2n}$; for the last equality, I used lemma 1. Since $\hat{\Pi}$ is $\sqrt{n}$–consistent and $\hat{\theta}$ is consistent, there exist some sequences $\delta_{1n} \downarrow 0$ and $\delta_{2n} \downarrow 0$ such that

$$\begin{aligned}
||\sqrt{nh_n}M_2(\hat{\Pi},\hat{\theta})|| &\leq ||\sqrt{nh_n}\big(M_{2n}(\hat{\Pi},\hat{\theta}) - M_2(\hat{\Pi},\hat{\theta})\big)|| + ||\sqrt{nh_n}M_{2n}(\hat{\Pi},\hat{\theta})|| \\
&\leq \sup ||\sqrt{nh_n}\big(M_{2n}(\Pi,\theta) - M_2(\Pi,\theta)\big)|| + o_p(1) \\
&\leq \sup ||\sqrt{nh_n}\big(M_{2n}(\Pi,\theta) - M_2(\Pi,\theta)\big) - \sqrt{nh_n}M_{2n}(\Pi_0,\theta_0)|| + ||\sqrt{nh_n}M_{2n}(\Pi_0,\theta_0)|| + o_p(1) \\
&= o_p(1) + O(\sqrt{nh_n}h_n^2) + O_p(1) + o_p(1)
\end{aligned}$$

with probability approaching to 1, where sup is taken over $||\Pi - \Pi_0|| < \delta_{1n}$ and $||\theta - \theta_0|| < \delta_{2n}$.[19] Therefore,

$$\begin{aligned}
&O(h_n^2) + O_p(\frac{1}{\sqrt{nh_n}}) = ||M_2(\hat{\Pi},\hat{\theta})|| = ||\Gamma_{21}(\hat{\Pi} - \Pi_0) + \Gamma_{22}(\hat{\theta} - \theta_0) + O(||\hat{\Pi} - \Pi_0||||\hat{\theta} - \theta_0||)|| \\
&= ||\Gamma_{22}(\hat{\theta} - \theta_0) + O_p(\frac{1}{\sqrt{n}}) + O_p(\frac{||\hat{\theta} - \theta_0||}{\sqrt{n}})|| \geq ||\Gamma_{22}(\hat{\theta} - \theta_0)|| - O_p(\frac{1}{\sqrt{n}}) \geq \sqrt{\lambda_2}||\hat{\theta} - \theta_0|| - O_p(\frac{1}{\sqrt{n}}),
\end{aligned}$$

where λ_2 is the smallest eigenvalue of $\Gamma_{22}'\Gamma_{22}$. Since the full rank of Γ_{22} implies that $\lambda_2 > 0$, it follows that $\sqrt{nh_n}||\hat{\theta} - \theta_0|| = O_p(1) + O(\sqrt{nh_n}h_n^2) + O_p(\sqrt{h_n})$. □

Proposition 3 *The proposed estimators have the asymptotic expansion of theorem 1.*

[19] Any $\delta_{1n} \downarrow 0$ satisfying $\sqrt{n}\delta_{1n} \to \infty$ and $r_n\delta_{1n} \to 0$ will do, where r_n is defined in lemma 5.

Proof: Since the asymptotic expansion of $\hat{\Pi}(\tau_1)$ follows from standard quantile regression (see e.g. Koenker (2005, p 122)), its proof will be omitted and I only consider $\hat{\theta}(\tau_1, \tau_2)$ here. In the following proof, the asymptotic expansion $\sqrt{n}(\hat{\Pi} - \Pi_0) = \Gamma_{11} M_{1n}(\Pi_0) + o_p(1)$ will be taken for granted.

From differentiability of $M_2(\Pi, \theta)$, consider the expansion

$$M_2(\Pi, \theta) = \Gamma_{21}(\Pi - \Pi_0) + \Gamma_{22}(\theta - \theta_0) + O(||\Pi - \Pi_0||||\theta - \theta_0||).$$

Define

$$L_{2n}(\theta) = \Gamma_{22}(\theta - \theta_0) + M_{2n}(\Pi_0, \theta_0).$$

Then,

$$\begin{aligned}
M_{2n}(\Pi, \theta) &= M_2(\Pi, \theta) + \big(M_{2n}(\Pi, \theta) - M_2(\Pi, \theta)\big) \\
&= L_{2n}(\theta) + \big(M_{2n}(\Pi, \theta) - M_2(\Pi, \theta)\big) - M_{2n}(\Pi_0, \theta_0) + O(||\Pi - \Pi_0||) + O(||\Pi - \Pi_0||||\theta - \theta_0||) \\
&= L_{2n}(\theta) + \Big(M_{2n}(\Pi, \theta) - E\big(M_{2n}(\Pi, \theta)\big) - M_{2n}(\Pi_0, \theta_0) + E\big(M_{2n}(\Pi_0, \theta_0)\big)\Big) \\
&\quad + E\big(M_{2n}(\Pi, \theta)\big) - M_2(\Pi, \theta) - E\big(M_{2n}(\Pi_0, \theta_0) + O(||\Pi - \Pi_0||) + O(||\Pi - \Pi_0||||\theta - \theta_0||). \quad (22)
\end{aligned}$$

In fact, equation (22) shows that $M_{2n}(\Pi, \theta)$ is decomposed into four different components; the linear term $L_{2n}(\theta)$, the stochastic equicontinuity term $\Big(M_{2n}(\Pi, \theta) - E\big(M_{2n}(\Pi, \theta)\big)\Big) - \Big(M_{2n}(\Pi_0, \theta_0) - E\big(M_{2n}(\Pi_0, \theta_0)\big)\Big)$, the bias term $\Big(E\big(M_{2n}(\Pi, \theta)\big) - M_2(\Pi, \theta)\Big) - E\big(M_{2n}(\Pi_0, \theta_0)\big)$, and the remainer term. Therefore, for any estimator θ_n such that $\sqrt{nh_n}(\theta_n - \theta_0) = O_p(1)$,

$$\begin{aligned}
\sqrt{nh_n}||M_{2n}(\hat{\Pi}, \theta_n) - L_{2n}(\theta_n)|| &\leq \sqrt{nh_n}||\big(M_{2n}(\hat{\Pi}, \theta_n) - M_2(\hat{\Pi}, \theta_n)\big) - M_{2n}(\Pi_0, \theta_0)|| + O_p(\sqrt{h_n}) \\
&\leq \sup \sqrt{nh_n}||\big(M_{2n}(\Pi, \theta) - M_2(\Pi, \theta)\big) - M_{2n}(\Pi_0, \theta_0)|| + o_p(1) = O(\sqrt{nh_n}h_n^2) + o_p(1),
\end{aligned}$$

where sup is taken over $||\Pi - \Pi_0|| < \delta_{1n}$ and $||\theta - \theta_0|| < \delta_{2n}$ for some sequences $\delta_{1n} \downarrow 0$ and $\delta_{2n} \downarrow 0$; the second inequality holds with probability approaching to 1 due to consistency, and the

last equality is by equation (21). In particular, undersmoothing ensures that

$$\sqrt{nh_n}||M_{2n}(\hat{\Pi}, \theta_n) - L_{2n}(\theta_n)|| = o_p(1). \tag{23}$$

Now, consider an (infeasible) estimator $\tilde{\theta}$ such that

$$\tilde{\theta} - \theta_0 = \arg \min_{\theta \in \mathbb{R}^{1+k_x}} ||L_{2n}(\theta)|| = -\Gamma_{22}^{-1} M_{2n}(\Pi_0, \theta_0).$$

Note here that although I minimized over $\mathbb{R}^{1+k_x}$, $\tilde{\theta} - \theta_0$ will belong to Θ as n increases, because θ_0 is in the interior of Θ. In the following, I will show that $\hat{\theta}$ is distributionally equivalent to $\tilde{\theta}$. First, note that $\sqrt{nh_n}(\hat{\theta} - \theta_0) = O_p(1)$ and $\sqrt{nh_n}(\tilde{\theta} - \theta_0) = O_p(1)$. Therefore, by equation (23),

$$\sqrt{nh_n}||L_{2n}(\hat{\theta})|| = \sqrt{nh_n}||M_{2n}(\hat{\Pi}, \hat{\theta})|| + o_p(1) = o_p(1).$$

Now, note that $L_{2n}(\hat{\theta}) = L_{2n}(\tilde{\theta}) + \Gamma_{22}(\hat{\theta} - \tilde{\theta})$ by definition, where $L_{2n}(\tilde{\theta})$ is in fact equal to 0. Therefore, it follows that

$$o_p(1) = \sqrt{nh_n}||\Gamma_{22}(\hat{\theta} - \tilde{\theta})|| \geq \sqrt{nh_n}\sqrt{\lambda_2}||(\hat{\theta} - \tilde{\theta})|| \geq \sqrt{\lambda_2}\big|||\sqrt{nh_n}(\hat{\theta} - \theta_0)|| - ||\sqrt{nh_n}(\tilde{\theta} - \theta_0)|||\big|,$$

where $\lambda_2 > 0$ is the smallest eigenvalue of $\Gamma_{22}'\Gamma_{22}$. Therefore,

$$\sqrt{nh_n}(\hat{\theta} - \theta_0) = \sqrt{nh_n}(\tilde{\theta} - \theta_0) + o_p(1),$$

which completes the proof. □

D Proof of Proposition 2

Proposition 2 follows from lemmas 9 and 10. □

Lemma 9

$$\frac{1}{nb_{1n}}\sum_{i=1}^{n} S_i S_i' k\big(\frac{-\hat{R}_i(\tau_1)}{b_{1n}}\big) = \frac{1}{nb_{1n}}\sum_{i=1}^{n} S_i S_i' k\big(\frac{-R_i(\tau_1)}{b_{1n}}\big) + o_p(1),$$
$$\frac{1}{nb_{2n}^2}\sum_{i=1}^{n} S_i S_i' k\big(\frac{-\hat{R}_i(\tau_1)}{b_{2n}}\big)k\big(\frac{-\hat{\epsilon}_i(\tau_1,\tau_2)}{b_{2n}}\big) = \frac{1}{nb_{2n}^2}\sum_{i=1}^{n} S_i S_i' k\big(\frac{-R_i(\tau_1)}{b_{2n}}\big)k\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big) + o_p(1).$$

Proof: Since two statements are similar, we only prove the second one. Let $\Delta_1 = \Pi - \Pi_0,\ \Delta_2 = \theta - \theta_0$ and $\hat{\Delta}_1 = \hat{\Pi} - \Pi_0,\ \hat{\Delta}_2 = \hat{\theta} - \theta_0$. By the mean value theorem, we can write

$$\begin{aligned}
\frac{1}{nb_{2n}^2}\sum_{i=1}^{n} S_i S_i' k\big(\frac{-\hat{R}_i(\tau_1)}{b_{2n}}\big)k\big(\frac{-\hat{\epsilon}_i(\tau_1,\tau_2)}{b_{2n}}\big) &= \frac{1}{nb_{2n}^2}\sum_{i=1}^{n} S_i S_i' k\big(\frac{-R_i(\tau_1)}{b_{2n}}\big)k\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big) \\
&+ \frac{1}{nb_{2n}^2}\sum_{i=1}^{n} S_i S_i' k'\big(\frac{W_i'\tilde{\Delta}_1 - R_i(\tau_1)}{b_{2n}}\big)k\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big)\frac{W_i'\hat{\Delta}_1}{b_{2n}} \\
&+ \frac{1}{nb_{2n}^2}\sum_{i=1}^{n} S_i S_i' k\big(\frac{-R_i(\tau_1)}{b_{2n}}\big)k'\big(\frac{S_i'\tilde{\Delta}_2 - \epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big)\frac{S_i'\hat{\Delta}_2}{b_{2n}} \\
&+ \frac{1}{nb_{2n}^2}\sum_{i=1}^{n} S_i S_i' k'\big(\frac{W_i'\tilde{\Delta}_1 - R_i(\tau_1)}{b_{2n}}\big)k'\big(\frac{S_i'\tilde{\Delta}_2 - \epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big)\frac{W_i'\hat{\Delta}_1}{b_{2n}}\frac{S_i'\hat{\Delta}_2}{b_{2n}} \qquad (24)
\end{aligned}$$

for some $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$.

We claim that the last three terms in (24) are all $o_p(1)$, which easily follows from the stronger bandwidth requirements when W_i has unbounded support. Therefore, we only consider the case where $b_{1n},\ b_{2n}$ satisfy the weaker bandwidth requirements but W_i has bounded support.

Consider the last term in (24) since the other two terms are easier. Since $||\frac{\hat{\Delta}_1}{b_{2n}}|| = O_p(\frac{1}{\sqrt{n}b_{2n}}) = o_p(1)$ and $\frac{\hat{\Delta}_2}{b_{2n}} = O_p(\frac{1}{\sqrt{nh_n}b_{2n}}) = o_p(1)$, for some sequence $\delta_n \downarrow 0$, we have

$$\begin{aligned}
&||\frac{1}{nb_{2n}^2}\sum_{i=1}^{n} S_i S_i' k'\big(\frac{W_i'\tilde{\Delta}_1 - R_i(\tau_1)}{b_{2n}}\big)k'\big(\frac{S_i'\tilde{\Delta}_2 - \epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big)\frac{W_i'\hat{\Delta}_1}{b_{2n}}\frac{S_i'\hat{\Delta}_2}{b_{2n}}|| \\
&\leq \sup_{||\frac{\hat{\Delta}_1}{b_{2n}}||\leq\delta_n,\ ||\frac{\hat{\Delta}_2}{b_{2n}}||\leq\delta_n} ||\frac{1}{nb_{2n}^2}\sum_{i=1}^{n} S_i S_i' k'\big(\frac{W_i'\tilde{\Delta}_1 - R_i(\tau_1)}{b_{2n}}\big)k'\big(\frac{S_i'\tilde{\Delta}_2 - \epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big)\frac{W_i'\hat{\Delta}_1}{b_{2n}}\frac{S_i'\hat{\Delta}_2}{b_{2n}}|| \\
&\leq \frac{1}{nb_{2n}^2}\sum_{i=1}^{n} ||S_i||^3||W_i|| \sup_{|t|\leq||W_i||\delta_n} |k'\big(\frac{-R_i(\tau_1)}{b_{2n}} + t\big)| \sup_{|t|\leq||S_i||\delta_n} |k'\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}} + t\big)|\delta_n^2
\end{aligned}$$

with probability approaching to 1. Note here that

$$\frac{1}{nb_{2n}^2}\sum_{i=1}^{n}||S_i||^3||W_i|| \sup_{|t|\leq||W_i||\delta_n}|k'\big(\frac{-R_i(\tau_1)}{b_{2n}}+t\big)| \sup_{|t|\leq||S_i||\delta_n}|k'\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}+t\big)|$$
$$\leq C\frac{1}{nb_{2n}^2}\sum_{i=1}^{n}||S_i||^3 \sup_{|t|\leq C\delta_n}|k'\big(\frac{-R_i(\tau_1)}{b_{2n}}+t\big)| \sup_{|t|\leq||S_i||\delta_n}|k'\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}+t\big)| \quad (25)$$

for some $C>0$, because W_i has bounded support. Therefore, it suffices to shows that the RHS in (25) is $O_p(1)$. Note here that

$$\sup_{|t|\leq C\delta_n}|k'\big(\frac{-R_i(\tau_1)}{b_{2n}}+t\big)|$$
$$=\max_{c_j\in\mathcal{S}}\big(|k'(c_j)|1\{|c_j+\frac{R_i(\tau_1)}{b_{2n}}|<C\delta_n\},\ |k'(\frac{-R_i(\tau_1)}{b_{2n}}+C\delta_n)|,\ |k'(\frac{-R_i(\tau_1)}{b_{2n}}-C\delta_n)|\big)$$

and that

$$\sup_{|t|\leq||S_i||\delta_n}|k'\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}+t\big)|$$
$$=\max_{c_j\in\mathcal{S}}\big(|k'(c_j)|1\{|c_j+\frac{\epsilon_i(\tau_1,\tau_2)}{b_{2n}}|<||S_i||\delta_n\},\ |k'(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}+||S_i||\delta_n)|,\ |k'(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}-||S_i||\delta_n)|\big).$$

Therefore, plugging in these expressions to the RHS of (25), taking expectations, and using the usual change of variables technique finishes the proof. □

Lemma 10 *If $\epsilon_i(\tau_1,\tau_2)$ and $R_i(\tau_1)$ have a joint density conditional on S_i (e.g. when there is at least one continuous instrument),*

$$\frac{1}{nb_{1n}}\sum_{i=1}^{n}S_iS_i'k\big(\frac{-R_i(\tau_1)}{b_{1n}}\big)=E\Big(S_iS_i'f_{R(\tau_1)|S}(0|S_i)\Big)+o_p(1),$$
$$\frac{1}{nb_{2n}^2}\sum_{i=1}^{n}S_iS_i'k\big(\frac{-R_i(\tau_1)}{b_{2n}}\big)k\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big)=E\Big(S_iS_i'f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}(0,0|S_i)\Big)+o_p(1).$$

Proof: Since the two statements are similar, we only consider the second one. Note first that

$$\frac{1}{nb_{2n}^2}\sum_{i=1}^{n} S_i S_i' k\big(\frac{-R_i(\tau_1)}{b_{2n}}\big)k\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big) = E\Big(\frac{1}{b_{2n}^2} S_i S_i' k\big(\frac{-R_i(\tau_1)}{b_{2n}}\big)k\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big)\Big)+O_p(\frac{1}{\sqrt{n}b_{2n}}), \tag{26}$$

because letting ξ_{tsi} be the t–s element of $S_i S_i'$, squaring and taking expectations yields

$$\begin{aligned}
&\frac{1}{nb_{2n}^2}E\Big(\xi_{tsi}^2 k\big(\frac{-R_i(\tau_1)}{b_{2n}}\big)^2 k\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big)^2\frac{1}{b_{2n}^2}\Big)\\
&\qquad = \frac{1}{nb_{2n}^2}E\big(\xi_{tsi}^2\int k\big(\frac{-r}{b_{2n}}\big)^2 k\big(\frac{-e}{b_{2n}}\big)^2\frac{1}{b_{2n}^2} f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}(r,e|S_i)drde\big)\\
&\qquad = \frac{1}{nb_{2n}^2}E\big(\xi_{tsi}^2\int k(t)^2k(s)^2 f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}(-b_{2n}t,-b_{2n}s|S_i)dtds\big) = O(\frac{1}{nb_{2n}^2}).
\end{aligned}$$

Therefore, I only consider the expectation in the RHS of equation (26).

$$\begin{aligned}
E\Big(\frac{1}{b_{2n}^2}S_i S_i' k\big(\frac{-R_i(\tau_1)}{b_{2n}}\big)k\big(\frac{-\epsilon_i(\tau_1,\tau_2)}{b_{2n}}\big)\Big) &= E\Big(S_i S_i'\int k(t)k(s) f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}(-b_{2n}t,-b_{2n}s|S_i)dtds\Big)\\
&= E\Big(S_i S_i' f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}(0,0|S_i)\Big)\int k(t)dt\int k(s)ds\\
&+ b_{2n}E\Big(S_i S_i'\int k(t)k(s)\big(t\mathcal{D}_1 f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}(\bar{t},\bar{s}|S_i)+s\mathcal{D}_2 f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}(\bar{t},\bar{s}|S_i)\big)dtds\Big),
\end{aligned}$$

where $\mathcal{D}_j f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}$ is the partial derivative with respect to the j^{th} argument and $\bar{t}$, $\bar{s}$ denote the mean values. Since $\sup_{t,s}|\mathcal{D}_j f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}(t,s|S_i)| \le \phi(S_i)$, we have

$$\begin{aligned}
&b_{2n}||E\Big(S_i S_i'\int k(t)k(s)\big(t\mathcal{D}_1 f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}(\bar{t},\bar{s}|S_i)+s\mathcal{D}_2 f_{R(\tau_1),\epsilon(\tau_1,\tau_2)|S}(\bar{t},\bar{s}|S_i)\big)dtds\Big)||\\
&\qquad\qquad \le b_{2n}E\Big(||S_i||^2\phi(S_i)\int|k(t)||k(s)|(|t|+|s|)dtds\Big) = o(1). \quad \square
\end{aligned}$$

Table I: Experiments using the Angrist and Krueger data

p-values of the significance of instruments in the first stage quantile regression

	$\tau_1 =$ / F	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$\sigma = 0.00$	4.9071	0.0000*	0.0000*	0.0000*	1.0000	1.0000	1.0000	1.0000	1.0000
$\sigma = 0.01$	4.9116	0.0000*	0.0000*	0.0000*	0.0000	0.2641	0.3110	0.3792	0.1254
$\sigma = 0.02$	4.8659	0.0000*	0.0000*	0.0000*	0.0000	0.0031	0.1060	0.1550	0.0111
$\sigma = 0.03$	4.9384	0.0000	0.0000*	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\sigma = 0.04$	4.9266	0.0000	0.0000*	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000
$\sigma = 0.05$	4.8893	0.0000	0.0000*	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

	$\tau_1 =$ / F	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90
$\sigma = 0.00$	4.9071	1.0000	0.0000*	0.0000*	0.0000*	0.0000*	1.0000	1.0000	0.0000
$\sigma = 0.01$	4.9116	0.0000	0.0000*	0.0000*	0.0000	0.0000*	0.8133	0.9997	0.0000
$\sigma = 0.02$	4.8659	0.0000	0.0000*	0.0000*	0.0000	0.0000	0.5650	0.9840	0.0014
$\sigma = 0.03$	4.9384	0.0000	0.0000	0.0000	0.0000	0.0000	0.4630	0.9400	0.0020
$\sigma = 0.04$	4.9266	0.0000	0.0000	0.0000	0.0001	0.0000	0.3775	0.6384	0.0028
$\sigma = 0.05$	4.8893	0.0000	0.0000	0.0000	0.0021	0.0000	0.3010	0.4350	0.0173

Note:

H_0: $\Pi_2(\tau_1) = 0$, where $Q_{D_i^*|X_i,Z_i}(\tau_1) = X_i'\Pi_1(\tau_1) + Z_i'\Pi_2(\tau_1)$ with $D_i^* = D_i + \epsilon_i$.

D_i is the education variable of the Angrist–Krueger data, and ϵ_i is a random noise from $N(0, \sigma^2)$.

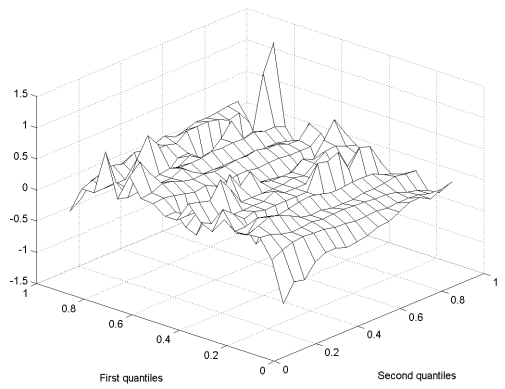
X_i contains 10 dummies indicating birth-years.

Z_i contains 30 instruments of birth-quarters interacted with birth-years.

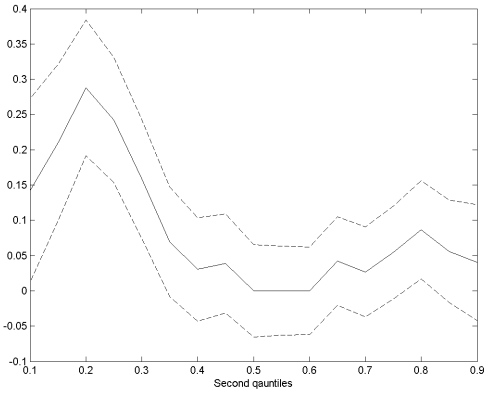
The asterisks $(*)$ indicate the cases where the Wald statitics divided by 30 were greater than 10.

The F values in the second column show the first stage F statistics testing the coefficients of Z_i in the standard mean regression of D_i^* on X_i and Z_i.

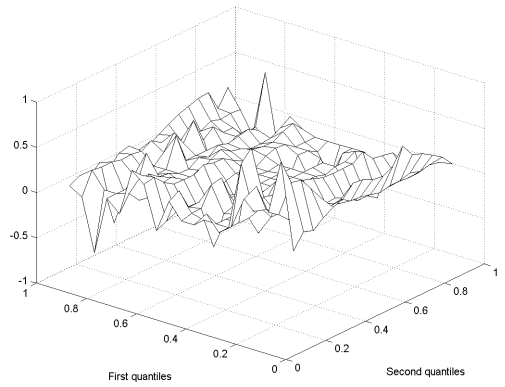
Figure I: Experiments using the Angrist and Krueger data



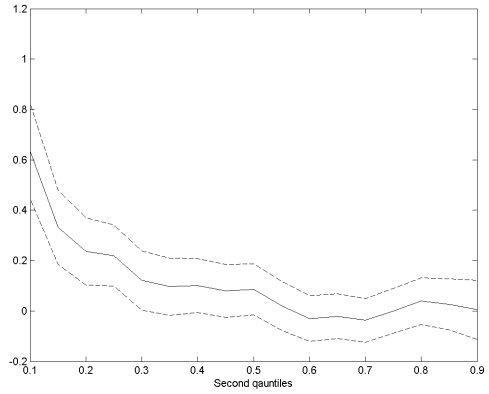
Point Estimates of $\alpha(\tau_1, \tau_2)$ with $\sigma = 0.01$



Confidence Intervals for $\alpha(0.20, \tau_2)$ with $\sigma = 0.01$



Point Estimates of $\alpha(\tau_1, \tau_2)$ with $\sigma = 0.02$



Confidence Intervals for $\alpha(0.20, \tau_2)$ with $\sigma = 0.02$

Note:

$Q_{Y_i|Z_i,X_i,V_i=\tau_1}(\tau_2) = Q_{D_i^*|Z_i,X_i}(\tau_1)\alpha(\tau_1, \tau_2) + X_i'\beta(\tau_1, \tau_2)$, where $D_i^* = D_i + \epsilon_i$.

$\epsilon_i \sim N(0, \sigma^2)$ is the same as those used in Table I.

Y_i is the log wage variable of the Angrist–Krueger data.

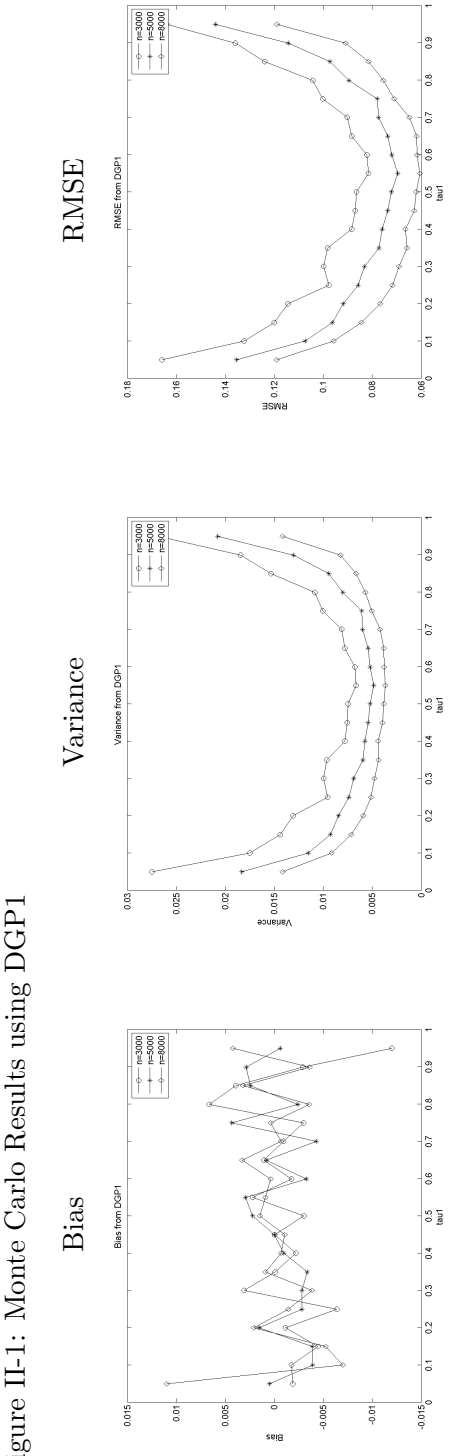


Figure II-1: Monte Carlo Results using DGP1

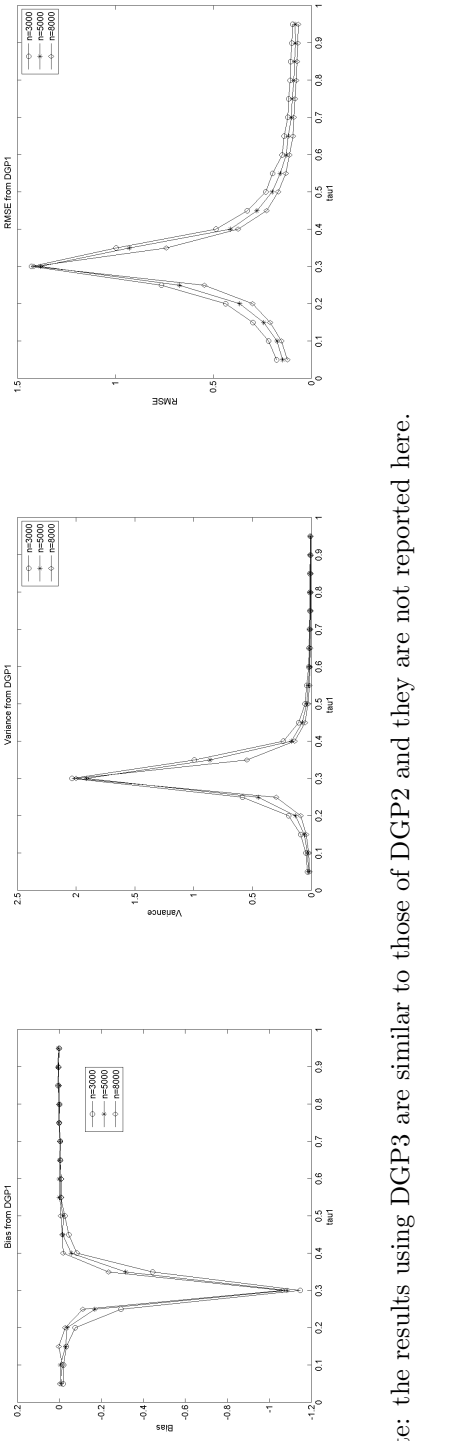


Figure II-2: Monte Carlo Results using DGP2

Note: the results using DGP3 are similar to those of DGP2 and they are not reported here.

Figure III-1: Sensitivity of RMSE to bandwidth choices (DGP1 with $n = 5000$)

$\tau_1 = 0.25,\ \tau_2 = 0.50$ $\tau_1 = 0.50,\ \tau_2 = 0.50$ $\tau_1 = 0.75,\ \tau_2 = 0.50$

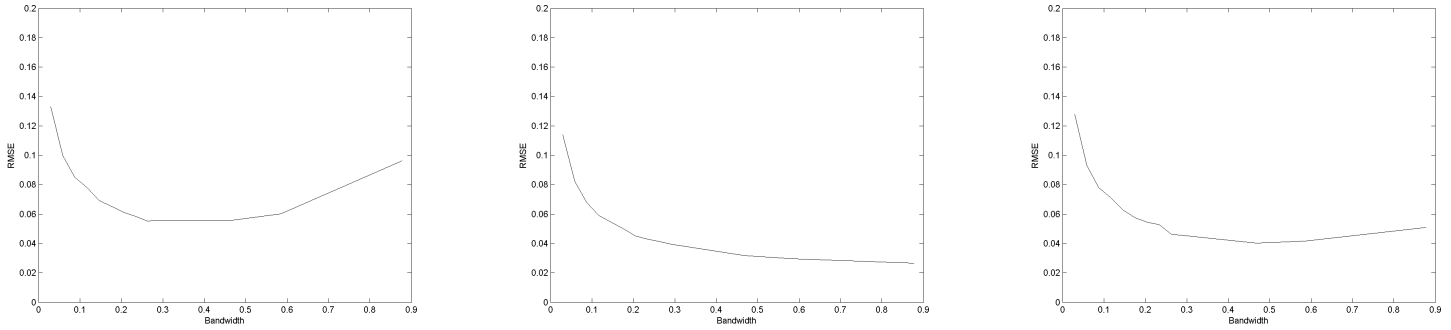


Figure III-2: Sensitivity of RMSE to bandwidth choices (DGP2 with $n = 5000$)

$\tau_1 = 0.30,\ \tau_2 = 0.50$ $\tau_1 = 0.50,\ \tau_2 = 0.50$ $\tau_1 = 0.75,\ \tau_2 = 0.50$

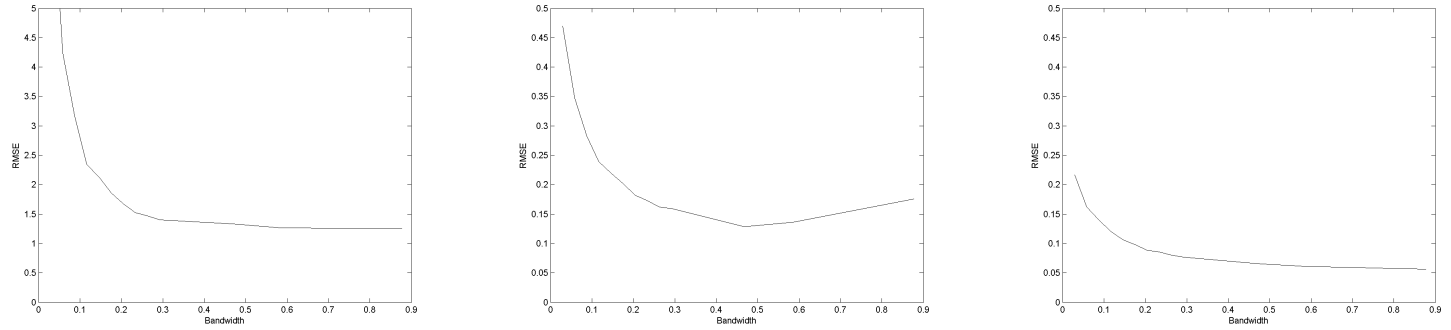


Figure IV: Rejection Rates of t-statistic testing the truth with nominal size 5% for the various values of τ_1

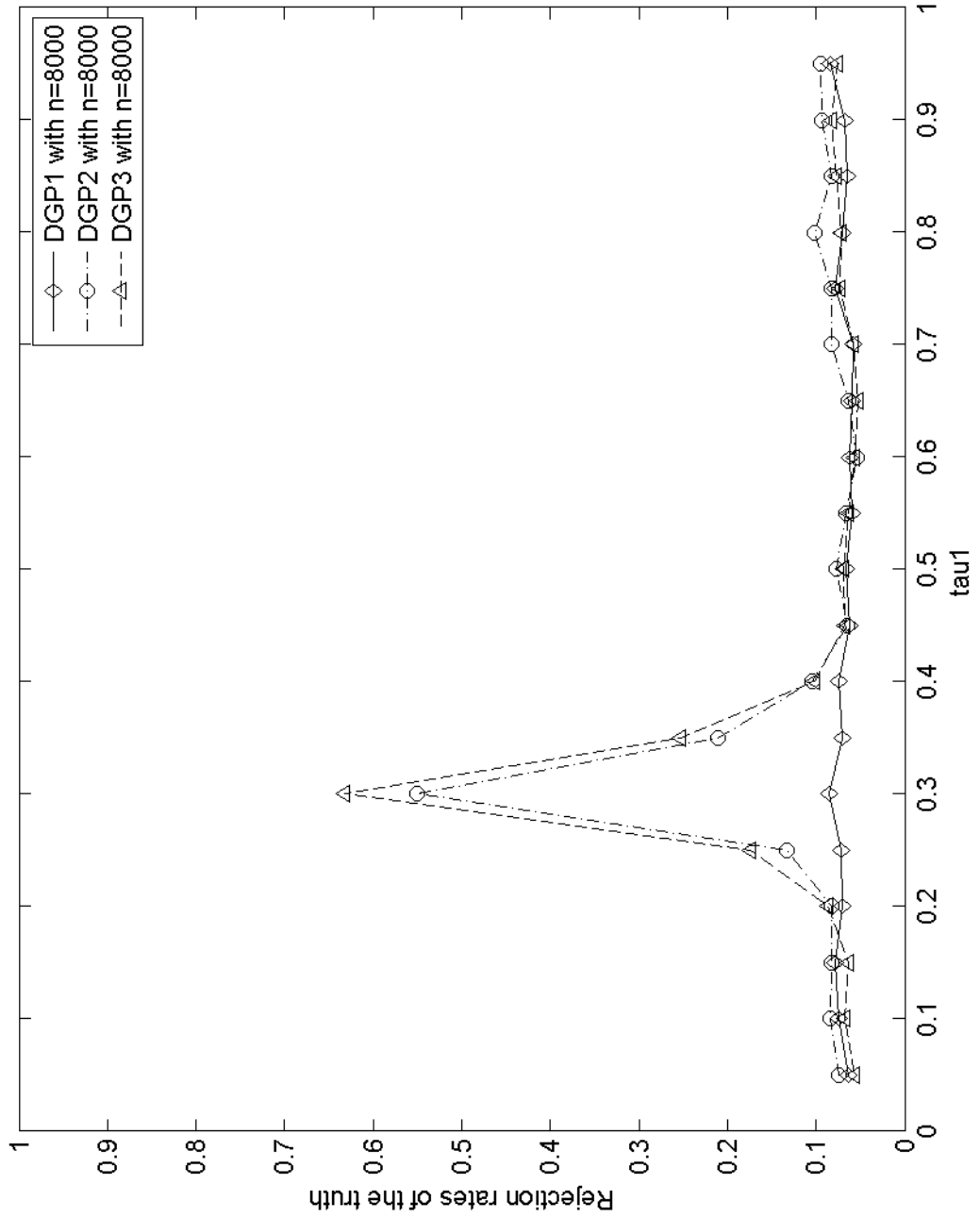
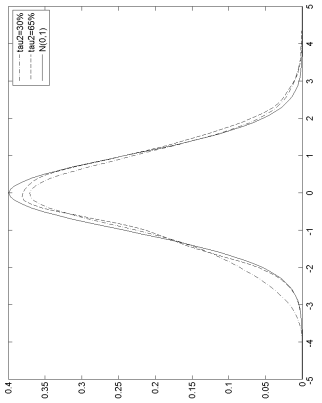
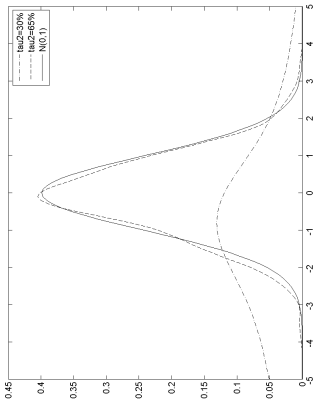


Figure V: Estimated Densities of t-statistics from Monte Carlo

DGP1 with $n = 8000$



DGP2 with $n = 8000$



DGP3 with $n = 8000$

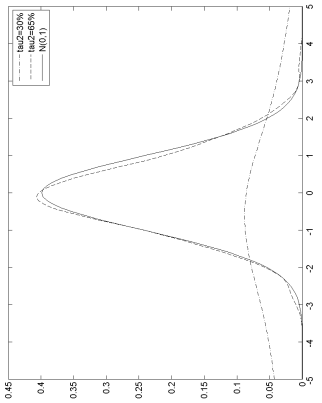
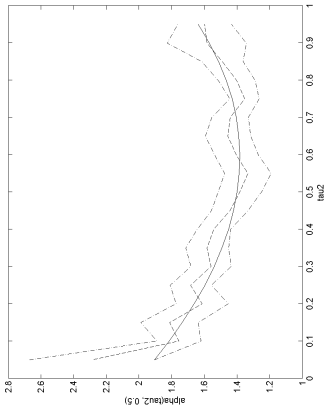
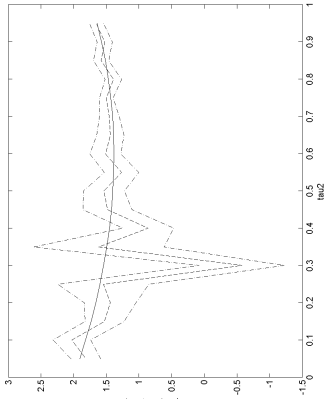


Figure VI: Examples of 95% Confidence Intervals

DGP1 with $n = 8000$



DGP2 with $n = 8000$



DGP3 with $n = 8000$