Dual billiard

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1 Definition and examples

The first volume of the Mathematical Intelligencer contains an article by Jurgen Moser "Is the solar system stable?" [24]. As a toy model for planetary motion, Moser proposed the system illustrated in figure 1 and called the dual (or outer) billiard. The dual billiard table P is a planar oval. Choose a point x outside P. There are two tangent lines from x to P; choose one of them, say, the right one from x's view-point, and reflect x in the tangency point z. One obtains a new point, y, and the transformation $T : x \mapsto y$ is the dual billiard map. Like the planetary motions, the dual billiard dynamics is easy to define but hard to analyze, in particular, it is difficult to reach conclusions about its global properties, such as boundedness or unboundedness of orbits.

In this article we survey results on the dual billiard problem obtained since the publication of Moser's article. We hope that the reader will share our fascination with this beautiful subject. We do not assume familiarity with a much better studied subject of the conventional, inner billiards; an interested reader is referred to [13, 17, 28].

The definition of the dual billiard map has a shortcoming: T is not defined if the tangency point z is not unique. This is the case if the dual billiard curve γ, the boundary of P, contains a straight segment, for example, if γ is a polygon. The

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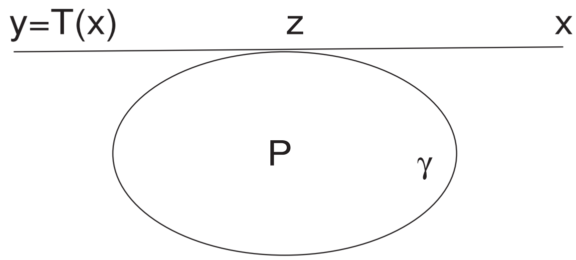

Figure 1: Defining the dual billiard map

dual billiard map and its iterations are not defined for the points on the extensions of straight segments of γ and their preimages under T. This set is a countable collection of lines and therefore a null set, hence one has an ample room to play the game of dual billiard. The situation resembles the inner billiard: if a billiard ball hits a corner of the billiard table then its motion is not defined beyond this point.

J. Moser also considered the dual billiard system in his influential book [23]. He learned about dual billiards from B. Neumann whose 1959 address on the subject to the Manchester Mathematical Colloquium entitled “Sharing ham and eggs” appeared in [25]. By now, there exists a substantial literature on dual billiards listed in the bibliography.

Let us consider examples.

Example 1. If the dual billiard table is a circle then every point moves along a concentric circle, that is, the concentric circles are invariant curves of the dual billiard map. Thus the dual billiard map about a circle is integrable (i.e., there is a conserved quantity): the radius of the circles is an invariant function. Since the dual billiard map commutes with affine transformations of the plane, the dual billiard about an ellipse is integrable as well. An outer version of the celebrated Birkhoff conjecture (concerning inner billiards) states that the only integrable dual billiard is the elliptic one. Like its inner counterpart, this conjecture is open.

Example 2. If the dual billiard table is a square then the motion of every point is periodic. The structure of orbits is depicted in figure 2 where every point of a tile marked n visits once all other tiles with the same marking (there are $4n$ of them) before returning back. One can similarly describe the dynamics of the dual billiard about a triangle or an affine-regular hexagon.

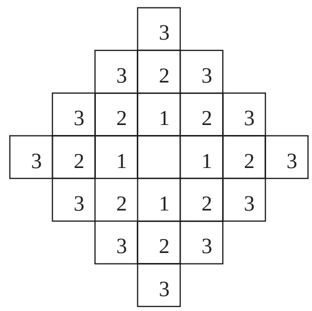

Figure 2: Dual billiard about a square

Example 3. Let the dual billiard table be a regular pentagon. This example was analyzed in [29, 32], see also [28]. The set of full measure, made of regular pentagons and decagons, consists of periodic orbits. In addition, there exist infinite orbits. One such orbit, or rather, its closure, is shown in figure 3. One cannot help noticing self-similarity of this set. Its Hausdorff dimension is to equal $\ln 6/\ln(\sqrt{5}+2) = 1.24....$ Computer experiments show a similar behavior for other regular n-gons (except $n = 3, 4, 6$) but, so far, a rigorous analysis is available only in the cases $n = 5, 8$; cf. figure 3 for the case of a regular octagon. See [1, 3, 12, 19, 21, 22] for related study of piecewise rotations.

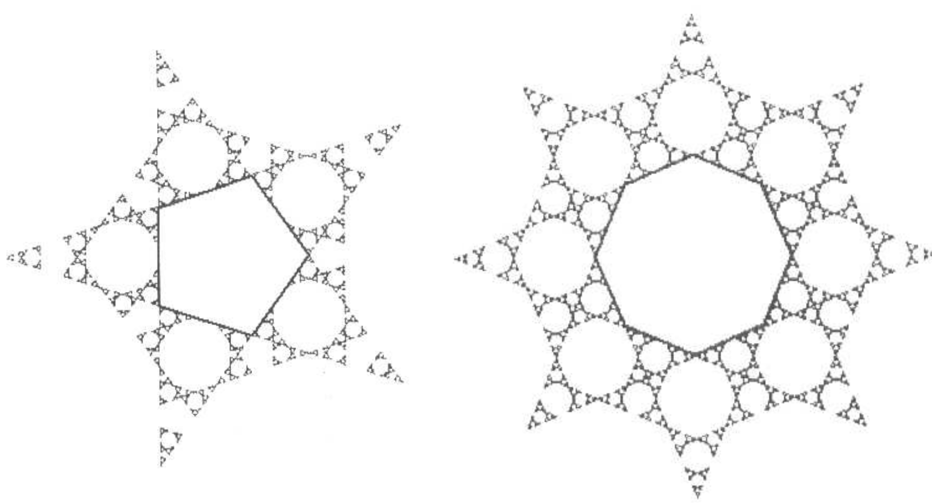

Figure 3: Dual billiards about regular pentagon and octagon

Example 4. An interesting example of a dual billiard table is a semi-circle. A numerical study of this case reveals a very complicated behavior: periodic trajectories and surrounding elliptic "islands" (large white ovals in figure 4) coexist with chaotic orbits (black set). There is a strong computer evidence that some orbits, and even domains, escape to infinity; these escaping domains are seen in figure 4 as small white ovals, positioned between large elliptic islands.

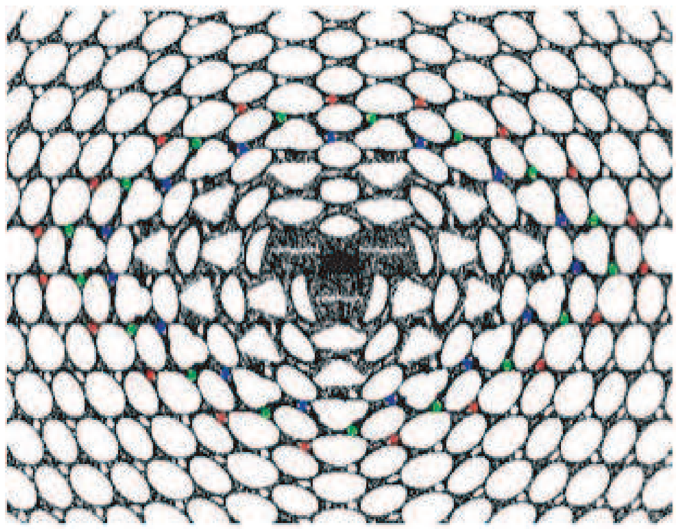

Figure 4: Dual billiard about a semicircle

We finish this section with a mechanical interpretation of the dual billiard system as an impact oscillator, due to Ph. Boyland [4]. Consider a harmonic oscillator on the line, that is, a particle whose coordinate, as a function of time, is a linear combination of $\sin t$ and $\cos t$. There is a 2π–periodically moving massive wall to the left of the particle whose position $p(t)$ satisfies the differential equation $p''(t) + p(t) = r(t)$ where $r(t)$ is a non-negative periodic function which is L^2 orthogonal to $\sin t$ and $\cos t$ [1]. When the particle collides with the wall, an elastic reflection occurs so that the speed of the particle relative to the wall instantaneously changes sign. This is illustrated in figure 5, borrowed from [4].

This mechanical system is isomorphic to the dual billiard about a closed convex curve $\gamma(t)$, parameterized by the angle made by its tangent line with the horizontal direction, whose curvature radius is $r(t)$. Choose an origin O inside γ and let $p(t)$

[1] The reader will easily show that this condition is necessary since $\int_0^{2\pi}(p''(t)+p(t))\sin t\; dt = \int_0^{2\pi}(p''(t)+p(t))\cos t\; dt = 0$

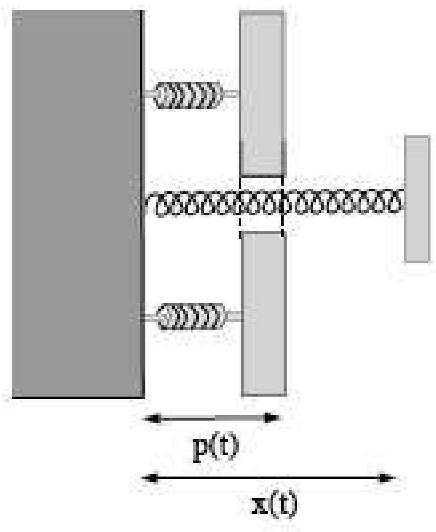

Figure 5: Impact oscillator

be the support function, that is, the distance from O to the tangent line at $\gamma(t)$. Elementary differential geometry tells us that $p''(t)+p(t)=r(t)$, see, e.g., [26].

Let x be a point outside of γ, and let the plane rotate with constant angular speed about the origin. Consider the projections of x and γ on the horizontal line. The projection of the point x is a harmonic oscillator on the line, and the right end point of the projection of γ is "the wall" $p(t)$. When the oscillator and the wall collide, the tangent line from x to γ is horizontal; for the elastic reflection to occur in the projection, the point x should reflect in the tangency point – see figure 6.

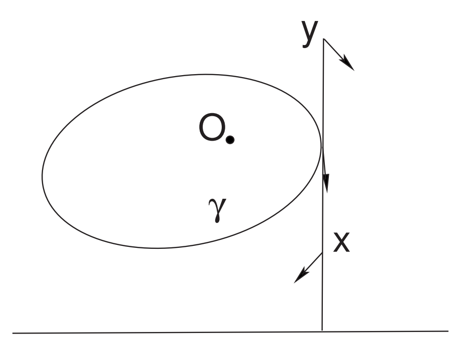

Figure 6: Dual billiard as an impact oscillator

2 Area preserving property

Irrespectively of the shape of the the dual billiard table, the dual billiard map enjoys the fundamental area preserving property. Let us explain why.

Choose infinitesimally close points X and X' on the dual billiard curve. For a positive number r, consider the tangent segments to γ of length r. The end points of these segments trace the curves AA' and BB', see figure 7. The dual billiard map T takes AA' to BB'. Now repeat the construction replacing r by $r - \varepsilon$ where ε is an infinitesimal. We obtain two infinitesimal quadrilaterals $AA'C'C$ and $BB'D'D$, and the map T takes one to another. Let δ be another infinitesimal, the angle between AB and $A'B'$.

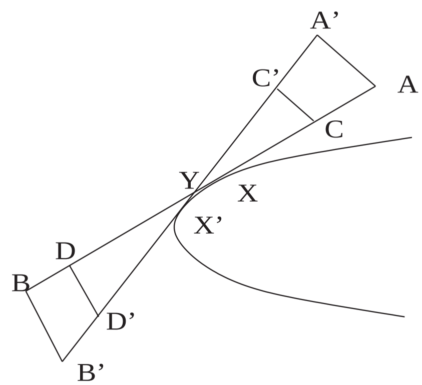

Figure 7: Area preserving property of the dual billiard map

Let us compute the areas of the two quadrilaterals modulo ε^2 and δ^2. One has:

$$\text{Area } AYA' = \delta r^2/2; \text{Area } CYC' = \delta(r-\varepsilon)^2/2 = \delta r^2/2 - \delta\varepsilon r,$$

and hence Area $AA'C'C = \delta\varepsilon r$. Likewise, Area $BB'D'D = \delta\varepsilon r$, and the area preserving property follows.

This property has numerous consequences. The phase space of the dual billiard map T is the exterior of the oval P, topologically, a cylinder. This cylinder is foliated by the positive tangent half-lines to γ. The map T is an area preserving twist map; the latter means that the differential dT rotates the tangent vectors to the leaves in the positive sense, see figure 8.

The theory of area preserving twist maps is well developed, see, e.g., [18]. One of the consequences of this theory concerns periodic orbits of the dual billiard map. Such an orbit is an n-gon, circumscribed about γ whose sides are bisected

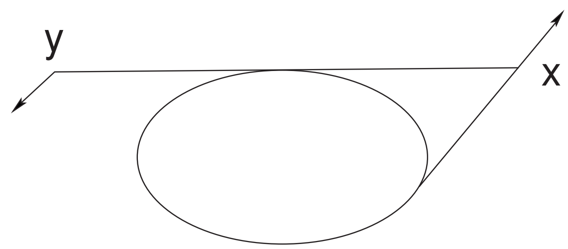

Figure 8: Twist condition for the dual billiard map

by the tangency points, see figure 9. A periodic trajectory has a topological characteristic called the rotation number, the number of turns made by the respective circumscribed polygon around the dual billiard table. A dual billiard version of the celebrated Birkhoff theorem asserts that for every $n \geq 3$ and every integer rotation number $1 \leq r \leq n/2$ there exist at least two distinct n-periodic orbits with the rotation number r.

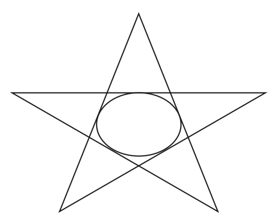

Figure 9: A 5-periodic orbit of the dual billiard map with the rotation number 2

In fact, periodic trajectories of the dual billiard map correspond to circumscribed polygons of extremal area. This is illustrated in figure 10: if the side AB is not bisected by the tangency point then an infinitesimal rotation of the segment to the new position $A'B'$ changes the area in the linear approximation (this is essentially the same argument as in figure 7). One of the n-periodic orbits with a fixed rotation number corresponds to the circumscribed n-gon of minimal area; the second one is of mini-max type.

Suppose that the dual billiard map has an invariant curve, say, Γ. Can one recover the dual billiard curve γ from Γ? The following construction does the job. Consider the 1-parameter family of lines that cut off a segment of fixed area c from Γ, and let γ be the envelop of this family. This envelop may have singularities,

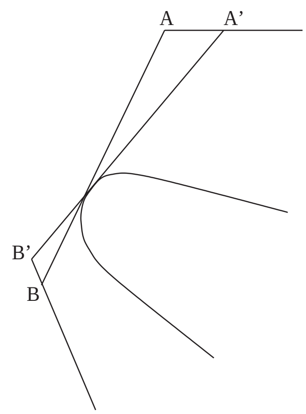

Figure 10: Periodic orbits correspond to area extrema

generically, semi-cubical cusps (see, e.g., [9] for a study of these singularities); assume however that γ is smooth. Then the dual billiard map about γ preserves the curve Γ; a proof of this fact is, essentially, in figure 7. Note that this area construction depends on the area c: there is a 1-parameter family of dual billiards with a given invariant curve[2].

Note also that the area construction resembles a more classical string construction for the inner billiards: to recover a billiard table γ from an invariant curve Γ of the billiard map one wraps a closed non-stretchable string around the curve and moves it around as shown in figure 11 on the right (see. e.g., [28]). Applied to an ellipse, the string construction produces a confocal ellipse; this fact is known as the Graves theorem [2].

The invariant curve Γ does not have to be smooth. For example, one can start with a square, then the dual billiard curve will consist of four arcs of hyperbolas. D. Genin [10] discovered recently that if the area parameter c is small enough then this dual billiard system exhibits a hyperbolic behavior inside this invariant square Γ (and outside of four 4-periodic regular octagons). For inner billiards, numerous examples of convex domains are known that enjoy hyperbolic dynamics, starting with the celebrated Bunimovich stadium, see [28] for a survey.

[2]This construction is also known in the flotation theory where a segment of constant area represents the submerged part of a floating body; the constant c is the density. See [11] on flotation theory

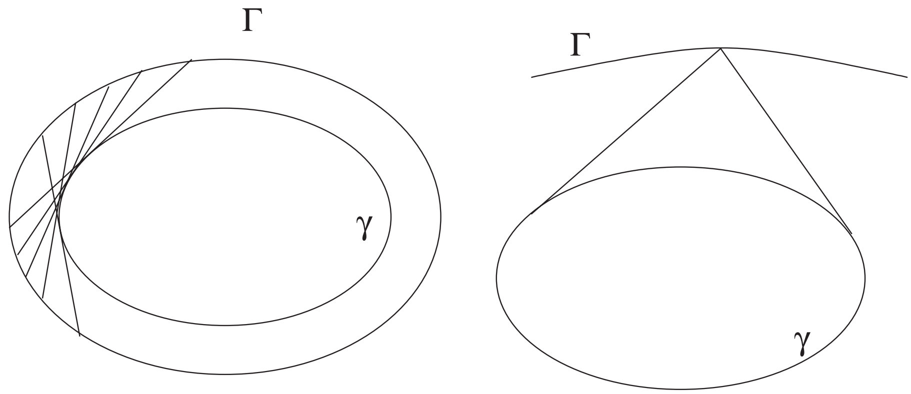

Figure 11: Area and string constructions

3 Duality between inner and outer billiards

The reader has noticed a duality of sorts between the inner and outer billiards. For example, a periodic billiard trajectory is a polygon of extremal perimeter length, inscribed in the billiard curve (see, e.g., [28]), while a periodic dual billiard trajectory is a circumscribed polygon of extremal area. Another manifestation of this duality is shown in figure 11. How does one explain this length-area duality[3]?

The situation becomes more clear if one replaces the plane by the unit sphere. On the sphere, one has duality between points and oriented lines (i.e., great circles): to a pole there corresponds its oriented equator, see figure 12. Note that the spherical distance AB equals the angle between the lines a and b.

Duality preserves the incidence relation: if a point A lies on a line b then the dual point B lies on the dual line a. Duality extends to smooth curves: a curve γ determines a 1-parameter family of tangent lines, and each line determines the dual point. The resulting 1-parameter family of points is the dual curve γ^*. If one applies this construction to γ^* then one obtains the curve that is antipodal to γ.

Consider an instance of the billiard reflection in a curve γ, see figure 13. The law of billiard reflection says: the angle of incidence equals the angle of reflection. In terms of the dual picture, this means that $AL = LB$, and hence the dual billiard reflection about the dual curve γ^* takes A to B. Thus the inner and outer billiards are conjugated by the spherical duality.

We can also explain the length-area duality. Consider a polygon of extremal

[3]And justifies the term "dual billiard"

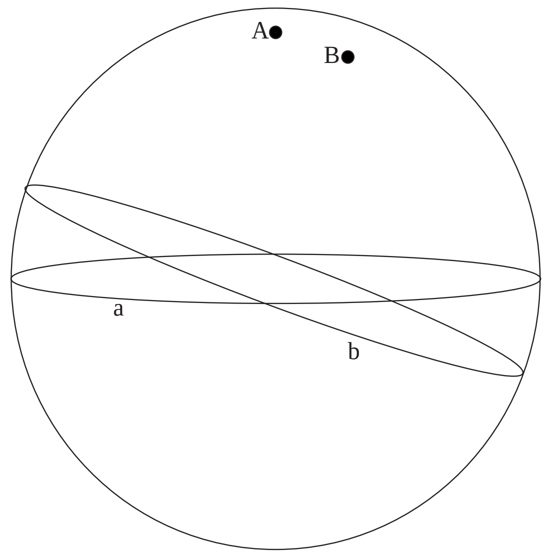

Figure 12: Spherical duality

perimeter, inscribed in a curve γ. The dual polygon is circumscribed about the dual curve γ^* and has an extremal sum of angles. The latter is related to the area of the polygon via the Gauss-Bonnet theorem: the sum of exterior angles of a polygon equals 2π minus its area. This explains why area extrema are responsible for dual billiard periodic trajectories. One may consider the plane as a sphere of infinite radius. In this limit, the sum of angles of a polygon becomes a constant but the area retains its role as the generating function of the dual billiard map, whose extrema correspond to periodic orbits.

4 Behavior at infinity. Rational and quasi-rational polygons

A property that is peculiar to the dual billiard in the plane is a simple limiting motion far away from the table, observed in computer experiments. A bird's eye view of a dual billiard curve γ is almost a point and the map T is almost the reflection in this point. More precisely, after rescaling, the distance between a point x and its second iteration $T^2(x)$ is very small, and the evolution of a point under the second iteration T^2 appears a continuous motion. This motion happens along a piece-wise smooth centrally symmetric curve Γ and satisfies the second

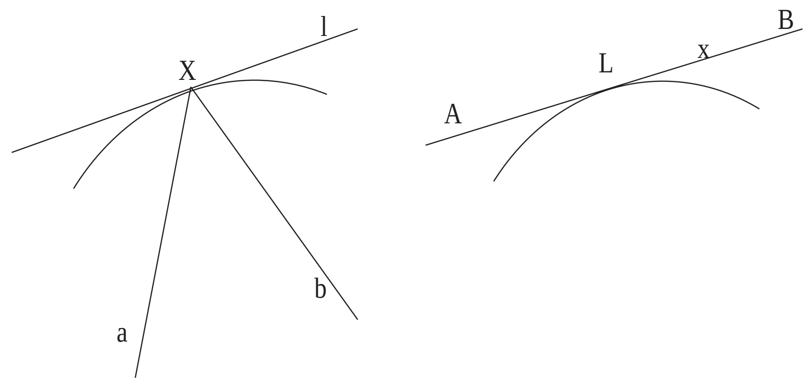

Figure 13: Duality between inner and outer billiards

Kepler law: the area swept by the position vector of a point depends linearly on time (the unit of time being one iteration of the map T^2). Figure 14 features some dual billiard curves γ and the respective trajectories "at infinity" Γ. The last curve Γ is made of two parabolas intersecting at right angles, it corresponds to a semi-circle γ, cf. Example 4 in the first section.

To explain these observations, assume that $\gamma(t)$ is a parameterized smooth curve. Consider the tangent line to $\gamma(t)$. There is another tangent line, parallel to that at $\gamma(t)$; let $v(t)$ be the vector that connects the tangency points of the former and the latter, see figure 15.

For points very far away from the dual billiard table, the angle at vertex B in figure 15 is very small, and the tangent direction to the trajectory at infinity $\Gamma(t)$ is parallel to the vector $v(t)$. Thus we need to solve the differential equation $\Gamma'(t) \sim v(t)$. If a solution exists, it is unique, up to homothety. In fact, one can solve the equation explicitly:

$$\Gamma(t) = \frac{v'(t)}{v(t) \times v'(t)}$$

where $\times$ denotes the cross-product, that is, the determinant of two vectors. Indeed, a straightforward computation (left to the reader) reveals that Γ, defined by the above formula, satisfies $v \times \Gamma' = 0$, as needed.

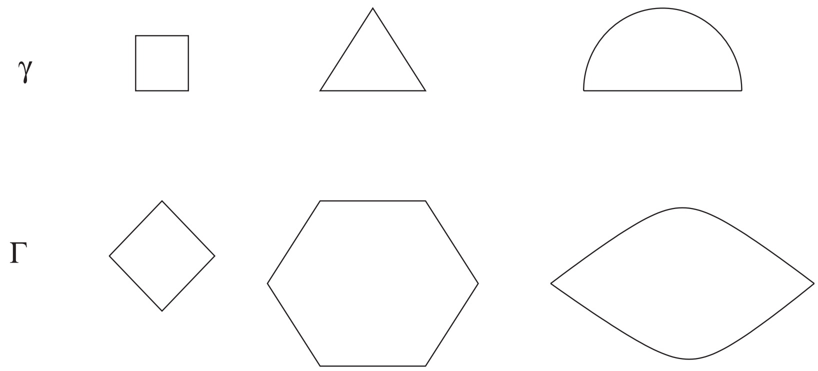

Figure 14: Trajectories of the dual billiard map at infinity

An explicit formula for $\Gamma(t)$ makes it possible to explain the Kepler law: the velocity of the motion along Γ is $v(t)$, and the rate of change of the sectorial area is $v(t) \times \Gamma(t) = 1$; of course, the value of the constant does not make much sense since everything is defined only up to scaling.

The reader is challenged to prove that if γ is centrally symmetric then the correspondence $\gamma \mapsto \Gamma$ is a duality, that is, applied twice, it yields the original curve γ.

We see that the dual billiard dynamics at infinity is approximated by a continuous motion along curves, homothetic to Γ. This motion has an integral (a conserved quantity): a homogeneous function whose level curves are these curves, homothetic to Γ. Thus the dual billiard map at infinity is a small perturbation of an integrable mapping. Assume that γ is sufficiently smooth (C^5 will do) and has everywhere positive curvature. Then one has a KAM (Kolmogorov, Arnold, Moser) theory type theorem that the dual billiard map has invariant curves arbitrarily far from γ. This result was described by Moser in [23, 24]; a detailed proof was given by R. Douady [7]. A T-invariant curve serves as a wall that no orbit of the dual billiard map can cross, and hence all its orbits stay bounded. It is unknown whether this remains true for dual billiard curves that are less smooth or whose curvature has zeros.

If the dual billiard curve γ is a polygon then the trajectory at infinity Γ is a centrally symmetric $2k$-gon, and the vectors $\pm v_1, \dots, \pm v_k$ are diagonals of γ. To every side of Γ there corresponds "time", the ratio of the length of this side to the magnitude of the respective vector v. One obtains a collection of "times" $(t_1, \dots, t_k)$, defined up to a common factor. The polygon is called quasi-rational if

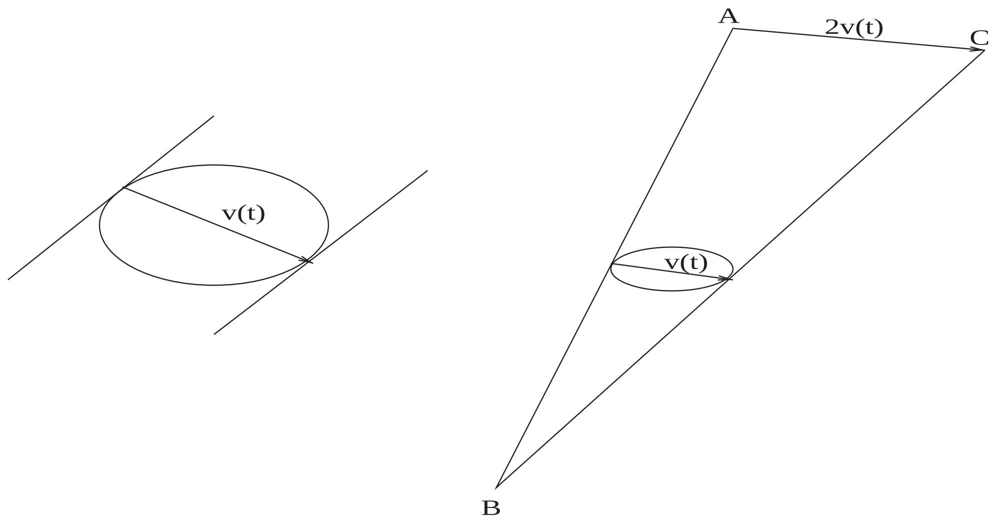

Figure 15: Explaining the behavior at infinity

all these numbers are rational multiples of each other.

An example of a quasi-rational polygon is a lattice polygon whose vertices have integer coordinates. Another example is a regular polygon: the numbers t_i are all equal in this case.

A partial answer to Moser's question [24] is given by the following theorem [27, 20, 15]: if the dual billiard table is a quasi-rational polygon then every orbit of the dual billiard map T is bounded. In this situation one has an analog of invariant curves: these are T-invariant necklaces of polygons around the dual billiard table connected to each other at their common vertices. In figure 3, one can see the first such necklace: it consists of 5 regular decagons surrounding the fractal "pentagram" for the left table and of 8 regular octagons around the fractal 8-ended star for the right one.

A corollary of this theorem is that if γ is a lattice polygon then all dual billiard orbits are periodic. Indeed, the orbit of a point is discrete and, by the above theorem, bounded. One would expect an easy proof of this property of lattice polygons; we are not aware of one.

In conclusion of this section, let us mention that, until very recently, it was not known whether the dual billiard about a polygon always has a periodic orbit. In summer of 2004, a participant of the Penn State REU program, C. Culter, proved that, for every polygonal dual billiard, periodic orbits exist and, moreover, as far as the measure is concerned, periodic points constitute a positive proportion of the whole plane [5] [4].

5 Dual billiard in the hyperbolic plane

We discussed the dual billiard system in the Euclidean plane and on the sphere. One can equally well consider dual billiards in the hyperbolic plane H^2. It is convenient to use the Klein-Beltrami (or projective) model of hyperbolic geometry. Then H^2 is represented by the interior of the unit circle ("circle at infinity"), straight lines – by the chords of this circle, and the distance between points x and y is given by the formula

$$d(x, y) = \ln[a, x, y, b]$$

where a and b are the intersection points of the line xy with the circle and

$$[a, x, y, b] = \frac{(y - a)(b - x)}{(x - a)(b - y)}$$

is the cross-ratio.

The first steps of the study of dual billiards in the hyperbolic plane are made in [8, 30, 35]. In this case, the dual billiard map T extends to a continuous map $t : S^1 \to S^1$ of the circle at infinity. This circle map contains much information about the dual billiard map. Let $\rho \in \mathbf{R}/\mathbf{Z}$ be the Poincaré rotation number of t (see, e.g., [18] for a definition and main properties). The rotation number ρ depends continuously on the dual billiard table.

Assume first that the dual billiard curve is sufficiently smooth and strictly convex. As we saw, in the Euclidean plane this would imply that all dual billiard orbits stay bounded. In the hyperbolic plane, the situation is quite different. Assume that ρ is rational and the circle map t has a hyperbolic periodic orbit. Then there exists a domain in H^2 that escapes to infinity under the dual billiard map, more specifically, is attracted to a hyperbolic periodic orbit at infinity. Moreover, this behavior is stable with respect to small perturbations of the dual billiard table.

If the dual billiard curve is a circle in the hyperbolic plane then the dual billiard map is integrable: its invariant curves are concentric circles, just as in Example 1 at the beginning of this article. What about an elliptic dual billiard curve γ?

[4] For inner polygonal billiards, even for obtuse triangles, the existence of periodic trajectories is an open problem. A remarkable progress has been recently made by R. Schwartz who proved the existence of periodic billiard trajectories in all triangles with the obtuse angle not greater than 100 degrees

The dual billiard map is still integrable, and the invariant curves are ellipses from the pencil of conic generated by γ and the circle at infinity[5]. One can derive the classical Poncelet porism of projective geometry from this integrability, see [30].

Consider now the case of polygonal dual billiards in the hyperbolic plane. Let γ be a convex n-gon. One can prove that $\rho \geq 1/n$. An n-gon is called large if $\rho = 1/n$ and the circle map t has a hyperbolic n-periodic point; see figure 16 for an example. The set of large polygons is open in the natural topology.

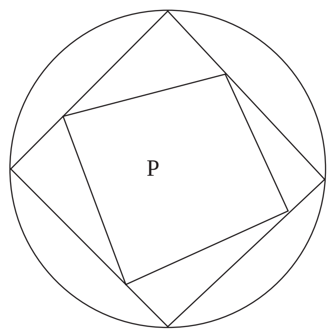

Figure 16: A large quadrilateral

As far as the stability properties of the dual billiard orbits, large polygons in the hyperbolic plane are on the opposite end of the spectrum from smooth strictly convex curves in the Euclidean plane: it is proved in [8] that every dual billiard orbit about a large polygon escapes to infinity.

The class of large triangles can be described explicitly. Consider a triangle with sides a_1, a_2 and a_3 and semi-perimeter s. This triangle is large if and only if

$$\sqrt{\sinh s \ \sinh(s-a_1) \ \sinh(s-a_2) \ \sinh(s-a_3)} > \frac{1}{2}.$$

The left hand side of this formula coincides with Heron's formula for the area of a Euclidean triangle.

Example 5. Let the dual billiard table P be a regular n-gon with right angles ($n \geq 5$). Such polygons tile the hyperbolic plane, see figure 17. Similarly to Example 2 in Section 1, all orbits of the dual billiard map T are periodic: T cyclically permutes the tiles that form concentric "necklaces" around P. The

[5]A pencil consists of conics passing through four fixed points; in the case at hand, these points are complex

rotation number is given by the formula:

$$\rho(P) = \frac{n - \sqrt{n(n-4)}}{2n}$$

(in a sense, this formula holds for $n = 4$ as well: a square tiles the Euclidean, not the hyperbolic, plane, and the dual billiard map "at infinity" is just a central symmetry with the rotation number $1/2$).

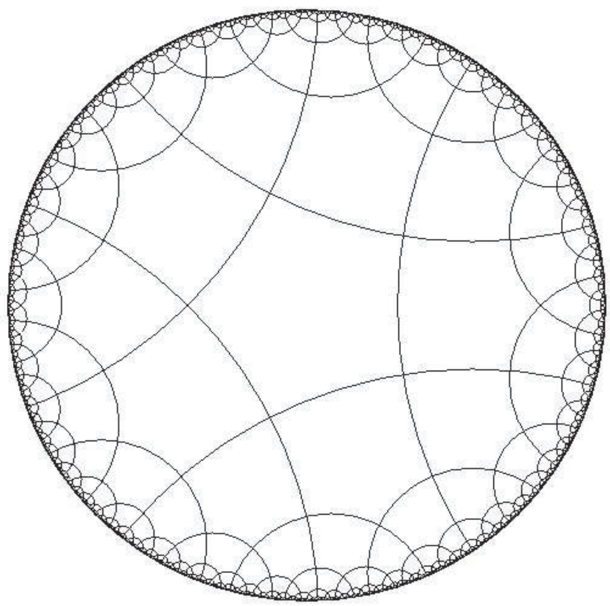

Figure 17: Tiling of the hyperbolic plane by regular right-angled pentagons

We do not know whether there exist polygons in the hyperbolic plane for which all orbits of the dual billiard map are bounded but not all orbits are finite. Such polygons would be analogs of quasirational, but not lattice, polygons in the Euclidean setup.

6 Multi-dimensional dual billiard

The inner billiard is defined in any dimension. The dual billiard can be defined in an even-dimensional space (the plane is even-dimensional, after all). Identify

$\mathbf{R}^{2n}$ with $\mathbf{C}^n$ and let J be the operator of multiplication by $\sqrt{-1}$. A dual billiard table is a bounded convex domain with smooth boundary M^{2n-1}, the dual billiard hypersurface. One would be able to define the dual billiard map if there was a unique tangent line at every point of M. The problem is, there are too many such tangent lines.

This difficulty is resolved as follows. Let N be the outer normal direction to M at point z. Then $J(N)$ is tangent to M at z, and we obtain a well defined oriented tangent line $\ell(z)$ at every point $z \in M$. One can prove that through every point x outside of M there pass exactly two such tangent lines to M, one oriented from M and another to M, just as in the plane. The dual billiard map is defined as follows: find a point $z \in M$ so that $\ell(z)$ passes through x and reflect x in z to obtain a new point $y = T(x)$, cf. figure 1.

As an indication that this is "the right" definition, one has an analog of the area preserving property. The space $\mathbf{C}^n$ carries a symplectic structure, a non-degenerate skew symmetric bilinear form, given by the formula

$$\omega(u, v) = J(u) \cdot v$$

where u and v are tangent vectors. The above defined line $\ell(z)$ is the symplectic orthogonal complement to the tangent hyperplane $T_z M$ (it is called the characteristic direction at point z).

For every dual billiard table, the dual billiard map T preserves the symplectic structure. As in the plane, this has numerous consequences, for example, the existence of periodic orbits; see [28, 29, 32, 36]. However, by and large, multi-dimensional dual billiards remain terra incognita.

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