ON THE DISTRIBUTION OF $\alpha p$ MODULO 1

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§1. Introduction. In [4] we have given a simple method of estimating trigonometrical sums over prime numbers. Here we show how the argument can be adapted in order to give estimates for the distribution of $\alpha p$ modulo 1 which are sharper than those obtained by I. M. Vinogradov [5], [6]. Vinogradov uses the sieve of Eratosthenes to relate the sum

$$\sum_{p \leq N} f(p)$$

to the bilinear form

$$\sum_{d_1, \ldots, d_s \leq \theta} \mu(d_1) \ldots \mu(d_s) \sum_{m_1, \ldots, m_s, d_1 \ldots d_s \leq N} f(m_1 \ldots m_s d_1 \ldots d_s),$$

the function $\mu$ being the Mobius function. When $d_1 \ldots d_s$ is small compared with $N$ this can be treated in a fairly straightforward manner. However, in order to treat the terms with $d_1 \ldots d_s$ close to $N$, Vinogradov has to introduce an argument of a rather recondite combinatorial nature.

Our fundamental idea is to avoid these combinatorial difficulties by using instead the (not unrelated) identity

$$\sum_n f(1, n) = \sum_{d \leq u} \sum_r \sum_n \mu(d)f(rd, n) - \sum_{m > u} \sum_n \sum_{d \mid m} \mu(d)f(m, n), \quad (1)$$

which holds for any double sequence $f(m, n)$ of complex numbers for which the first series on the right converges absolutely. The identity is a trivial consequence of the observation

$$\sum_{d \mid m} \mu(d) = 0 \quad (1 < m \leq u).$$

We remark in passing that special cases of (1) are important underlying ingredients in each of the recent estimates [2, Chapter 16], [1, Chapter 25], [3], obtained by
analytic means, for sums of the form
\[ \sum_{n \leq N} \Lambda(n)e(\alpha n), \]
where \( \Lambda \) is von Mangoldt's function.

We first of all prove the following theorem, which can be compared with
Theorem 3 of Chapter IX of [5].

**Theorem 1.** Suppose that \((a, q) = 1, |\alpha - a/q| \leq q^{-2}, H \geq 1, N \geq 1, \mathcal{L} = \log 2HN \) and
\[ D = \max_{m \leq HN} \left( \sum_{d \leq H} 1 \right). \]

Then
\[ \sum_{h=1}^{H} \left| \sum_{n=1}^{N} \Lambda(n)e(\alpha hn) \right| \leq \mathcal{L}^{2}(HNq^{-\frac{1}{2}} + HN^{\frac{1}{2}} + (HNq)^{\frac{1}{4}} + DH^\frac{1}{4}N^\frac{1}{4}) \quad (2) \]

We then use Theorem 1 to establish

**Theorem 2.** Suppose that \((a, q) = 1, |\alpha - a/q| \leq q^{-2}, 0 \leq \Delta < \frac{1}{2}, \mathcal{L} = \log (Nq/\Delta) \) and \( \varepsilon \) is any fixed positive number. Define the function \( f_\Delta(\theta) \) to be 1 when \(-\Delta \leq \theta < -\Delta \) or \( \Delta \leq \theta < \frac{1}{2} \), and to be periodic with period 1. Then
\[ \sum_{n=1}^{N} \Lambda(n)(f_\Delta(\alpha n - \beta) - 2\Delta) \leq \mathcal{L}^{\frac{3}{2}}(Nq^{-\frac{1}{2}} + N^{\frac{1}{2}} + (\Delta Nq)^{\frac{1}{4}} + \Delta^{2/5} N^{\frac{1}{4}} (Nq/\Delta)^{\varepsilon}). \quad (3) \]

Consequential upon this we have

**Corollary 2.1.** Suppose that \( \alpha \) is irrational and \( \|\gamma\| \) denotes the distance of \( \gamma \) from a nearest integer. Then there are infinitely many prime numbers \( p \) such that
\[ \|\alpha p - \beta\| \leq p^{-\frac{2}{5}}(\log p)^{\frac{3}{5}}. \quad (4) \]

**Corollary 2.2.** Suppose that \( 0 \leq \gamma < \gamma + \delta \leq 1, \mathcal{L} = \log(2Nq/\delta) \) and \( M \) is the number of prime numbers \( p \) with \( p \leq N \) and \( \gamma \leq \{\alpha p\} < \gamma + \delta \). Then
\[ M - \delta \pi(N) \leq \mathcal{L}^{\frac{3}{2}} \left( Nq^{-\frac{1}{2}} + N^{\frac{1}{2}} + (Nq)^{\frac{1}{4}} \right) + \delta^{2/5} N^{\frac{1}{4}} (Nq/\delta)^{\varepsilon}. \quad (5) \]

When \( \delta \) is small this is appreciably sharper than the theorem of Chapter XI of [5].

§2. Lemmata.

**Lemma 1 (Vinogradov).** Suppose that \( X \geq 1, Y \geq 1, (a, q) = 1, |\alpha - a/q| \leq q^{-2}. \) Then
\[ \sum_{x \leq X} \min(Y, \|ax\|^{-1}) \leq XYq^{-1} + (X + q) \log 2q, \quad (6) \]
ON THE DISTRIBUTION OF $\alpha p$ MODULO 1

$$\sum_{x \leq X} \min(Y, \|x\beta^{-1}\|^{-1}) \ll XYq^{-1} + Y + (X + q) \log 2q,$$
(7)

$$\sum_{x \leq X} \min(XY/x, \|x\beta^{-1}\|^{-1}) \ll (XYq^{-1} + X + q) \log 2XYq.$$  (8)

These inequalities are essentially Lemmas 8a, 8b of Chapter I of Vinogradov [5], and are easily obtained by writing $\theta = q^2 \alpha - qa$ and $x = hq + r$ with $0 \leq h \leq X/q$ and $1 \leq r \leq q$, and, if necessary, considering the cases $h = 0$, $1 \leq r \leq q/2$ and $hq + r > q/2$ separately.

**Lemma 2.** Assume the hypothesis of Lemma 1 and write $l = \log 2XYq$ and

$$S = \sum_{x \leq X} \max_{z \leq Y} \left| \sum_{y \in Z} a_x b_y e(xy) \right|,$$
(9)

where the $a_x$ and $b_y$ are complex numbers. Then

$$S \ll L^2 \left( \sum_{x \leq X} |a_x|^2 \sum_{y \leq Y} |b_y|^2 \right)^{1/2} (XYq^{-1} + X + Y + q)^{1/2}.$$

**Proof.** Let $L = \lfloor \log 2Y \log 2 \rfloor$ and $\mathcal{A}$ denote the set of numbers

$$\mathcal{A} = \left\{ \frac{g_1}{2} + \frac{g_2}{4} + \ldots + \frac{g_L}{2^L} : g_j = 0 \text{ or } 1 \right\}.$$

The elements of $\mathcal{A}$ are spaced at most $1/4Y$ apart. Hence for each $x$ with $1 \leq x \leq X$ there is an element $\gamma = \gamma(x)$ of $\mathcal{A}$ such that the maximum on the right of (9) is attained with $Z = Z(x) = 2Y\gamma$. Choose $g_j = g_j(x) = 0$ or 1 so that

$$\gamma = \frac{g_1}{2} + \frac{g_2}{4} + \ldots + \frac{g_L}{2^L},$$

and let

$$h_0 = 0, \quad h_j = h_j(x) = \frac{g_1}{2} + \frac{g_2}{4} + \ldots + \frac{g_j}{2^j} \quad (j \geq 1).$$
(10)

Then $0 = h_0 \leq h_1 \leq \ldots \leq h_L = \gamma$ and

$$\max_{z \leq Y} \left| \sum_{y \in Z} a_x b_y e(xy) \right| \leq \sum_{j=0}^{L-1} \left| \sum_{2Yh_j < y \leq 2Yh_{j+1}} a_x b_y e(xy) \right|.$$

Hence, by Cauchy’s inequality,

$$S^2 \leq L \left( \sum_{x \leq X} |a_x|^2 \right) \sum_{x \leq X} \sum_{j=0}^{L-1} \sum_{2Yh_j < y \leq 2Yh_{j+1}} b_y e(xy).$$

By (10), $h_j = m_j 2^{-j}$ where $0 \leq m_j = m_j(x) < 2^j$, and $h_{j+1} = h_j + g_{j+1} 2^{-j-1}$ with $g_{j+1}(x) = 0$ or 1. Therefore

$$S^2 \leq L \left( \sum_{x \in X} |a_x|^2 \right) \sum_{j=0}^{L-1} \sum_{m=0}^{2^{j+1}} \sum_{x \in X} \sum_{2Ym2^{-j} < y < 2Ym2^{-j} + Y2^{-j}} \sum \sum b_y \bar{b}_z e(\alpha x(y - z))$$

$$\leq L \left( \sum_{x \in X} |a_x|^2 \right) \sum_{j=0}^{L-1} \sum_{m=0}^{2^{j+1}} \sum_{2Ym2^{-j} < y < 2Ym2^{-j} + Y2^{-j}} \sum |b_y|^2 \sum \min (X, \|xh\|^{-1}).$$

On assuming, as we may, that $b_y = 0$ when $y > Y$, we obtain the desired conclusion from (6).

**Lemma 3.** Let

$$T = \sum_{x \in X} \max_{z \leq X\cdot Y} \left| \sum_{y \leq Z} a_x e(\alpha x y) \right|. \quad (11)$$

Then, on the hypothesis of Lemma 2,

$$T \leq l(XYq^{-1} + X + q) \max_{x \in X} |a_x|.$$

**Proof.** This follows at once from

$$T \leq \left( \max_{x \in X} |a_x| \right) \sum_{x \in X} \min (X Y x^{-1}, \|x\|^{-1}),$$

and (8).

§3. **Proof of Theorem 1.** When $H > q$ or $q > N$ the theorem is immediate from Lemma 2. Therefore it suffices to show that if

$$J \leq J' \leq 2J, \quad J' \leq H, \quad H \leq q \leq N, \quad (12)$$

then

$$\sum_{J \leq j \leq J'} \sum_{n=1}^{N} \Lambda(n) e(\alpha j n) \ll \mathcal{L}^7 \left( JNq^{-1/2} + JN^{3/4} + (JNq)^{1/2} + DJ^{3/5} N^{4/5} \right).$$

Let

$$u = \min (N^{2/5} J^{-1/5}, q, Nq^{-1}), \quad (13)$$

and in (1) take $f(m, n)$ to be $\Lambda(n) e(\alpha j mn)$ when $u < n \leq N/m$ and 0 otherwise. Then

$$\sum_{n=1}^{N} \Lambda(n) e(\alpha j n) = S_1 - S_2 - S_3 + O(N^{1/2}), \quad (14)$$

where

$$S_1 = \sum_{d \mid u} \sum_{r \leq N/d} \sum_{n \leq N/dr} \mu(d) \Lambda(n) e(\alpha j drn),$$

$$S_2 = \sum_{d \mid u} \sum_{n \leq u} \sum_{r \leq N/da} \mu(d) \Lambda(n) e(\alpha j drn),$$

$$S_3 = \sum_{m > u} \sum_{u < n \leq N/m} \sum_{d \mid m} \mu(d) \Lambda(n) e(\alpha j mn).$$
We first of all treat $S_3$. Let
\[ c_j = \exp(-i \arg S_3) \quad \text{when} \quad S_3 \neq 0, \quad c_j = 1 \quad \text{when} \quad S_3 = 0. \]
Then
\[ \sum_{J \leq j < J'} |S_3| = \sum_M S(M) \quad (15) \]
where $M$ takes the values $u, 2u, 4u, \ldots$ with $M < N/u$ and
\[ S(M) = \sum_{J \leq j < J'} \sum_{M < m \leq 2M} \sum_{u < n \leq N/m} \sum_{d|m} \mu(d) c_j \Lambda(n) e(axmn). \]
We estimate the sum $S(M)$ in two different ways according to the size of $M$. When $u \leq M < N^{1/2}$ we take $j/M = x$ and $n = y$ in Lemma 2, and when $N^{1/2} < M < N/u$ we take instead $m = x$ and $jn = y$. Thus, when $u \leq M < N^{1/2}$ we obtain
\[ S(M) \ll \sum_{JM < x \leq 4JM} \max_{Z \leq N/M} \left| \sum_{y \leq Z} d_3(x) \Lambda(y) e(axy) \right| \ll L^6(JM^{-1/2} + JM^{1/2} N^{1/2} + J^{1/2} N^{-1/2} + (JNq)^{1/2}), \]
so that, by (13),
\[ \sum_{u \leq M < N^{1/2}} S(M) \ll L^7(JM^{-1/2} + JN^{3/4} + (JNq)^{1/2} + J^{3/5} N^{4/5}). \]  
(16)

Similarly, when $N^{1/2} < M < N/u$ we have
\[ S(M) \ll \sum_{M < x \leq 2M} \max_{Z \leq 2NJ/M} \left| \sum_{y \leq Z} d(x) \left( \sum_{J \leq j < J'} \sum_{u < n \leq N/j} c_j \Lambda(n) \right) e(axy) \right| \ll L^4(JNq^{-1/2} + (JNM)^{1/2} + JNM^{-1/2} + (JNq)^{1/2}), \]
so that, by (13),
\[ \sum_{N^{1/2} \leq M < Nu^{-1}} S(M) \ll L^5(JNq^{-1/2} + JN^{3/4} + (JNq)^{1/2} + J^{3/5} N^{4/5}). \]

This with (15) and (16) gives a suitable estimate for
\[ \sum_{J \leq j < J'} |S_3|. \]

It remains to consider $S_1$ and $S_2$. By Lemma 3,
\[ \sum_{J \leq j < J'} |S_2| \ll L \sum_{x \leq J' u^2} \max_{Z \leq J'/u^2} \left| \sum_{y \leq Z} \left( \sum_{J \leq j < J'} \sum_{u < n \leq N/j} 1 \right) e(axy) \right| \ll \sum_{x \leq J' u^2} \max_{J \leq j < J'} \left( \sum_{u < n \leq N/j} 1 \right). \]
By (12) and (13) this is
\[ \ll L^2(JNq^{-1/2} + DJ^{3/5} N^{4/5} + qJ^{1/2}(N/q)^{1/2}). \]

Similarly
\[ \sum_{j} |S_j| \ll \sum_{x \in J} \left( \sum_{j_x} \right) \max_{z \in J/N/x} \left\{ \frac{1}{w} \sum_{y \leq z} e(xy) dw \right\} \]
\[ \ll L^2(JNq^{-1} + Ju + q) \max_{x \in J} \left( \sum_{j_x} \right) \]
\[ \ll L^2(JNq^{-1/2} + JN^{3/4} + (JNq)^{1/2}). \]

§4. Proof of Theorem 2. The function \( f_\alpha(\theta) - 2\Delta \) is well known to have the expansion
\[ \sum_{1 \leq |n| \leq H} \frac{\sin 2\pi h \Delta}{\pi h} e(h\theta) + O\left( \min \left( 1, \frac{1}{H\|\theta + \Delta\|} \right) + \min \left( 1, \frac{1}{H\|\theta - \Delta\|} \right) \right). \]

Hence
\[ \sum_{n=1}^{N} \Lambda(n)(f_\alpha(xn - \beta) - 2\Delta) \ll \sum_{h=1}^{H} \min \left( \Delta, \frac{1}{h} \right) \left| \sum_{n=1}^{N} \Lambda(n) e(xhn) \right| \]
\[ + \mathcal{L} \sum_{n=1}^{N} \left( \min \left( 1, \frac{1}{H\|xn - \beta + \Delta\|} \right) + \min \left( 1, \frac{1}{H\|xn - \beta - \Delta\|} \right) \right). \quad (17) \]

We take \( H = 1 + [Nq/\Delta] \). Then, by (7), we have
\[ \mathcal{L} \sum_{n=1}^{N} \left( \min \left( 1, \frac{1}{H\|xn - \beta + \Delta\|} \right) + \min \left( 1, \frac{1}{H\|xn - \beta - \Delta\|} \right) \right) \ll \mathcal{L}^2(Nq^{-1} + 1). \]

Thus it remains to estimate the first expression on the right of (17). Let
\[ S(u) = \sum_{h < u} \left| \sum_{n=1}^{N} \Lambda(n) e(xhn) \right|. \]

Then
\[ \sum_{h=1}^{H} \min \left( \Delta, \frac{1}{h} \right) \left| \sum_{n=1}^{N} \Lambda(n) e(xhn) \right| = \frac{S(H)}{H} + \int_{\Delta^{-1}}^{H} \frac{S(u)}{u^2} du, \]
and the desired conclusion follows easily from Theorem 1.

§5. Proofs of the corollaries. To prove Corollary 2.1, let \( C \) be a sufficiently large constant and let \( a/q \), with \( q \) arbitrarily large and \((a, q) = 1\), be a convergent to the continued fraction for \( x \). Then \(|x - a/q| \leq q^{-2}\). Take \( N = q^2 \), \( \Delta = C(\log N)^8 N^{-1/4} \).
and $\varepsilon = 1/100$ in Theorem 2. Hence
\[
\sum_{n=1}^{N} \Lambda(n)(\alpha n - \beta - 2\Delta) \leq \mathcal{O}(N^{3/4} + C^{1/2} N^{5/8}) + C^{2/5} N^{7/10 + 2\varepsilon},
\]
so that, by the prime number theorem
\[
\sum_{p \leq N} f_{\Delta}(\alpha p - \beta) \log p > \Delta N.
\]

The second corollary is trivial when $\delta = 1$ (for then $\gamma = 0$). Thus we may suppose that $\delta < 1$. Now on taking $\beta = \gamma + \delta/2$, $\Delta = \delta/2$ the hypothesis of Theorem 2 is satisfied. Moreover $f_{\Delta}(\alpha n - \beta) = 1$, if, and only if, there is an integer $h$ such that
\[-\Delta \leq \alpha n - \beta - h < \Delta,\]
i.e. such that $\gamma \leq \alpha n - h < \gamma + \delta$. But then $h \leq \alpha n < h + 1$ so that $h = \lfloor \alpha n \rfloor$. Therefore, by Theorem 2,
\[
\sum_{p \leq N} \log p - \delta \sum_{p \leq N, \gamma \leq \{\alpha p\} < \gamma + \delta} \log p \leq \mathcal{O}(N^{-1/2} + N^{3/4} + (\delta Nq)^{1/2}) + \delta^{2/5} N^{4/5}(Nq/\delta)^{4},
\]
and the conclusion now follows by partial summation.

References


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