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Analysis of a Stochastic Implicit Interface Model for an Immersed Elastic Surface in a Fluctuating Fluid

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Abstract

We present some mathematical analyses of a recently proposed stochastic implicit interface model for an elastic surface immersed in an incompressible viscous fluid subject to fluctuation forces. We derive suitable a priori estimates and establish the well-posedness of pathwise solutions and provide uniform control on the solutions in probability.

1. Introduction

With the growing scientific and technological interest in microfluidic devices and complex fluids, the study of effects of microscopic fluctuations on macroscopic properties is becoming an important research topic, especially for systems that involve structures or interfaces immersed in a fluid environment. For instance, it is widely known that the effects of thermal fluctuations are very important for various biological processes and functions, such as a lipid bilayer vesicle immersed in a solution [40]. A number of approaches have been developed in recent years to model the interactions of an immersed interface in a fluid environment subject to fluctuation forces, such as discrete particle-based Brownian dynamics [17], Stokesian dynamics [4] and dissipative particle dynamics [22], interfacial Langevin equations [27], and the stochastic immersed boundary methods (SIBMs) [2,28]. In an earlier work [12], based on the fluctuation-dissipation theorem and motivated by the SIBMs, we derived a stochastic implicit interface model (SIIM) for an immersed interface in an incompressible viscous fluid subject to fluctuating forces. Within the framework, the small scale fluctuations are modeled by random noises in the macroscopic field equations, while the immersed interface is described implicitly by a phase field function or a level-set function. To consider the near-equilibrium

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fluctuations, we arrived at a system of stochastic PDEs with the phase field or level set functions being governed by a transport equation, while the random fluctuation term appear only in the momentum equation for the fluid velocity field in a suitable functional form. The implicit interface representation in the SIIM is a particularly attractive feature for the design of effective numerical methods to simulate fluctuating interfaces that take complex geometric shapes and are subject to topological changes.

In many microfluidic studies, one is working with an incompressible viscous fluid in the low Reynolds number regime. In such cases, if we let u be the fluid velocity, p the pressure, and ϕ the phase field/level-set function for the immersed interface, then one particular form of the SIIM is given by

$$\begin{cases} u_t - \mu\triangle u + \nabla p = (\nabla E[\phi]\nabla\phi) * \zeta + f, \\ \nabla \cdot u = 0, \\ \phi_t + (u * \zeta) \cdot \nabla\phi = 0, \end{cases}$$

together with appropriate initial and boundary conditions for u and ϕ. Here, μ is the viscosity. $E = E[\phi]$ represents an interfacial energy with $\nabla E[\phi]$ being a simple notation for the variational derivative of $E[\phi]$. $\nabla E[\phi]\nabla\phi$ accounts for the effect of the interface deformation on the fluid. ζ is a spatial averaging kernel or, equivalently, a spatial mollification function, and $*$ denotes the spatial convolution. f is a suitably defined stochastic force. Spatial mollification has been naturally used in the stochastic immersed boundary method [2]. We refer to Section 2 for a brief discussion on the derivation the above SIIM model. Without loss of generality, we neglect the effect of other external body forces in the momentum equation. To get a simpler mathematical expression of the stochastic force term and to avoid the extra complications associated with the boundary effects, we consider periodic boundary conditions for both u and ϕ throughout the paper.

Given the nonlinear nature of the SIIM and the contributions due to the noise term, it is of both theoretical and practical interest to study its well-posedness. By confining ourselves to the low Reynolds number regime, where we can work with the linear Stokes equations instead of the full nonlinear Navier–Stokes equations, we bypass many analytical issues often associated with the stochastic Navier–Stokes equations (see [19,34,35] and the references cited therein). This allows us to focus on the nonlinear coupling between the elastic interface and the underlying fluid environment, instead. In addition, due to the spatial averaging (smoothing) effect of the mollification (which is often performed in the practical application of the related immersed boundary methods, see [2,28,38]), it turns out that suitable controls of nonlinearity can be achieved, despite the weak regularity in time and the lack of explicit dissipation terms in the transport equation. In the present work, we establish suitable energy laws which allow us to prove the basic well-posedness theorems for both the deterministic and the stochastic implicit interface models (SIIM). We note that not only was there no well-posedness result on the SIIM in the past literature, even the result for the deterministic case given in our work is new, in the sense that the well-posedness of solutions have often been shown in the existing literature with the help of extra damping terms in the transport equation in order to treat the full nonlinearity in the coupled system [13,32]. In addition, uniform bounds on the solutions are derived in a suitable probabilistic sense for the stochastic models. We also illustrate that our techniques can be used for the SIIMs corresponding to various forms of the interfacial energy functionals, which can represent either conventional interface tension or bending elastic energy. The analytical results given here provide a sound mathematical basis for the validity of the stochastic systems upon which further mathematical studies can be carried out. For example, similar analyses can be applied to some numerical algorithms for the approximations of the SIIM. Other issues, such as the an appropriate sharp interface limit and long time asymptotic behavior of the solutions, can also be further examined.

The paper is organized as follows, we review the SIIM along with the related physical background and mathematical problems in Section 2; a consistent derivation of the model using a mollified velocity field is also briefly presented. We then present the analysis of the corresponding deterministic model and prove its well-posedness (Theorem 1) in Section 3, where a result is shown without imposing any extra damping directly to the transport equation for the phase field/level set function. The analysis of the stochastic model is given in Section 4, including the main existence and uniqueness theorems for a SIIM with a phase field surface tension energy (Theorem 2) and for one with a phase field bending elasticity energy (Theorem 4). An example of probabilistic uniform bounds on the stochastic solutions is also proved (Theorem 3). Some final remarks are given in Section 5.

2. The stochastic implicit interface model

To motivate the study of the SIIM and its various examples considered in this paper, we first consider some application problems. It is known that many interfacial problems can be effectively modeled by implicit interface representations, such as the level set methods and phase field or diffuse interface methods [1,36]. For instance, the phase field method has been a convenient approach for modeling many interfacial problems. In general, a phase field method uses a smooth phase field function to distinguish two regions, or phases, in the physical domain. For instance, a phase field function might take a value of 1 in one phase and -1 in the other. The interface between the two phases is characterized by a thin transition layer around the zero level set of the phase field function, thus leading to a diffuse interface description.

An advantage of using an implicit interface formulation is that it avoids explicitly tracking or constructing the interface between the two phases. The dynamics and energetics of the interfaces are encoded in the dynamics and variational calculus of the phase field functions [24]. As a particular motivation to the current work, we recall a recent example given by the phase field vesicle–fluid interaction model for the deformation of a bio-membrane vesicle in fluid [16]. The thermal fluctuations of fluid-like vesicles immersed in a solution have prompted a number of theoretical and experimental studies [11,40,41,43]. In the absence of fluctuation effects, a vesicle membrane Γ, formed by a lipid bilayer, has an equilibrium configuration often characterized by the minimizer of the elastic bending energy [37]. Although more general forms of the bending energy can be considered, for illustration we focus on the following form [42]:

$$E_b = \int_\Gamma \frac{\kappa}{2}(H - c_0)^2 \mathrm{d}S, \tag{1}$$

where Γ is the surface of vesicle membrane, H is the mean curvature of Γ, c_0 is the spontaneous curvature and κ is the bending modulus. In our earlier works, a phase field function ϕ is used to represent the vesicle configuration as a labeling function with ϕ taking a value of nearly $+1$ inside the vesicle membrane, and -1 outside. The width of the thin transition layer is characterized by a small positive parameter ϵ. The elastic bending energy (1) is given by [14],

$$E_b[\phi] = \frac{\kappa}{2\epsilon} \int_\Omega \left(\epsilon \Delta \phi + \frac{1}{\epsilon} \phi (1 - \phi^2) \right)^2 \mathrm{d}x, \tag{2}$$

where Ω is the computational domain that encloses the interface of interest. For the sharp interface analysis of the functional, we refer to [15,39,45]. An even simpler but better known energy functional is given by

$$E[\phi] = \int_\Omega \left(\frac{\epsilon}{2} |\nabla \phi|^2 + \frac{1}{4\epsilon} (\phi^2 - 1)^2 \right) \mathrm{d}x, \tag{3}$$

which has often been used to model the dimensionless interfacial tension (interfacial area) [1].

Given an interfacial energy $E = E[\phi]$ (or $E = E_b[\Phi]$) defined for the phase field function ϕ, the deterministic phase field interface–fluid interaction model is derived through the least action principle in [16]. For the fixed time interval $[0, T_0]$, the least action principle [5] asserts the fluid flow particle trajectory $x(t, \alpha)$ minimizes the action functional,

$$A[x(t, \alpha)] = \int_0^{T_0} \left(\int_\Omega \frac{1}{2} |u(t, x)|^2 \mathrm{d}x - E[\phi] \right) \mathrm{d}t.$$

The fluid velocity field $u(t, x)$ is defined, in an implicit fashion, by

$$\begin{cases} \dot{x}(t, \alpha) = u(t, x(t, \alpha)) \\ x(0, \alpha) = \alpha, \end{cases}$$

and the phase field function is transported by the fluid flow via

$$\phi_t + u \cdot \nabla \phi = 0,$$

which is implicitly determined by $x(t, \alpha)$ as well. Via suitable variations to the trajectory $x = x(t, \alpha)$, the deterministic phase field vesicle–fluid interaction equations [13,16] can then be derived as:

$$\begin{cases} u_t + u \cdot \nabla u = \mu \Delta u + \nabla p + \nabla E[\phi] \nabla \phi, \\ \phi_t + u \cdot \nabla \phi = 0, \\ \nabla \cdot u = 0. \end{cases} \tag{4}$$

For convenience, $\nabla E[\phi]$ is used here and throughout the paper as a simple notation for the variational derivative of $E[\phi]$ with respect to the variation in ϕ, though we allow the functional E to depend not only on ϕ but also on its spatial derivatives, so $\nabla E[\phi]$ should also account for variations to the derivatives of ϕ (see examples provided later). Note that the general form of the equation (4) is independent of the specific choice of the energy $E[\phi]$. The derivation is thus applicable to more general settings where the interfacial energy $E = E[\phi]$ may represent other physical energies, such as surface tension, or it may be given in a level-set formulation rather than in a diffuse interface representation [12].

The deterministic equation (4) formally has a dissipative energy law,

$$\frac{d}{\mathrm{d}t}\left(\int_\Omega \frac{1}{2}|u(t,x)|^2\mathrm{d}x + E[\phi]\right) = -\mu\int_\Omega |\nabla u(t,x)|^2\mathrm{d}x.$$

Assume that ϕ_0 is a local minimizer of $E[\phi]$, and the Hessian of $E[\phi]$ at ϕ_0 is positive definite. Then $(u,\phi)\equiv(0,\phi_0)$ gives a stable steady state of the deterministic system (4).

To account for the effect of thermal fluctuation, in [12], we considered the stochastic extension of (4), which is referred to here as the stochastic implicit interface model, following the framework of the SIBMs [2,29]. The derivation of SIIM can be done in much more general terms, which we demonstrate here, by the transport via a mollified velocity field. We note that the use of a spatially mollified (averaged) velocity, a technique going back to Leray [31] for the analysis of Navier–Stokes equations, has been a standard practice in the immersed boundary methods. For simplicity, we assume that an averaging (smoothing) kernel ζ satisfies that ζ is a nonnegative, compactly supported, and radially symmetric kernel, that is $\zeta(x)=\zeta(|x|)$. Additional assumptions on ζ are made in the later theorems for the simplicity in technical derivations.

Let $*$ denote the standard convolution operator. We define ϕ formally via the following transport equation with the mollified velocity $u*\zeta$:

$$\begin{cases} \phi_t + (u*\zeta)\cdot\nabla\phi = 0, \\ \phi(0)=\phi_0. \end{cases} \tag{5}$$

Then, the mollified flow map $z(t,\alpha)$ is determined by

$$\begin{cases} \dot{z}(t,\alpha) = \displaystyle\int_\Omega u(t,y)\zeta(z(t,\alpha)-y)\mathrm{d}y, \\ z(0,\alpha)=\alpha. \end{cases}$$

As we are mainly concerned with boundary conditions, the kernel ζ is periodically extended and Ω then denotes the basic periodic cell. Since u is divergence free, we see that $z(t,\alpha)$ is a volume preserving map. The transport equation (5) then implies that $\phi(t,x)=\phi_0(z^{-1}(t,x))$. We have

Lemma 1. *The variation of ϕ is given by $\delta\phi = -\nabla\phi\cdot\delta z$.*

Proof. $\phi(t,x) = \phi_0(z^{-1}(t,x))$ implies

$$\delta\phi = \nabla_\alpha \phi_0(z^{-1}(t,x)) \cdot \delta z^{-1}(t,x).$$

Since $z^{-1}(t, z(t,\alpha)) = \alpha$,

$$\delta z^{-1}(t, z(t,\alpha)) + \nabla_x z^{-1}(t, z(t,\alpha))\delta z(t,\alpha) = 0.$$

It follows immediately that

$$\delta z^{-1}(t,x) = -\nabla_x z^{-1}(t,x)\delta z.$$

Therefore,

$$\begin{aligned}\delta\phi &= -\nabla_\alpha \phi_0(z^{-1}(t,x))\, \nabla_x z^{-1}(t,x)\, \delta z \\ &= -\nabla_x \phi(t,x)\delta z.\end{aligned}$$

Lemma 2. *The variation of* z *is given by* $\delta z = \delta x * \zeta$*, thereby* $\delta\phi = -\nabla\phi \cdot (\delta x * \zeta)$.

Proof. In $\dot z(t,\alpha) = \int_\Omega u(t,y)\zeta(z(t,\alpha) - y)\mathrm{d}y$, fix $\alpha = \alpha_0$. We carry out the computation in the Lagrangian coordinate.

$$\begin{aligned}\frac{d}{\mathrm{d}t}\delta z(t,\alpha_0) &= \int_\Omega \frac{d}{\mathrm{d}t}\delta x(t,\alpha)\zeta(z(t,\alpha_0) - x(t,\alpha))\mathrm{d}\alpha \\ &\quad + \int_\Omega \dot x(t,\alpha)\nabla\zeta(z(t,\alpha_0) - x(t,\alpha))(\delta z(t,\alpha_0) - \delta x(t,\alpha))\mathrm{d}\alpha \\ &= \frac{d}{\mathrm{d}t}\int_\Omega \delta x(t,\alpha)\zeta(z(t,\alpha_0) - x(t,\alpha))\mathrm{d}\alpha \\ &\quad - \int_\Omega \delta x(t,\alpha)\nabla\zeta(z(t,\alpha_0) - x(t,\alpha))(\dot z(t,\alpha_0) - \dot x(t,\alpha))\mathrm{d}\alpha \\ &\quad + \int_\Omega \dot x(t,\alpha)\nabla\zeta(z(t,\alpha_0) - x(t,\alpha))(\delta z(t,\alpha_0) - \delta x(t,\alpha))\mathrm{d}\alpha \\ &= \frac{d}{\mathrm{d}t}\int_\Omega \delta x(t,\alpha)\zeta(z(t,\alpha_0) - x(t,\alpha))\mathrm{d}\alpha \\ &\quad - \dot z(t,\alpha_0)\int_\Omega \delta x(t,\alpha)\nabla\zeta(z(t,\alpha_0) - x(t,\alpha))\mathrm{d}\alpha \\ &\quad + \delta z(t,\alpha_0)\int_\Omega \dot x(t,\alpha)\nabla\zeta(z(t,\alpha_0) - x(t,\alpha))\mathrm{d}\alpha \\ &= \frac{d}{\mathrm{d}t}\int_\Omega \delta x(t,\alpha)\zeta(z(t,\alpha_0) - x(t,\alpha))\mathrm{d}\alpha - \dot z(t,\alpha_0)I_1 + \delta z(t,\alpha_0)I_2.\end{aligned}$$

We then convert the integrals I_1 and I_2 back to Eulerian coordinates. Note that $\mathrm{div}\delta x = 0$, we have

$$\begin{aligned}I_1 &= -\int_\Omega \delta x(t,y)\nabla_y\zeta(z(t,\alpha_0) - y)\mathrm{d}y \\ &= \int_\Omega \mathrm{div}\, \delta x(t,y)\zeta(z(t,\alpha_0) - y)\mathrm{d}y \\ &= 0\end{aligned}$$

and

$$
\begin{aligned}
I_2 &= -\int_{\Omega} u(t, y)\nabla_y \zeta(z(t,\alpha_0) - y)\mathrm{d}y \\
&= \int_{\Omega} \operatorname{div} u(t, y)\zeta(z(t,\alpha_0) - y)\mathrm{d}y \\
&= 0.
\end{aligned}
$$

Since $\delta z(0) = \delta x(0) = 0$, we conclude that

$$
\delta z(t, \alpha_0) = \int_{\Omega} \delta x(t, \alpha)\zeta(z(t, \alpha_0) - x(t, \alpha))\mathrm{d}\alpha
$$

holds for any α_0. That is,

$$
\delta z = \delta x * \zeta \text{ and } \delta\phi = -\nabla\phi \cdot (\delta x * \zeta).
$$

Consider the level set or phase field function ϕ being transported via the mollified fluid velocity and the fluid being in the low Reynolds number regime; we have the fluid velocity field u and ϕ satisfying the following coupled system of equations,

$$
\begin{cases}
u_t = \mu\triangle u + (\nabla E[\phi]\nabla\phi) * \zeta + \nabla p, \\
\nabla \cdot u = 0, \\
\phi_t + (u * \zeta) \cdot \nabla\phi = 0,
\end{cases}
\tag{6}
$$

where $\nabla E[\phi]$ is a simple notation for the variational derivative of $E[\phi]$.

To account for the near-equilibrium effect of the stochastic fluctuation, we have the following equations for the SIIM: the phase field or level set function is transported via a mollified velocity field, together with a suitable noise term being added to the momentum equation. That is,

$$
\begin{cases}
u_t = \mu\triangle u + (\nabla E[\phi]\nabla\phi) * \zeta + \nabla p + f, \\
\nabla \cdot u = 0, \\
\phi_t + (u * \zeta) \cdot \nabla\phi = 0,
\end{cases}
\tag{7}
$$

where the noise f is delta-correlated in time and in Fourier expansion of the spatial variables, such that

$$
\langle P_k \hat{f}_k(t) \otimes \overline{P_{k'} \hat{f}_{k'}(t')} \rangle = 8\pi^2 \mu K_b T |k|^2 P_k \delta_{k,k'} \delta(t - t') \tag{8}
$$

for any t, t' and the wave vectors k, k'. Here, K_b is the Boltzmann constant, T is the temperature, $\delta_{k,k'}$ and $\delta(t - t')$ are the standard Dirac-Delta functions and the operators

$$
P_k = I - \frac{k \otimes k}{|k|^2} \tag{9}
$$

for all wave vectors k are projections to the divergence-free spaces. The derivation of this particular noise term f and the fact that there is no noise term in the transport equation are consequences of the fluctuation-dissipation theorem as demonstrated in [12], which remains valid for the case of transport via the mollified velocity. One may check that the Boltzmann distribution corresponds to an invariant measure induced by the above coupled nonlinear stochastic system. Moreover, the noise can also be represented in more familiar terms [30], for instance, in 2D case, it could be expressed as $\nabla \cdot S$ where $S = (s_{i,j}(t, x))$ is a 2×2 random stress tensor, with the correlation structure

$$\langle s_{i,j}(t, x) s_{m,l}(t', x') \rangle = \delta(t - t')\delta(x - x')(\delta_{im}\delta_{jl} + \delta_{il}\delta_{jm}).$$

Again, we may elect to use either the phase field or the level set formulation for the interfacial energy term $E = E[\phi]$ in the above equations of SIIM, see [12] for examples of other forms of the model equations.

3. Analysis of the deterministic model

In this section, we prove the well-posedness of the deterministic model (6) with a periodic boundary condition. Our approach is based on a Galerkin approximation, and the key is to derive proper a priori estimates of the solution. Such estimates depend on the nonlinear force density $\nabla E[\phi]\nabla\phi$ and the regularity of kernel ζ. For notational convenience, we often use the notation $f_\zeta \triangleq f * \zeta$ to represent the convolution when there is no ambiguity.

Let us first work with the energy,

$$E[\phi] = \int_\Omega \frac{\epsilon}{2}|\nabla\phi|^2 + \frac{1}{4\epsilon}(\phi^2 - 1)^2 \mathrm{d}x,$$

which has a relatively simpler form, to illustrate the idea. Such an energy is often used to model the surface tension of an immersed structure. The corresponding force density is

$$(\nabla E[\phi]\nabla\phi) * \zeta = \left(-\epsilon\Delta\phi\nabla\phi + \frac{1}{\epsilon}(\phi^3 - \phi)\nabla\phi\right) * \zeta.$$

Since the variation of the second term in the energy

$$\left(\frac{1}{\epsilon}(\phi^3 - \phi)\nabla\phi\right) * \zeta = \nabla\left(\left(\frac{1}{4\epsilon}(\phi^2 - 1)^2\right) * \zeta\right) \tag{10}$$

is a gradient term, it can be included in the pressure, so we have

$$\begin{cases} u_t = \mu\Delta u - (\epsilon\Delta\phi\nabla\phi) * \zeta + \nabla p, \\ \nabla \cdot u = 0, \\ \phi_t + (u * \zeta) \cdot \nabla\phi = 0. \end{cases} \tag{11}$$

Moreover, for any polynomial $f = f(x)$,

$$\begin{aligned}\frac{d}{\mathrm{d}t}\int_\Omega f(\phi)\,\mathrm{d}x &= \int_\Omega f'(\phi)\partial_t\phi\,\mathrm{d}x\\ &= -\int_\Omega f'(\phi)\nabla\phi\cdot(u*\zeta)\,\mathrm{d}x\\ &= -\int_\Omega \nabla f(\phi)\cdot(u*\zeta)\,\mathrm{d}x\\ &= 0\end{aligned}$$

Particularly,

$$\frac{d}{\mathrm{d}t}\int_\Omega \frac{1}{4\epsilon}(\phi^2-1)^2\,\mathrm{d}x = 0.$$

Thus, we may effectively control the energy $E = E[\phi]$ as long as proper estimates on the Dirichlet integral type energy

$$\int_\Omega \frac{\epsilon}{2}|\nabla\phi|^2\,\mathrm{d}x \tag{12}$$

can be derived from the model equations.

First, let us state the main analytical result concerning the well-posedness of equation (11).

Theorem 1. (Well-posedness theorem for a deterministic model) *Let Ω be a unit cell of periodic domain in $\mathbb{R}^N$ with $N \leqq 3$. Under the condition that the kernel $\zeta \in C^2$, the initial condition $u_0 \in H_d(\Omega)$, $\phi_0 \in H^2(\Omega)$, there exists a unique pair of functions u and ϕ with*

1. *$u \in L^2(0, T; H_d(\Omega))$ and $\partial_t u \in L^2(0, T; H_d^{-1}(\Omega))$,*
2. *$\phi \in L^2(0, T; H^2(\Omega))$ and $\partial_t\phi \in L^2(0, T; L^2(\Omega))$,*

such that (u, ϕ) is a weak solution to the system (11) with the initial condition $u(0, x) = u_0(x)$ and $\phi(0, x) = \phi_0(x)$.

In the above theorem, $H_d(\Omega)$ is the subspace of divergence free vector fields in $[H^1(\Omega)]^N$ whose average is the zero vector, and $H_d^{-1}(\Omega)$ is the dual space of $H_d(\Omega)$. As will be shown later, there exists a stress τ such that the force term $\nabla E[\phi]\nabla\phi = \nabla\cdot\tau$, so the average of u is thus a constant vector in time and space. Without loss of generality, we restrict the initial condition to have average zero, so that the average of u remains a zero vector at a later time, which would allow us to conveniently use $\|\nabla v\|_{L^2}$ as a norm of $v \in H_d(\Omega)$. We also note that previous analyses of the well-posedness studies of the phase-field Navier–Stokes equations are often carried out for systems with extra damping in order to treat the full non-linearity of the coupled system. [13,18,32]. For the case of a pure transport of a director field, (that is, ϕ being vector-valued) and without applying any velocity mollification, the local well-posedness of smooth enough solutions and the global well-posedness of weak solutions for initial conditions that are sufficiently close to the equilibrium state have been established in [33], even with the presence of the full nonlinear convective term $u \cdot \nabla u$ in the momentum equation. The key idea there is to obtain a suitable decay estimate for the transport equation through a change of variable. As our objective is to eventually work with the stochastic version of (11), a simpler proof of the above theorem is presented here which works for general initial conditions.

To better organize the proof, we first prepare several formal estimates on the solution then prove the theorem by applying the Galerkin procedure and passing to the limit.

3.1. Formal estimates on the solutions

We begin with a couple of formal estimates for the transport equation and for the nonlinear force density in the momentum equation. We use $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty}$ to denote norms in the spatial variables. The following lemma has the same spirit as the discussion in [6].

Lemma 3. (Estimate of transport equation) *Let $u = u(t, x)$ be a divergence-free velocity field. If ϕ is defined by*

$$\begin{cases} \phi_t + u \cdot \nabla \phi = 0, \\ \phi(0) = \phi_0(x), \end{cases}$$

then, there exists some constant C such that

1. *for any $t \in (0, T)$,*

$$\|\phi_t(t)\|_{L^2} \leqq \|u(t)\|_{L^\infty} \cdot \|\nabla\phi(t)\|_{L^2},$$

2. *for $u \in L^1(0, T; W^{1,\infty}(\Omega))$,*

$$\|\nabla\phi(t)\|_{L^2}^2 \leqq \|\nabla\phi_0\|_{L^2}^2 \exp\left(C \int_0^t \|\nabla u(s)\|_{L^\infty}\, ds\right),$$

3. *for $u \in L^1(0, T; W^{2,\infty}(\Omega))$,*

$$\|\Delta\phi(t)\|_{L^2}^2 \leqq \|\Delta\phi_0\|_{L^2}^2 \exp\left(C \int_0^t \|\Delta u\|_{L^\infty} + \|\nabla u\|_{L^\infty}\, ds\right).$$

Proof. The first result is obvious. Now, differentiating $\phi_t + u \cdot \nabla\phi = 0$, we have

$$\nabla\phi_t + u \cdot \nabla(\nabla\phi) = -\nabla u \cdot \nabla\phi.$$

Then, multiplying the above equation by $\nabla\phi$ and integrating by parts, we get

$$\begin{aligned} \frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\phi\|_{L^2}^2\right) &= -\int_\Omega \nabla u \cdot \nabla\phi \cdot \nabla\phi \, \mathrm{d}x \\ &\leqq C \cdot \|\nabla u\|_{L^\infty} \|\nabla\phi\|_{L^2}^2. \end{aligned}$$

Similarly,

$$\triangle\phi_t + u\cdot\nabla\triangle\phi = -\triangle u\cdot\nabla\phi - 2\nabla u\cdot(\nabla\nabla\phi),$$

which implies

$$\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|\triangle\phi\|_{L^2}^2\right) \leqq C\cdot(\|\triangle u\|_{L^\infty} + \|\nabla u\|_{L^\infty})\|\triangle\phi\|_{L^2}^2.$$

Then the Gronwall inequality implies results 2 and 3 of the lemma.

In the above and hereafter, we use the notation C to denote a generic positive constant, as long as there is no ambiguity.

Lemma 4. (Estimate on nonlinear force density) *There exists some constant C, such that*

$$\|(\triangle\phi\nabla\phi)*\zeta\|_{H^{-1}} \leqq C\cdot\|\nabla\phi\|_{L^2}^2.$$

Proof. First, we note that $\triangle\phi\nabla\phi = \nabla\cdot\tau$, where $\tau = \nabla\phi\otimes\nabla\phi - \frac{1}{2}|\nabla\phi|^2$.

For any test function $\eta\in H_d^1(\Omega)$, we have

$$\begin{aligned}\left|\int_\Omega(\triangle\phi\nabla\phi)*\zeta\cdot\eta\,\mathrm{d}x\right| &= \left|\int_\Omega\nabla\cdot\tau\;\eta_\zeta\,\mathrm{d}x\right|\\ &= \left|\int_\Omega\tau\cdot\nabla\eta_\zeta\,\mathrm{d}x\right|\\ &\leqq \|\nabla\eta_\zeta\|_{L^\infty}\int_\Omega|\tau|\,\mathrm{d}x\\ &\leqq C\cdot\|\eta\|_{H_d^1}\cdot\|\nabla\phi\|_{L^2}^2.\end{aligned}$$

Then, the conclusion follows.

3.2. Proof of the well-posedness theorem

Proof. 1. Let $S_n = span\{\omega_k(x),\ k\leqq n\}$, the finite dimensional space spanned by the eigenfunctions $\{\omega_k(x)\}$ of the Stokes operator with periodic condition on the domain Ω. Π_n is the L^2 projection to S_n. We construct the approximate equation as

$$\begin{cases}\partial_t u = \mu\triangle u + \Pi_n[(\nabla E[\phi]\cdot\nabla\phi)*\zeta],\\ \partial_t\phi + u_\zeta\cdot\nabla\phi = 0,\\ \nabla\cdot u = 0.\\ u(0,x) = \Pi_n u_0(x),\\ \phi(0,x) = \phi_0(x).\end{cases}\tag{13}$$

The velocity field u is naturally restricted in S_n as an approximate solution. The divergence-free condition $\nabla\cdot u = 0$ implies $\nabla\cdot(u*\zeta) = 0$, so $u*\zeta$ is also a volume preserving map. We then use the approximate u and the transport equation to get the approximate solutions for ϕ. Given the regularity on ϕ_0, the transport of ϕ is via a mollified velocity field and Lemma 3, by a standard Picard iteration, the local solution exits (see [33] for similar discussion). Furthermore, we have

$$\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|u\|_{L^2}^2 + E[\phi]\right) = -\mu \int_\Omega |\nabla u|^2 \mathrm{d}x. \tag{14}$$

So,

$$\|u(t)\|_{L^2} \leqq \|u(0)\|_{L^2}, \quad \forall\, 0 \leqq t < \infty,$$

which implies the globe existence of $u \in S_n$ and hence the corresponding ϕ.

2. For each S_n, let u_n and ϕ_n be the solution to the system (13). By the energy law (14), we have that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$ and $L^2(0, T; H_d^1(\Omega))$.
Next, by the properties of the kernel ζ, we have

$$\|\nabla u_n * \zeta\|_{L^\infty} \leqq C'\|u_n\|_{L^2}.$$

Then, according to Lemma 3, there exist some constants C, C' such that

$$\begin{aligned}\|\nabla\phi_n(t)\|_{L^2}^2 &\leqq \|\nabla\phi_0\|_{L^2}^2 \exp\left(C\int_0^t \|\nabla(u_n * \zeta)(s)\|_{L^\infty}\,\mathrm{d}s\right)\\ &\leqq \|\nabla\phi_0\|_{L^2}^2 \exp\left(CC'\int_0^t \|u_n(s)\|_{L^2}\,\mathrm{d}s\right).\end{aligned}$$

So, ϕ_n is uniformly bounded in $L^\infty(0, T; H^1(\Omega))$.
By repeating a similar argument, we may get that ϕ_n is also uniformly bounded in $L^\infty(0, T; H^2(\Omega))$. This in turn implies that $\partial_t\phi_n$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$.
Following from Lemma 4, we have $(\triangle\phi\nabla\phi) * \zeta$ is uniformly bounded in $L^\infty(0, T; H_d^{-1}(\Omega))$. So, $\partial_t u_n$ is uniformly bounded in $L^2(0, T; H_d^{-1}(\Omega))$.
Consequently, by the Aubin–Lions compactness result, there exists a subsequence $\{(u_{n_k}, \phi_{n_k})\}$ and (u, ϕ) such that
   - $u_{n_k} \rightharpoonup u$ weakly in $L^2(0, T; H^1(\Omega))$;
   - $u_{n_k} \to u$ strongly in $L^2(0, T; L^p(\Omega))$, for $1 \leqq p < 6$;
   - $\phi_{n_k} \rightharpoonup \phi$ weakly in $L^2(0, T; H^2(\Omega))$;
   - $\phi_{n_k} \to \phi$ strongly in $L^2(0, T; W^{1,p}(\Omega))$, for $1 \leqq p < 6$.
3. Let $\alpha = \alpha(t) \in C^1([0, T])$, for any test functions $\nu = \nu(x) \in S_n$ and $\eta = \eta(x) \in L^2(\Omega)$, we have

$$\begin{aligned}\int_0^T \alpha\left(\partial_t u_{n_k},\ \nu\right)\mathrm{d}t = &-\int_0^T \alpha\left(\nabla u_{n_k},\ \nabla\nu\right)\,\mathrm{d}t\\ &-\epsilon\int_0^T\int_\Omega \alpha\triangle\phi_{n_k}\nabla\phi_{n_k}\,\nu_\zeta\,\mathrm{d}x\,\mathrm{d}t,\\ \int_0^T \alpha(\partial_t\phi_{n_k},\ \eta)\,\mathrm{d}t = &\int_0^T \alpha\int_\Omega (u_{n_k} * \zeta)\,\nabla\phi_{n_k}\,\eta\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Let $n_k \to \infty$, we get

$$\int_\Omega u_t \nu \,\mathrm{d}x = -\mu \int_\Omega \nabla u \cdot \nabla \nu \mathrm{d}x - \epsilon \int_\Omega (\triangle\phi\nabla\phi) * \zeta \cdot \nu \,\mathrm{d}x,$$
$$\int_\Omega \phi_t \eta \mathrm{d}x + \int_\Omega (u * \zeta) \cdot \nabla\phi\, \eta \,\mathrm{d}x = 0.$$

Since $\nu = \nu(x)$ is arbitrarily chosen in S_n, by a density argument the above equations remain valid for any $\nu \in H_d(\Omega)$, and one can also check that the initial condition is satisfied.

4. For uniqueness, assume that there exist two solutions (u_1, ϕ_1) and (u_2, ϕ_2) with the same initial condition. Let $\tilde{u} = u_1 - u_2$, $\tilde{\phi} = \phi_1 - \phi_2$. Then $\tilde{u}, \tilde{\phi}$ satisfy the equation

$$\begin{cases} \tilde{u}_t = \mu\nabla\tilde{u} + \nabla\tilde{p} - \epsilon[(\triangle\tilde{\phi}\nabla\phi_1)_\zeta + (\triangle\phi_2\nabla\tilde{\phi})_\zeta], \\ \tilde{\phi}_t + \tilde{u}_\zeta\nabla\phi_1 + (u_2 * \zeta)\nabla\tilde{\phi} = 0. \end{cases} \tag{15}$$

Multiplying the first equation in (15) by $\tilde{u}$ and the second equation by $-\epsilon\triangle\tilde{\phi}$, then integrating over the whole domain, we have

$$\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|\tilde{u}\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla\tilde{\phi}\|_{L^2}^2\right) \leqq -\mu\|\nabla\tilde{u}\|_{L^2}^2 - \epsilon\int_\Omega \triangle\phi_2\nabla\tilde{\phi}\tilde{u}_\zeta \,\mathrm{d}x$$
$$+\epsilon\int_\Omega \nabla\tilde{\phi}\triangle\tilde{\phi}(u_2 * \zeta)\,\mathrm{d}x$$

There exists constant C such that,

$$\left|\int_\Omega \triangle\phi_2\nabla\tilde{\phi}\tilde{u}_\zeta \,\mathrm{d}x\right| \leqq \frac{1}{2}\|\nabla\tilde{\phi}\|_{L^2}^2 + \frac{1}{2}\|\triangle\phi_2\|_{L^2}^2\|\tilde{u}_\zeta\|_{L^\infty}^2$$
$$\leqq \frac{1}{2}\|\nabla\tilde{\phi}\|_{L^2}^2 + \frac{C}{2}\|\triangle\phi_2\|_{L^2}^2\|\tilde{u}\|_{L^2}^2.$$
$$\left|\int_\Omega \nabla\tilde{\phi}\triangle\tilde{\phi}\,(u_2 * \zeta)\,\mathrm{d}x\right| = \left|\int_\Omega \nabla\cdot\tilde{\tau}\,(u_2 * \zeta)\,\mathrm{d}x\right|$$
$$\leqq \|\nabla(u_2 * \zeta)\|_{L^\infty}\cdot\|\tilde{\tau}\|_{L^2}^2$$
$$\leqq C\cdot\|u_2\|_{L^2}\cdot\|\nabla\tilde{\phi}\|_{L^2}^2,$$

where $\tilde{\tau} = \nabla\tilde{\phi}\otimes\nabla\tilde{\phi} - \frac{1}{2}|\nabla\tilde{\phi}|^2$.
Recall that u_1 and u_2 are uniformly bounded in $L^\infty(0, T; L^2(\Omega))$ and ϕ_1 and ϕ_2 are uniformly bounded in $L^\infty(0, T; H^2(\Omega))$. Then there exists some constant C', such that

$$\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\tilde{u}\|_{L^2}^2 + \frac{\epsilon}{2}\|\tilde{\phi}\|_{L^2}^2\right) \leqq C'\cdot\left(\frac{1}{2}\|\tilde{u}\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla\tilde{\phi}\|_{L^2}^2\right).$$

So the Gronwall inequality implies that if the two solutions share the same initial data, then the two solutions remain identical afterwards.

Remark 1. As we mentioned, we chose a relatively simple form of the phase field energy corresponding to surface area/tension to illustrate the idea of the proof, but the techniques work for other forms of the phase field or level set energies as well. The main ingredients are as follows: First, we have a dissipative energy law (14) which is independent of the choice of energy. The energy law provides uniform bounds of $\|u(t)\|_{L^2}^2$ in time. Second, we can differentiate the transport equation of ϕ in space, then the evolution equations of the spatial derivatives of ϕ are derived. Such derivatives are bounded by initial data and the fluid velocity field (Lemma 3). For different energy, we have different nonlinearity in $\nabla E[\phi]\nabla\phi$. As long as ζ is smooth enough, it is always possible to differentiate the transport equation as needed in order to obtain the bounds on the high order spatial derivatives of ϕ. Finally, all those bounds provide compactness, so we may pass to the limit to get the existence of a solution.

Remark 2. Through a boot-strapping argument, if the initial conditions are smooth, then we can see that the solutions discussed in the theorems are actually globally smooth classical solutions to the coupled systems. This is a new result and it is also consistent with the local well-posedness results outlined in [33] for the case with the nonlinear convective term in the momentum equation and without any velocity mollification in the transport equation.

4. Analysis of SIIM

In [12], we extended (6) to SIIM which we rewrite as follows:

$$\begin{cases} u_t = \mu\triangle u + \nabla p + (\nabla E[\phi]\nabla\phi) * \zeta + \sigma Q^{\frac{1}{2}}\dfrac{\mathrm{d}W}{\mathrm{d}t}, \\ \nabla\cdot u = 0, \\ \phi_t + (u * \zeta)\nabla\phi = 0, \end{cases} \tag{16}$$

where μ is the viscosity, and σ is a small constant. The noise $\frac{\mathrm{d}W}{\mathrm{d}t}$ is a space-time white noise. Let $\{\lambda_k, e_k(x)\}$ be the eigenvalues and eigenfunctions of the $-\triangle$ operator. The covariance operator Q then depends on λ_k, and

$$Q^{\frac{1}{2}}\frac{\mathrm{d}W}{\mathrm{d}t} = \sum_k \sqrt{\lambda_k}\frac{\mathrm{d}\beta_k}{\mathrm{d}t}\cdot e_k(x)$$

where $\{\beta_k(t)\}$ are independent standard Wiener processes.

Particularly, for the unit cell of the periodic domain, we have the eigenfunctions $e_k(x) = \mathrm{e}^{i2\pi kx}$, and $\lambda_k = 4\pi^2|k|^2$. We collect all the constants into σ, then

$$Q^{\frac{1}{2}}\frac{\mathrm{d}W}{\mathrm{d}t} = \sum_k |k|\frac{\mathrm{d}\beta_k}{\mathrm{d}t}\cdot e_k(x),$$

and $\{\beta_k\}$ are the standard independent complex vector-valued Wiener processes with $\beta_k = \bar{\beta}_{-k}$. We will focus our discussion on this special case from now on.

We denote the semigroup generated by $\mu\triangle$ operator by $S(t)$. Let P be the L^2 projection to the divergence-free space. We can formally rewrite the first equation in (16) as an integral equation,

$$u(t) = S(t)u_0 + \int_0^t S(t-s)P(\nabla E[\phi]\nabla\phi)_\zeta \mathrm{d}s + \sigma \int_0^t S(t-s)PQ^{\frac{1}{2}}\mathrm{d}W(s).$$

Let

$$G(t) \triangleq \int_0^t S(t-s)PQ^{\frac{1}{2}}\mathrm{d}W(s),$$

and $v(t) = u(t) - \sigma G(t)$. Then,

$$v(t) = S(t)u_0 + \int_0^t S(t-s)P((\nabla E[\phi]\nabla\phi) * \zeta)\mathrm{d}s,$$

and

$$v(0) = u(0).$$

That is,

$$\begin{cases} v_t = \mu\triangle v + \nabla p + (\nabla E[\phi]\nabla\phi)_\zeta, \\ \nabla \cdot v = 0, \\ \phi_t + (v_\zeta + \sigma G_\zeta)\nabla\phi = 0, \end{cases} \tag{17}$$

together with initial condition,

$$\begin{cases} v(0) = u_0(x), \\ \phi(0) = \phi_0(x), \end{cases} \tag{18}$$

and periodic boundary conditions. The actual analysis is on equation (17). We interpret the solution to (17) as a solution to (16).

Remark 3. In our case, $G(t)$ has infinite variance. In fact,

$$\begin{aligned} G(t) &= \int_0^t S(t-s)P\, Q^{\frac{1}{2}}\, \mathrm{d}W(s) \\ &= \sum_k \left(P_k \int_0^t \mathrm{e}^{-4\pi^2\mu|k|^2(t-s)}|k|\, \mathrm{d}\beta_k(s) \right) \cdot e_k(x) \end{aligned}$$

where P_k is as used before in (9). So,

$$\begin{aligned} \mathbb{E}\|G(t)\|_{L^2}^2 &= \sum_k \mathbb{E} \left\| P_k \int_0^t \mathrm{e}^{-4\pi^2\mu|k|^2(t-s)}|k|\, \mathrm{d}\beta_k(s) \right\|^2 \\ &= \sum_k tr(P_k) \left(\int_0^t \mathrm{e}^{-8\pi^2\mu|k|^2(t-s)}|k|^2\, \mathrm{d}s \right) \\ &= C \sum_k \left(1 - \mathrm{e}^{-8\pi^2\mu|k|^2 t} \right) \\ &= \infty. \end{aligned}$$

However after the mollification, we have

$$\mathbb{E}\|G_\zeta(t)\|_{L^2}^2 \leqq C \cdot \|\zeta\|_{L^2}^2,$$

thus G_ζ is well defined.

4.1. Some technical estimates

Formally, we have the energy law

$$\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|v\|_{L^2}^2 + E[\phi]\right) = -\mu\|\nabla v\|_{L^2}^2 - \sigma \int_\Omega \nabla E[\phi]\nabla\phi\, G_\zeta(t)\,\mathrm{d}x. \qquad (19)$$

We thus seek estimates on the last term of the right-hand side, which requires some detailed estimations on G_ζ. Let us first present a lemma under some conditions on the mollifier ζ.

Lemma 5. *Let $T > 0$ and $m \geqq 1$ be given.*

1. *If $\zeta \in C^m$, then the Ornstein–Uhlenbeck process $\nabla^\alpha G_\zeta(t) \in C(0, T; L^2(\Omega))$ almost surely for $|\alpha| \leqq m$, and there exists a positive constant C such that*

$$\mathbb{E}\|G_\zeta\|_{H^m}^2 \leqq C\|\zeta\|_{H^m}^2.$$

2. *If in addition,*

$$\zeta = \zeta_1 * \zeta_2 \in C^{m+1}, \quad \textit{for} \quad \zeta_1 \in C^m,\ \zeta_2 \in C^1, \qquad (20)$$

then there exists a constant C, depending on ζ_2, such that when $|\alpha| \leqq m$,

$$\|\nabla^\alpha G_\zeta\|_{L^\infty} \leqq C\|\nabla^\alpha G_{\zeta_1}\|_{L^2}.$$

Proof. The continuity of the process is the direct consequence of Theorem 1 and its corollary in [23]. Moreover,

$$\begin{aligned}\mathbb{E}\|\nabla^\alpha G_\zeta(t)\|_{L^2}^2 &\leqq C \cdot \sum_k \left(\int_0^t \mathrm{e}^{-8\pi^2\mu|k|^2(t-s)}|k|^2\,\mathrm{d}s\right)\hat{\zeta}_k^2 \cdot |k|^{2|\alpha|}\\ &\leqq C\|\nabla^\alpha\zeta\|_{L^2}^2.\end{aligned}$$

And

$$\begin{aligned}\|\nabla^\alpha G_\zeta\|_{L^\infty} &= \|\nabla^\alpha G_{\zeta_1} * \zeta_2\|_{L^\infty}\\ &\leqq C\|\nabla^\alpha G_{\zeta_1}\|_{L^2}.\end{aligned}$$

So we have the results of the lemma.

Note that in practical simulations, the regularity of the function ζ may vary with the specific numerical schemes. The assumptions used in the above lemma are not essential, but they make the later technical derivations much simpler.

4.2. Well-posedness of SIIM

Similar to the discussion on deterministic model, we again focus our analysis on the special case where $E = E[\phi]$ is as in (3), and prove the existence of the solution for each realization of noise.

Theorem 2. (Well-posedness of pathwise solution) *Under the same constraints made as before on the initial conditions, the form of the noise, and the additional assumption* (20) *with $m = 2$ on the kernel ζ, for almost every realization of noise, there exists a weak solution on time interval* $(0, T)$ *to the system* (17).

Proof. 1. Here we give sketches of the proof because it is similar to the deterministic case. Let

$$W_n \triangleq span\{e_k = e_k(x) : |k| \leqq n\}$$

be the finite dimension space spanned by the Fourier modes.
Applying a Galerkin approximation and restrict v in the divergence-free subspace of W_n, we have the approximate equation of v,

$$\partial_t v = \mu \triangle v + P\Lambda_n(-\epsilon \triangle\phi\nabla\phi)_\zeta,$$

where Λ_n is the L^2 projection to W_n and P is the L^2 projection to the divergence-free space. (Note that P commutes with Λ_n).
ϕ is transported via

$$\phi_t + (v_\zeta + \sigma\Lambda_n G_\zeta)\nabla\phi = 0.$$

The energy law says,

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\left(\|v\|_{L^2}^2 + \epsilon\|\nabla\phi\|_{L^2}^2\right) = -\mu\|\nabla v\|_{L^2}^2 + \sigma\int_\Omega (\epsilon\triangle\phi\nabla\phi)\cdot\Lambda_n G_\zeta(t)\,\mathrm{d}x. \quad (21)$$

The key to showing the existence of a solution is obtaining a priori bounds via the energy law. Similar to the proof of the deterministic case, we write

$$\triangle\phi\nabla\phi = \nabla\cdot\tau,$$

where $\tau = \nabla\phi\otimes\nabla\phi - \frac{1}{2}|\nabla\phi|^2$.

$$\begin{aligned}\left|\int_\Omega \triangle\phi\nabla\phi\cdot\Lambda_n G_\zeta \mathrm{d}x\right| &= \left|\int_\Omega (\nabla\cdot\tau)\cdot\Lambda_n G_\zeta\,\mathrm{d}x\right| \\ &= \left|\int_\Omega \tau\cdot\nabla\Lambda_n G_\zeta\,\mathrm{d}x\right| \\ &\leqq \|\nabla\Lambda_n G_\zeta\|_{L^\infty}\int_\Omega |\tau|\,\mathrm{d}x\end{aligned}$$

Note that the convolution, differentiation and projection Λ_n commute with each other, hence

$$\begin{aligned}\|\nabla\Lambda_n G_\zeta\|_{L^\infty} &= \|(\nabla\Lambda_n G_{\zeta_1}) * \zeta_2\|_{L^\infty} \\ &\leqq C_1\cdot\|\nabla\Lambda_n G_{\zeta_1}\|_{L^2} \\ &\leqq C_2\cdot\|\nabla G_{\zeta_1}\|_{L^2}\end{aligned}$$

for some constants C_1 and C_2. Therefore,

$$\left|\int_\Omega \triangle\phi\nabla\phi\cdot\Lambda_n G_\zeta \mathrm{d}x\right| \leqq C_3\cdot\|\nabla G_{\zeta_1}\|_{L^2}\cdot\|\nabla\phi\|_{L^2}^2, \tag{22}$$

for some constant C_3.

By Lemma 5, $\nabla G_{\zeta_1}\in C(0,T;L^2(\Omega))$ almost surely. There exists a constant C which depends on the realization of the noise, ζ and T, but is independent of W_n, such that

$$\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|v\|_{L^2}^2+\frac{\epsilon}{2}\|\nabla\phi\|_{L^2}^2\right)\leqq-\mu\|\nabla v\|_{L^2}^2+\sigma\epsilon C\|\nabla\phi\|_{L^2}^2.$$

In each finite dimensional space W_n, as in Theorem 1, we can find an approximate solution v_n and corresponding ϕ_n. Then the Gronwall inequality implies that there exists some constant C, independent of W_n (but dependent on the realization of noise), such that

$$\frac{1}{2}\|v_n(t)\|_{L^2}^2+\frac{\epsilon}{2}\|\nabla\phi_n(t)\|_{L^2}^2\leqq \mathrm{e}^{Ct}\left(\frac{1}{2}\|v_0\|_{L^2}^2+\frac{\epsilon}{2}\|\nabla\phi_0\|_{L^2}^2\right)$$

for $0<t<T$.

Then, immediately, we have

- v_n is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$.
- ϕ_n is uniformly bounded in $L^\infty(0,T;H^1(\Omega))$.

By Lemma 3,

$$\|\nabla\phi_n(t)\|_{L^2}^2\leqq\|\nabla\phi_0\|_{L^2}^2\exp\left(C\int_0^t\|\nabla u_n*\zeta(s)\|_{L^\infty}+\sigma\|\nabla\Lambda_n G_\zeta(s)\|_{L^\infty}\mathrm{d}s\right).$$

$$\begin{aligned}\|\nabla\Lambda_n G_\zeta(t)\|_{L^\infty}&\leqq C\|\nabla\Lambda_n G_{\zeta_1}(t)\|_{L^2}\\&\leqq C\cdot\|\nabla G_{\zeta_1}\|_{L^2}.\end{aligned}$$

Therefore,

- $\nabla\phi_n$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$.
- Similarly, $\triangle\phi_n$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$.
- u_n is uniformly bounded in $L^2(0,T;H_d^1(\Omega))$.

Using the same argument as in the deterministic case, we have

- $\partial_t u_n$ is uniformly bounded in $L^2(0,T;H_d^{-1}(\Omega))$.
- $\partial_t\phi_n$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$.

The a priori bounds lead to compactness, so we can then pass to the limit to recover the solution. The detail is omitted here.

2. To prove the uniqueness of the solution, we assume that there exist two solutions (v_1,ϕ_1), (v_2,ϕ_2), which have the same initial data. Let $\tilde{v}=v_1-v_2$, and $\tilde{\phi}=\phi_1-\phi_2$. Then $\tilde{v}$ and $\tilde{\phi}$ satisfy the following equation,

$$\begin{cases}\tilde{v}_t=\mu\triangle\tilde{v}+\nabla\tilde{p}-\epsilon[(\triangle\tilde{\phi}\nabla\phi_1)_\zeta+(\triangle\phi_2\nabla\tilde{\phi})_\zeta],\\ \tilde{\phi}_t+\tilde{v}_\zeta\nabla\phi_1+((v_2*\zeta)+\sigma G_\zeta)\nabla\tilde{\phi}=0.\end{cases}\tag{23}$$

In the same way as the deterministic case, multiply the first equation in (23) by $\tilde{v}$, and the second equation by $-\epsilon\triangle\tilde{\phi}$. We have,

$$\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|\tilde{v}\|_{L^2}^2+\frac{\epsilon}{2}\|\nabla\tilde{\phi}\|_{L^2}^2\right)\leqq-\mu\|\nabla\tilde{u}\|_{L^2}^2-\epsilon\int_\Omega\triangle\phi_2\nabla\tilde{\phi}\tilde{v}_\zeta\,\mathrm{d}x$$
$$+\epsilon\int_\Omega\nabla\tilde{\phi}\triangle\tilde{\phi}(u_2*\zeta+\sigma G_\zeta)\,\mathrm{d}x.$$

Recall that $\nabla G_{\zeta_1}\in C(0,T;L^2)$. Using the same argument as in the deterministic case, we immediately derive the uniqueness of the pathwise solution.

4.3. Uniform bound in large probability

In the above proof, we used a Gronwall inequality to control the energy for a particular path, however we do not have any uniform bound on a family of paths. The following theorem asserts that indeed, large probability energy stays uniformly bounded for any finite time.

Theorem 3. *Under the same constraints made as before on the initial conditions, the form of the noise, and the additional assumption* (20) *with $m=2$ on the kernel ζ, there exist two constants C' and C'' such that*

$$\mathbb{P}\left(\frac{1}{2}\|v(t)\|_{L^2}^2+\frac{\epsilon}{2}\|\nabla\phi(t)\|_{L^2}^2\leqq\left(\frac{1}{2}\|u_0\|_{L^2}^2+\frac{\epsilon}{2}\|\nabla\phi_0\|_{L^2}^2\right)\mathrm{e}^{C't}\right)\geqq1-\sigma^2\cdot C''.$$

Proof. Let v_n and ϕ_n be the approximate solution, as in Theorem 2. Following from the estimates (21) and (22), we have

$$\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|v_n\|_{L^2}^2+\frac{\epsilon}{2}\|\nabla\phi_n\|_{L^2}^2\right)\leqq-\mu\|\nabla v_n\|_{L^2}^2+\sigma\epsilon\,C\cdot\|\nabla G_{\zeta_1}\|_{L^2}\|\nabla\phi_n\|_{L^2}^2.$$

∇G_{ζ_1} is a continuous martingale. By martingale inequality, for fixed constant $C_0>0$,

$$\mathbb{P}\left(\sup_{0<t<T}\|\sigma\nabla G_{\zeta_1}\|_{L^2}\geqq C_0\right)\leqq\frac{\sigma^2}{C_0^2}\sup_{0<t<T}\mathbb{E}(\|\nabla G_{\zeta_1}(t)\|_{L^2}^2)$$

By Lemma 5,

$$\sup_{0<t<T}\mathbb{E}(\|\nabla G_{\zeta_1}(t)\|_{L^2}^2)\leqq C_1\cdot\|\nabla\zeta_1\|_{L^2}^2,$$

where C_1 is independent of T. Therefore,

$$\mathbb{P}\left(\sup_{0<t<T}\sigma\|\nabla G_{\zeta_1}(t)\|_{L^2}\cdot\|\nabla\phi_n\|_{L^2}^2\leqq C_0\,\|\nabla\phi_n\|_{L^2}^2\right)>1-\frac{\sigma^2}{C_0^2}C_1\cdot\|\nabla\zeta_1\|_{L^2}^2. \tag{24}$$

Under the assumption $\sup_{0<t<T}\|\sigma\nabla G_{\zeta_1}(t)\|_{L^2}<C_0$,

$$\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|v_n\|_{L^2}^2+\frac{\epsilon}{2}\|\nabla\phi_n\|_{L^2}^2\right)\leqq-\mu\|\nabla v_n\|_{L^2}^2+\epsilon\,C\,C_0\|\nabla\phi_n\|_{L^2}^2. \tag{25}$$

Then a Gronwall inequality implies that, for two constants $C' = 2CC_0$ and $C'' = C_1\|\nabla\zeta_1\|_{L^2}^2/C_0^2$, which are independent of W_n, such that

$$\mathbb{P}\left(\frac{1}{2}\|v_n(t)\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla\phi_n(t)\|_{L^2}^2 \leqq \left(\frac{1}{2}\|u_0\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla\phi_0\|_{L^2}^2\right)\mathrm{e}^{C't}\right) \geqq 1 - \sigma^2 C''.$$

It is easy to check that a subsequence of v_n converges in $C([0,T]; L^2(\Omega))$, and a subsequence of ϕ_n converges in $C([0,T]; H^1(\Omega))$. By passing to the limit, we obtain the result.

Remark 4. From the proof, we can see that the constants C' and C'' are also independent of the final time T.

4.4. Analysis of SIIM with elastic bending energy

The proof for the theorem presented above is based on a relatively simple phase field energy (for an interface under constant interfacial tension). Indeed, we can extend it to other types of energies as well, such as the phase field and level set formulations of the various energies considered in [12], including those involving higher order derivatives or stronger nonlinearities. As mentioned in the introduction, we analyzed, in an earlier work [13], a vesicle–fluid interaction model in which we considered the elastic bending energy (1) in the phase field formulation (2). We can show that the discussion in the previous section is also applicable to this energy.

For convenience, Let $f_\epsilon(\phi) = \frac{1}{\epsilon^2}(\phi - \phi^3)$, $H_\epsilon(\phi) = \triangle\phi + f_\epsilon(\phi)$. For the elastic bending energy defined by (2), the force density term is given by

$$\nabla E_b[\phi]\nabla\phi = \kappa\epsilon\left(\triangle^2\phi + \triangle f_\epsilon(\phi) + H_\epsilon(\phi)\cdot f_\epsilon'(\phi)\right)\cdot\nabla\phi.$$

We first rewrite the force density in terms of a stress tensor, that is, $-\nabla E_b[\phi]\nabla\phi = \kappa\epsilon\nabla\cdot\tau$, where

$$\tau = 2H_\epsilon(\phi)\nabla^2\phi - \nabla(H_\epsilon(\phi)\nabla\phi) - \frac{1}{2}H_\epsilon(\phi)^2\cdot I.$$

Then we can start to estimate the last term in (19).

$$\left|\int_\Omega \nabla E_b[\phi]\nabla\phi\, G_\zeta(t)\,\mathrm{d}x\right| = \kappa\epsilon\left|\int_\Omega \tau\,\nabla G_\zeta(t)\,\mathrm{d}x\right|.$$

Note that

$$\begin{aligned}&\left|\int_\Omega \tau\,\nabla G_\zeta(t)\,\mathrm{d}x\right|\\ &\leqq 2\left|\int_\Omega H_\epsilon(\phi)\nabla^2\phi\,\nabla G_\zeta(t)\,\mathrm{d}x\right| + \left|\int_\Omega H_\epsilon(\phi)\nabla\phi\,\triangle G_\zeta(t)\,\mathrm{d}x\right|\\ &\quad + \left|\frac{1}{2}\nabla H_\epsilon(\phi)^2\,G_\zeta(t)\,\mathrm{d}x\right|\\ &= 2J_1 + J_2 + J_3.\end{aligned}$$

We now estimate each term individually.

$$
\begin{aligned}
J_1 &\leqq \left| \int_\Omega \triangle\phi \nabla^2 \phi \, \nabla G_\zeta(t)\, \mathrm{d}x \right| + \left| \int_\Omega f_\epsilon(\phi) \nabla^2 \phi \, \nabla G_\zeta(t)\, \mathrm{d}x \right| \\
&\leqq C_1 \cdot \|\nabla G_\zeta(t)\|_{L^\infty} (\|\triangle\phi\|_{L^2}^2 + \|f_\epsilon(\phi)\|_{L^2}^2).
\end{aligned}
$$

Since

$$
\partial_t f(\phi) + ((u_\zeta + G_\zeta) \cdot \nabla) f(\phi) = 0,
$$

we get $|f(\phi(t)| \leqq |f(\phi_0)|$. Therefore,

$$
\begin{aligned}
J_1 &\leqq C_1' \cdot \|\nabla G_\zeta\|_{L^\infty} (E_b[\phi(t)] + 1) \\
&\leqq \tilde{C}_1 \cdot \|\nabla G_{\zeta_1}\|_{L^2} (E_b[\phi(t)] + 1).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J_2 &\leqq C_2 \|\triangle G_\zeta\|_{L^\infty} \cdot (E_b[\phi] + 1) \\
&\leqq \tilde{C}_2 \|\triangle G_{\zeta_1}\|_{L^2} \cdot (E_b[\phi] + 1).
\end{aligned}
$$

Since G_ζ is divergence-free, we get

$$
J_3 = 0.
$$

So we have the following estimate

$$
\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|v\|_{L^2}^2 + E_b[\phi]\right) \leqq -\mu \|\nabla v\|_{L^2}^2 + \sigma\epsilon\kappa C \|\triangle G_{\zeta_1}\|_{L^2} \cdot (E_b[\phi] + 1)
$$

for some constant C. This then gives us a Gronwall type inequality so that for some generic constant C, we have

$$
\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\|v\|_{L^2}^2 + E_b[\phi] + 1\right) \leqq -\mu \|\nabla v\|_{L^2}^2 + C\sigma \|\triangle G_{\zeta_1}\|_{L^2}^2 \left(\frac{1}{2}\|v\|_{L^2}^2 + E_b[\phi] + 1\right).
$$

Assume that ζ is sufficiently regular. By the continuity of $\triangle G_\zeta(t)$, its realization on each time interval $(0\, T)$ is bounded, which implies that the solution is bounded. Now we can further prove the existence theorem.

Meanwhile, $\triangle G_\zeta$ is a martingale. The martingale inequality implies for large probability that $\sigma \|\triangle G_\zeta(t)\|_{L^2}^2$ is bounded. Therefore, as before, we can show that for large probability the solution remains bounded for any finite time. To sum up, we have the following theorem.

Theorem 4. *Under the same constraints as before on the initial conditions, the form of the noise, and the additional assumption* (20) *with* $m = 3$ *on the kernel* ζ*, for the energy given by the phase field elastic bending energy* (2)*, there exists a unique weak solution to the deterministic model* (6) *for any time interval* $(0,\ T)$*. In addition, the stochastic model* (17–18) *has a unique weak solution almost surely. Moreover, there exist two positive constants* C', C''*, such that*

$$
\mathbb{P}\left(\frac{1}{2}\|v(t)\|_{L^2}^2 + E_b[\phi(t)] + 1 \leqq \left(\frac{1}{2}\|u_0\|_{L^2}^2 + E_b[\phi_0] + 1\right) e^{C' t}\right) \geqq 1 - \sigma^2 \cdot C''.
$$

5. Conclusion

In this work, a recently proposed SIIM is rigorously analyzed. Results on the well-posedness of the models are established for both the deterministic and stochastic cases. The SIIM utilizes an implicit surface representation of an immersed interface which may undergo complex topological changes when it is subject to fluctuating forces. We note that through the works of many researchers in recent years, such implicit surface representation techniques have become very popular in the study of elastic interface and fluid interactions, for instance level set formulations [3,8–10,44] and phase field or diffuse interface formulations [1,7,13,16,20,21,25,26,46,47]. The analysis given here is still limited as we have not paid close attention to the minimum regularities required on the mollifiers, nor have we considered the long-time statistical properties of dynamic systems. The dependence of our analysis on the interfacial width parameter ϵ in the phase field context is largely overlooked as well in the current work. By considering only periodic boundary conditions, we have also neglected important boundary effects and interactions with the substrate [47]. The proposed SIIM gives an attractive alternative to model fluctuation effects on the fluid–interface interactions, but its effectiveness needs to be demonstrated through extensive numerical simulations. These and other interesting issues will be explored in the future.

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