Selling to a Group

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What mechanism is optimal (maximizes seller's revenue) given the constraints of incentive compatibility and individual rationality?
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(s.t. Incentive compatibility & Individual rationality)
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company pays $v_1$ for software $\approx$ company doesn’t buy software
What mechanism is optimal (maximizes seller’s revenue)?
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(s.t. Incentive compatibility & Individual rationality)
Sell at price $2$ if CTO agrees.

Not IR: $v_1 = 0$ then Utility of CEO = $-\frac{3}{2} \cdot 1$.

Sell at price $p$ if both agree.

Sell iff $w_1 v_1 + w_2 v_2 \geq \delta_{\text{max}}$.

$p \approx 0.35$ at $p \approx 0.78$.

$w_1 = \sqrt{\frac{3}{7}}, w_2 = 1 - w_1, \delta = 1 + w_2^2 \approx \frac{3}{9}$. 

$v_1 \sim U[0, 2], v_2 \sim U[0, 3]$. 

seller \rightarrow\text{software} \rightarrow CEO \leftarrow\text{company} \text{ money} \leftarrow CTO \rightarrow company
CEO

Sell at price $p$ if both agree

Sell iff $w_1 v_1 + w_2 v_2 \geq \delta_{max}$

$P[v_1 \geq p] \approx 0.35$ at $p \approx 0.78$

$v_1 = \sqrt{3/7}$, $w_2 = 1 - w_1$, $\delta = 1 + w_2^2$
Sell at price $\frac{3}{2}$ if CTO agrees

Not IR:

$\text{Utility of CEO} = -\frac{3}{2} \cdot \frac{1}{2}$

Sell at price $p$ if both agree

Sell iff $w_1 v_1 + w_2 v_2 \geq \delta$

$\text{max}_p \left[ v_1 \geq p \right] \approx 0.35 \text{ at } p \approx 0.78$

$w_1 = \sqrt{\frac{3}{7}}, w_2 = 1 - w_1, \delta = 1 + w_2^2$
Sell at price $\frac{3}{2}$ if CTO agrees
Not IR: $v_1 = 0$ then Utility of CEO = $-\frac{3}{2} \cdot \frac{1}{2}$
Sell at price $p$ if both agree.
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$\max_p p \mathbb{P}[v_1 \geq p] \mathbb{P}[v_2 \geq p] \approx 0.35$ at $p \approx 0.78$
Sell iff $w_1 v_1 + w_2 v_2 \geq \delta$

\[
\begin{align*}
w_1 &= \sqrt{\frac{3}{7}}, \quad w_2 = 1 - w_1, \quad \delta = 1 + \frac{w_2}{2}
\end{align*}
\]
Theorem

The following mechanism is optimal:

1. Allocation *maximizes* weighted sum of virtual values

2. Weights *minimize* weighted virtual surplus

3. *Transfer rule is “defined appropriately”*
Theorem

The following mechanism is optimal:

1. Allocation maximizes weighted sum of virtual values

\[ \text{Allocate} \iff \sum_i w_i^* \phi_i(v_i) \geq 0 \]

where \( \phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)} \)

2. Weights minimize weighted virtual surplus

3. Transfer rule is “defined appropriately”
Theorem

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1. Allocation maximizes weighted sum of virtual values

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2. Weights minimize weighted virtual surplus

   \[ w^* \in \arg \min_w \mathbb{E}[\max(\sum_i w_i \phi_i(v_i)), 0)] \]

3. Transfer rule is “defined appropriately”
Theorem

The following mechanism is optimal:

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2. Weights minimize weighted virtual surplus

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3. Transfer rule is “defined appropriately”

so that payment identity is satisfied
Proof step 1: revenue as virtual surplus

Lemma

Over all mechanisms with interim allocations $X_1, \ldots, X_n$:

optimal revenue $= \min_i \mathbb{E}[X_i(v_i)\phi_i(v_i)]$

Compare to if individual transfers were allowed:

optimal revenue $= \sum_i \mathbb{E}[X_i(v_i)\phi_i(v_i)]$

\[ \frac{5}{9} \]
Proof step 1: revenue as virtual surplus

Lemma

Over all mechanisms with interim allocations $X_1, \ldots, X_n$

\[
\text{optimal revenue} = \min_i \mathbb{E}[X_i(v_i)\phi_i(v_i)]
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Proof step 1: revenue as virtual surplus

**Lemma**

Over all mechanisms with interim allocations $X_1, \ldots, X_n$

$$\text{optimal revenue} = \min_i \mathbb{E}[X_i(v_i)\phi_i(v_i)]$$

Compare to if individual transfers were allowed

$$\text{optimal revenue} = \sum_i \mathbb{E}[X_i(v_i)\phi_i(v_i)]$$
Proof step 2: duality

\[
\begin{align*}
\max_{\text{allocation}} \quad \min_i \quad E[X_i(v_i)\phi_i(v_i)]
\end{align*}
\]
Proof step 2: duality

\[
\max_{\text{allocation}} \min_i \mathbb{E}[X_i(v_i)\phi_i(v_i)] = \min_{w_1, \ldots, w_n} \max_{\text{allocation}} \sum_i w_i \mathbb{E}[X_i(v_i)\phi_i(v_i)]
\]
Proof step 2: duality

$$\max_{\text{allocation}} \min_i \mathbb{E}[X_i(v_i)\phi_i(v_i)] = \min_{w_1,\ldots,w_n \text{ allocation}} \max \sum_i w_i \mathbb{E}[X_i(v_i)\phi_i(v_i)]$$
Proof step 2: duality

\[
\max_{\text{allocation}} \min_{i} \mathbb{E}[X_i(v_i)\phi_i(v_i)] = \min_{w_1,\ldots,w_n} \max_{\text{allocation}} \sum_i w_i \mathbb{E}[X_i(v_i)\phi_i(v_i)] = \min_{w_1,\ldots,w_n} \mathbb{E}[\max(\sum_i w_i \phi_i(v_i), 0)]
\]
Which agent has a higher weight?
Which agent has a higher weight? the “weaker” one
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Proposition

If $\phi_1$ is smaller than $\phi_2$ in the “hazard rate” order, then $w_1^* \geq w_2^*$. 
Which agent has a higher weight? the “weaker” one

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If $\phi_1$ is smaller than $\phi_2$ in the “hazard rate” order, then $w_1^* \geq w_2^*$. 

[Diagram showing interactions between seller, software company, CEO, and CTO with distributions for $v_1$ and $v_2$.]
Which agent has a higher weight? the “weaker” one

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![Diagram of seller and company interactions.]

- **Seller**
  - Sell at price 1 if CEO agrees

- **CEO**
  - Revenue = $\frac{7}{9}$

- **CTO**
  - Revenue = 0

- **Company**

  - Software
  - Money

  - $v_1 \sim U[0, 2]$
  - $v_2 \sim U[1, 3]$
Which agent has a higher weight? the “weaker” one

**Proposition**

If $\phi_1$ is smaller than $\phi_2$ in the “hazard rate” order, then $w_1^* \geq w_2^*$.

- **seller**
  - Sell at price 1 if **CEO** agrees
  - Revenue = $\frac{1}{2}$

- **company**
  - **CEO**
    - $v_1 \sim U[0, 2]$
  - **CTO**
    - $v_2 \sim U[1, 3]$

- Software

- Company money
Talk to us about these extensions and directions
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Posted pricing is not even *approximately optimal*

Pareto efficient mechanisms

What if only one agent has to agree to mechanism (or half of them)?

What if agents *pay out of pocket*, but in *fixed shares*?

Correlated/interdependent values?
Single seller, single product, “single” buyer

- Posting a price is not optimal
Single seller, single product, “single” buyer

- Posting a price is not optimal
- Pay more attention to “weaker” agents
Single seller, single product, “single” buyer

- Posting a **price** is **not optimal**
- Pay **more attention** to “weaker” agents

The End!