When Is Pure Bundling Optimal?

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Joint work with Jason Hartline (Northwestern)

September 26, 2019

Multi-product Monopolist's Optimal Selling Strategy?

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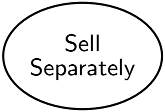

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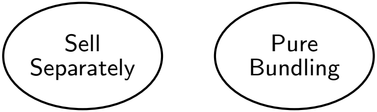
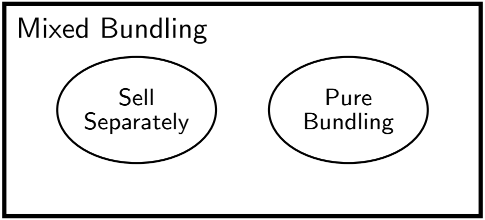


Multi-product Monopolist's Optimal Selling Strategy?

- **Sell Separately:** Offer each product for a price
- **Pure Bundling:** Offer only the grand bundle of all products
- **Mixed Bundling:** Offer a menu of bundles and prices



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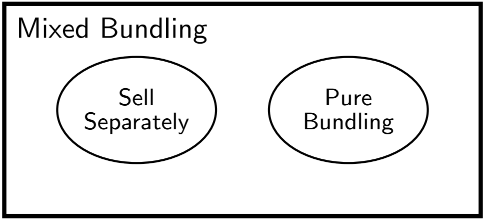

This paper: When is Pure Bundling Optimal?

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Single seller, products 1 to k, single buyer

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Price	**Bundle**
\$4	b
\$5	b'
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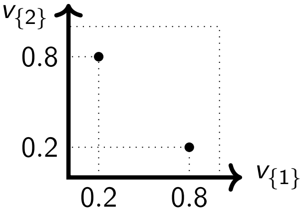

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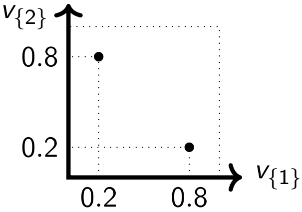

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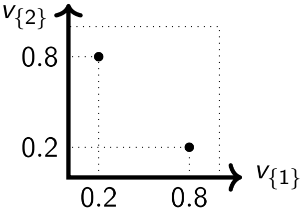

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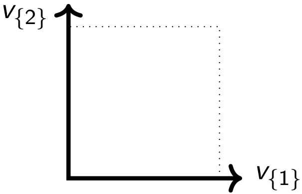

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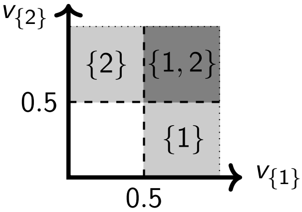

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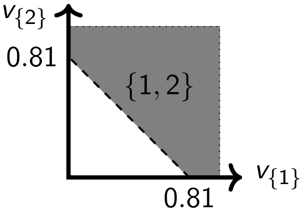

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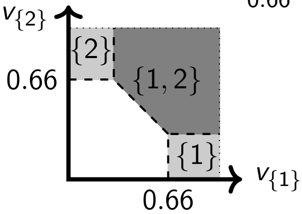

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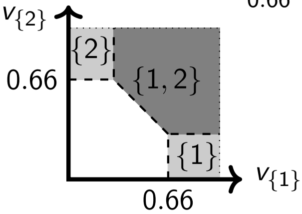

Stigler '63, Adams & Yellen '76: Bundle if values negatively correlated
McAfee et al. '89: Pure bundling generically not optimal

Special Case: Two Identical Products $v_{\{1\}} = v_{\{2\}}$

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Values $(v_1, v_{gb}) \sim \mu$

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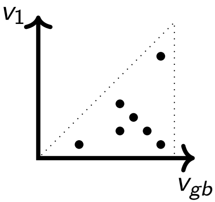

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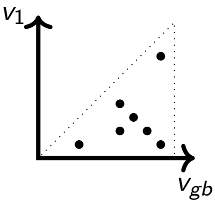

\$5	Two Units

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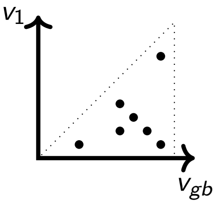

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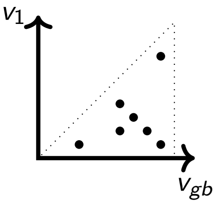

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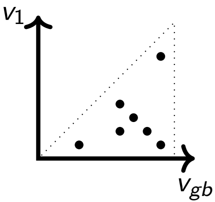

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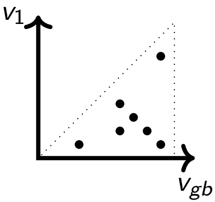

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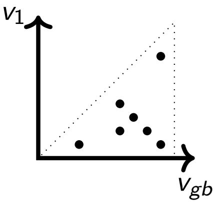

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- PB optimal if v_1/v_{gb} "stochastically nondecreasing" in v_{gb}
  - High v_{gb} implies high "relative utility"
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Intuition

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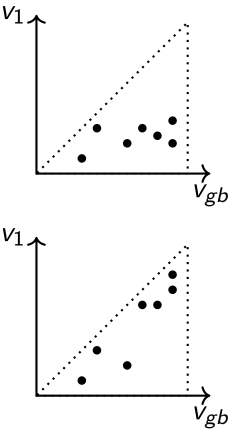

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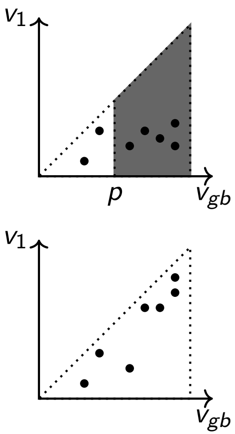

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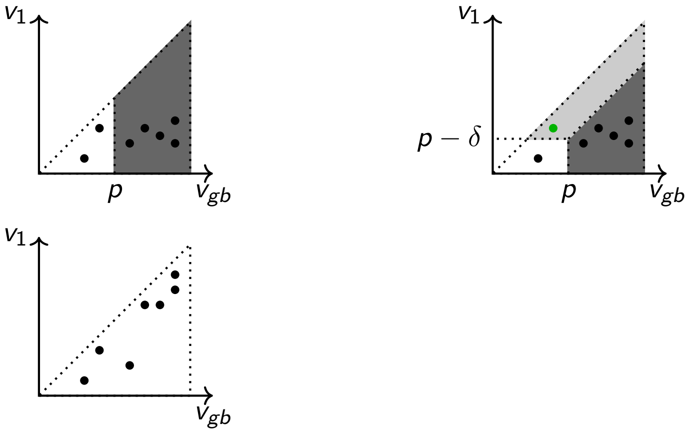

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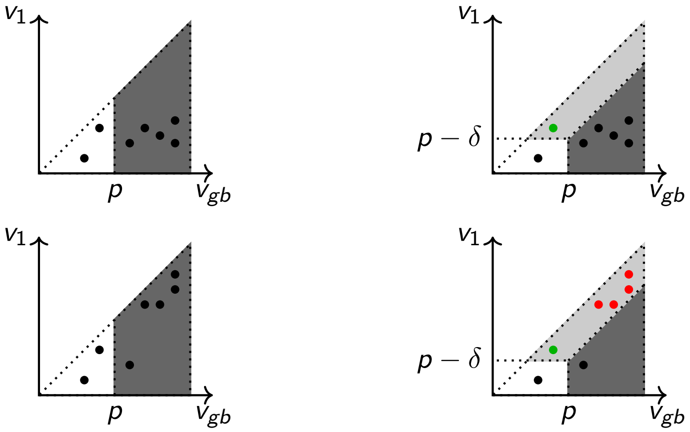
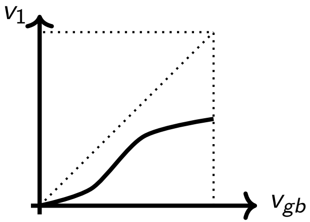


A Special Case: Types “on a Path”



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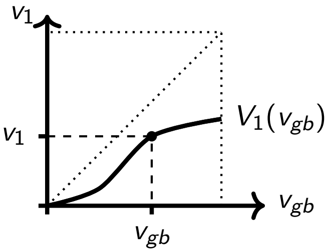

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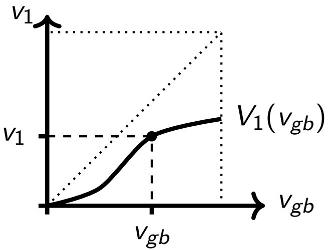

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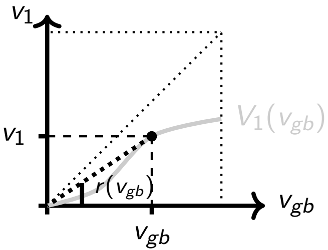
Ratio (relative utility) $r(v_{gb}) := v_1/v_{gb}$



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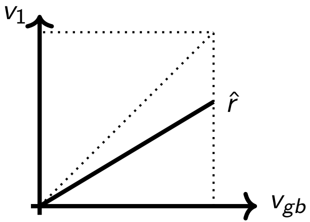

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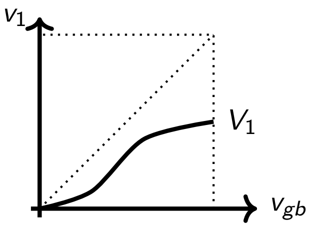

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Ratio (relative utility) $r(v_{gb}) := v_1/v_{gb}$

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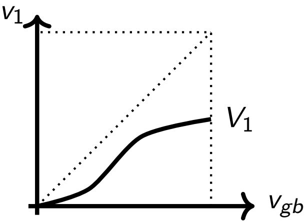
PB is optimal $\forall\mu$ over "Path" V_1 iff r monotone nondecreasing.



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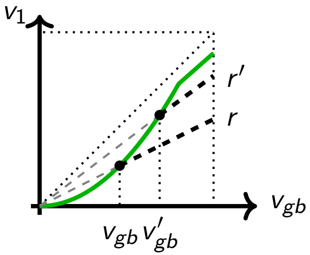

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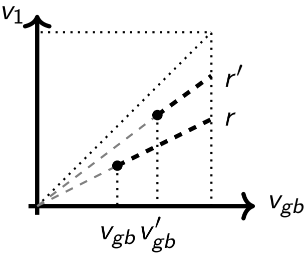

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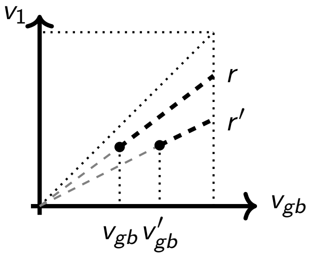

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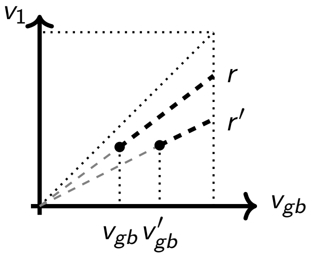

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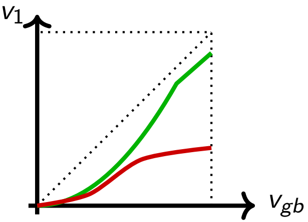

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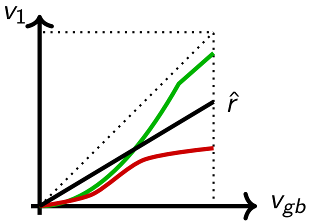
Stokey'79, Acquisti and Varian'05:

- PB optimal if r constant



Main Theorem (Two Identical Products)

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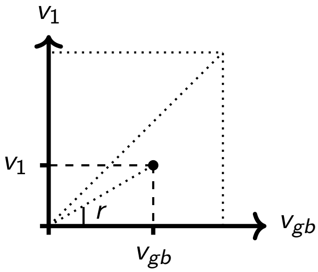

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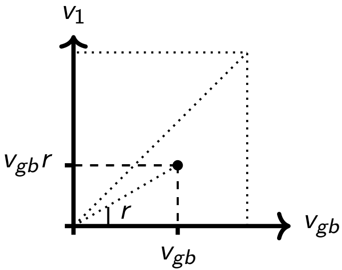

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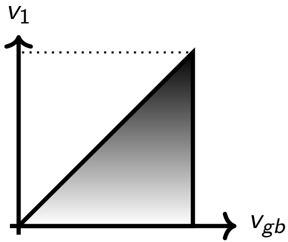

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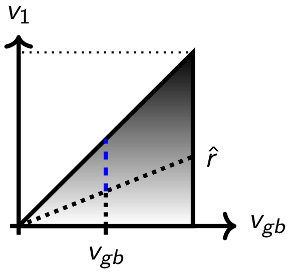
r stochastically nondecreasing in v_{gb}:

- $Pr(r \geq \hat{r} \mid v_{gb})$ nondecreasing in v_{gb} (stochastic dominance)



Main Theorem (Two Identical Products)

Ratio (relative utility) $r := v_1/v_{gb}$

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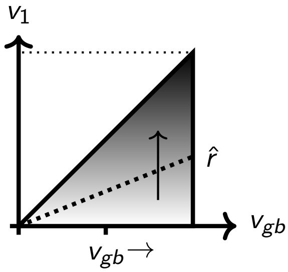

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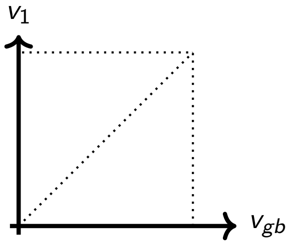

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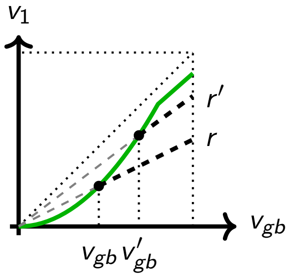

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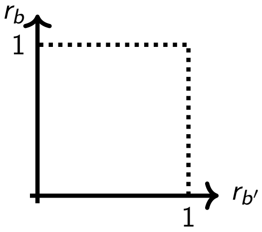

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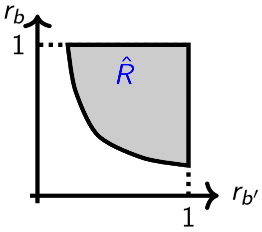

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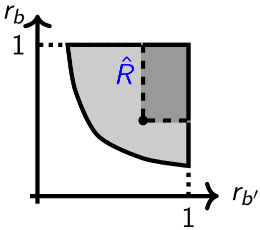

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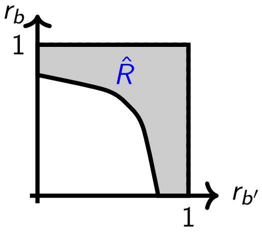

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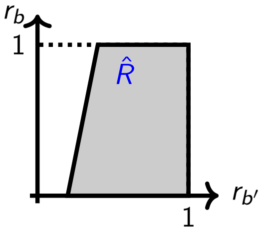

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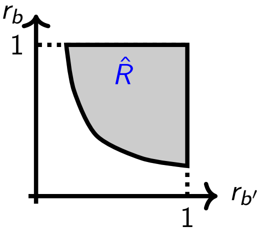

Example: Complementarities

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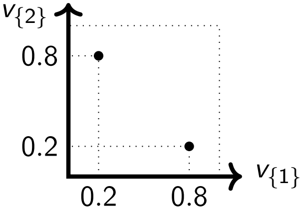
PB optimal if wealthier consumers consider products more substitutable

Recall Additive Example



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Additivity & perfect negative correlation $\Rightarrow v_{gb}$ constant

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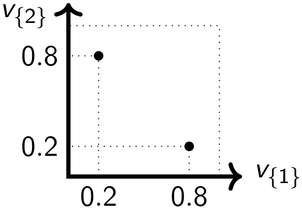

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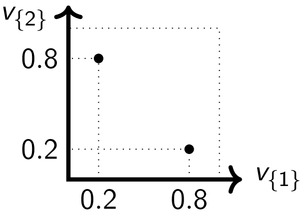

Folklore: Bundle if $v_{\{1\}}, v_{\{2\}}$ negatively correlated

- $v_{\{1\}}, v_{\{2\}}$: disutility from getting smaller bundle (compared to $\{1, 2\}$)
- **Reinterpretation:** Bundle if disutilities negatively correlated

Our result: Bundle if v_1/v_{gb} and v_{gb} positively correlated

- $1 - v_1/v_{gb}$: relative disutility from getting smaller bundle
- Bundle if relative disutility and v_{gb} negatively correlated

Envelope Analysis and Virtual Values

Single dimension: 

"virtual value" $\phi(v) = v$ - revenue loss

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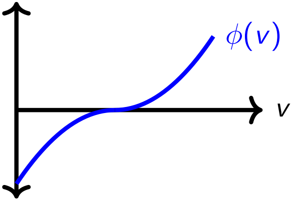

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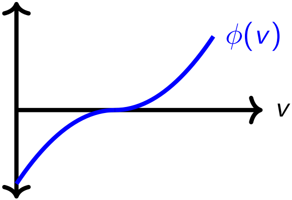

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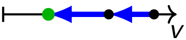

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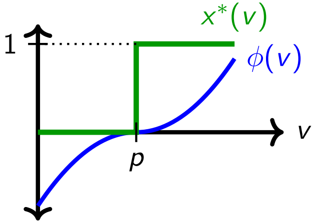

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Theorem (Myerson'81; Riley and Zeckhauser'83)

Posting a price for the item is the optimal mechanism

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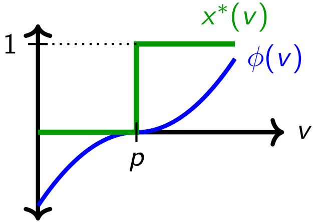

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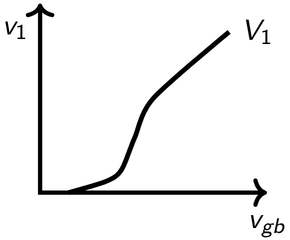
Revenue of any IC mechanism is $E_v[x_1(v) \cdot \phi_1(v) + x_{gb}(v) \cdot \phi_{gb}(v)]$



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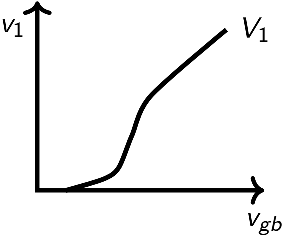

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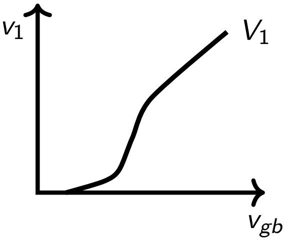

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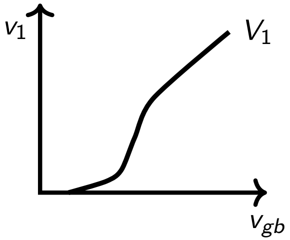

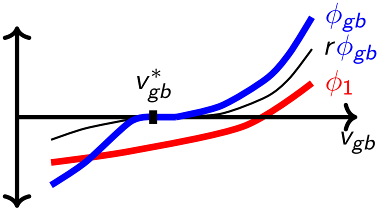

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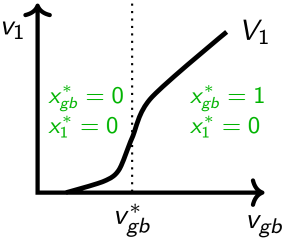
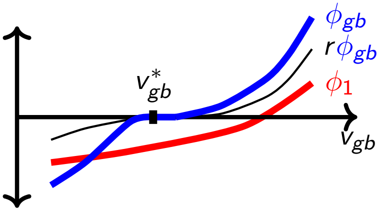
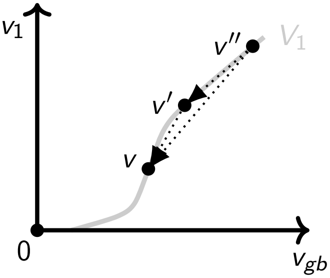




Beyond Regularity

If ratio r increasing, then only "downward" IC constraints bind



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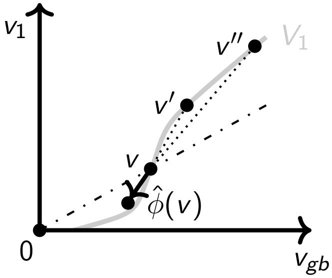

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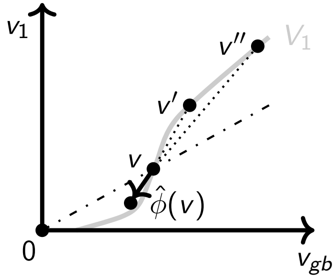
Thus $r\hat{\phi}_{gb} \geq \hat{\phi}_1$, and $x_1^* = 0$.



Beyond Paths: Orthogonalization

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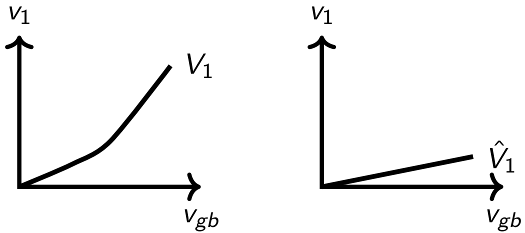
Two paths $V_1, \hat{V}_1$ (both with monotone ratio), same marginal F_{gb}



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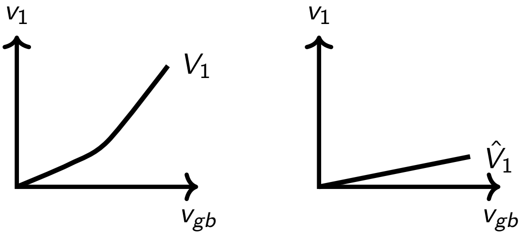
- Let $\pi^* = \max_p p(1 - F_{gb}(p))$, and p^* the price



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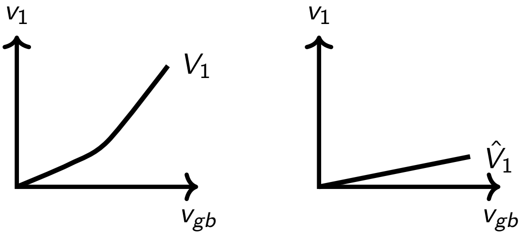

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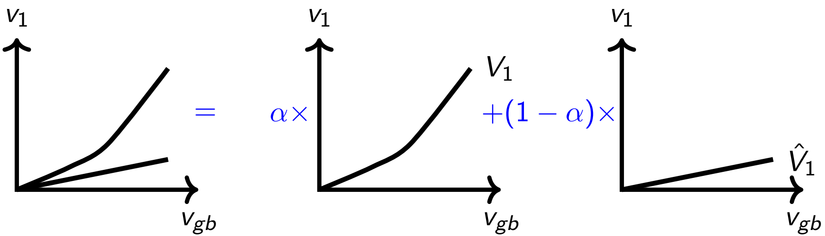

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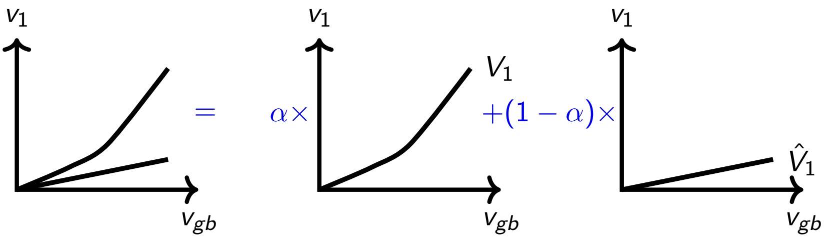

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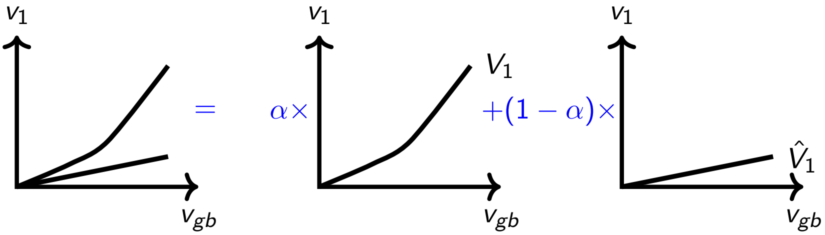
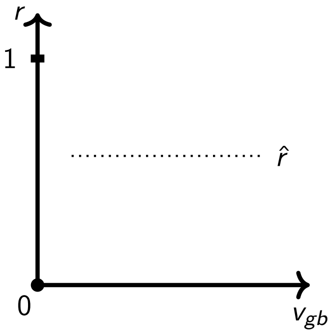


Question: When can a distribution be decomposed?

1. to ratio-monotone paths
2. with same marginal F_{gb}

When Can a Distribution be Decomposed?

r stochastically nondecreasing in v_{gb} ($Pr(r \geq \hat{r} \mid v_{gb}) \uparrow$ in v_{gb})



When Can a Distribution be Decomposed?

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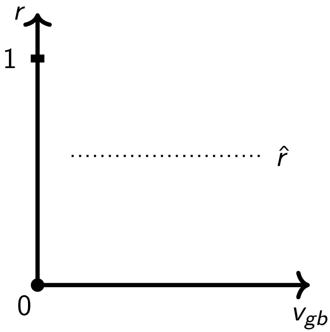
$\Leftrightarrow$ "contour lines" nondecreasing



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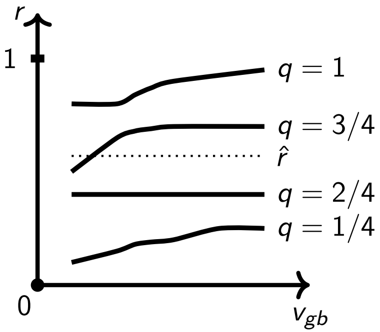

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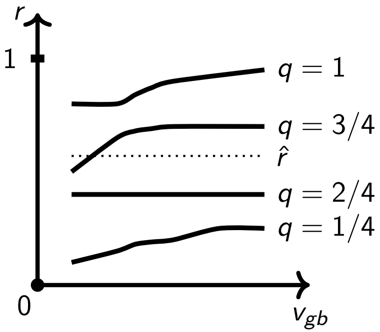
Decompose distribution μ into $\{\mu \mid q\}_{q \in [0,1]}$



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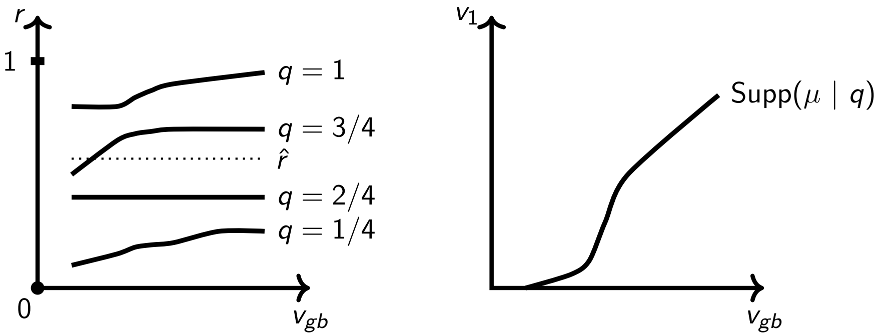

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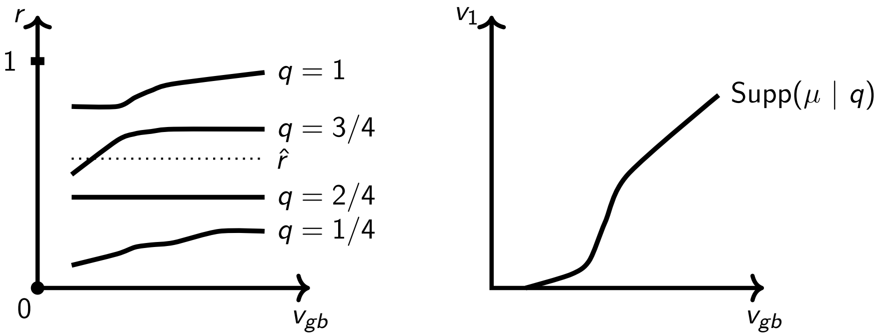
1. Support of each $\mu \mid q$ ratio-monotone
2. q independent from v_{gb}



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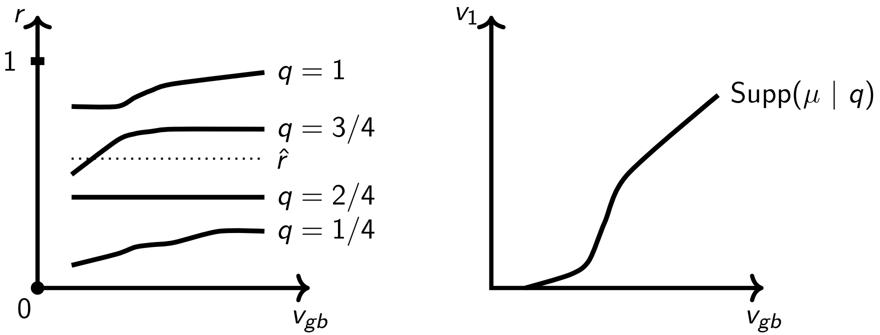
Strassen '65, Kamae et al. '77: generalization to higher dimensions



Pure Bundling not Optimal with Additive Values

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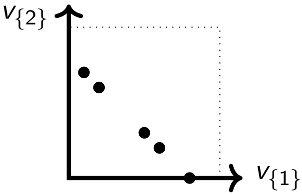
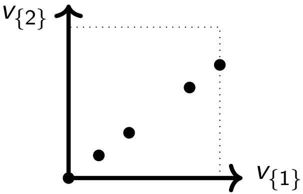
Proposition

PB optimal for all distributions over additive types if either

1. *All types have the same value for the grand bundle*
2. *All types have the same ratio of values*





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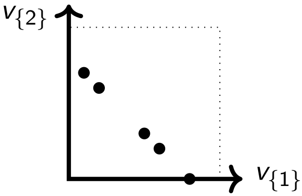

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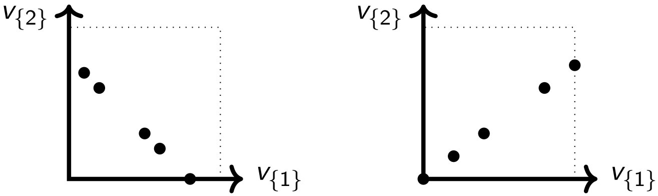
Consider a subset of types on a "path".

$$\frac{v_i}{v_1 + v_2}$$

must be non-decreasing for all i. Thus must be constant.



Related Work

Technically:

- Wilson '93, Armstrong '96: fixed paths
- Eso, Szentes '07; Pavan et al. '14
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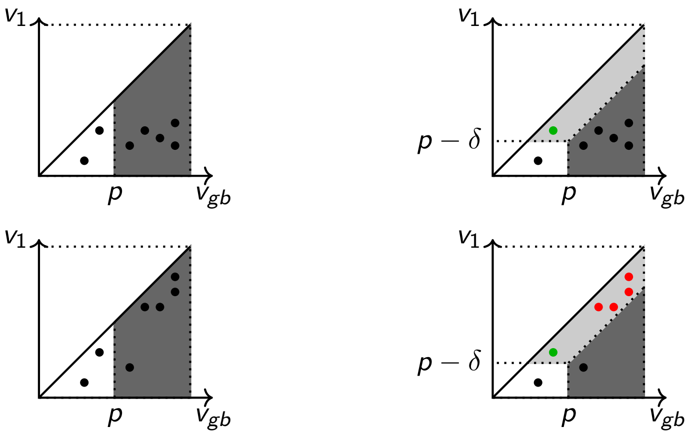
Bundling: Mostly additive values

- Fang and Norman '06: Pure bundling vs. selling separately
- Daskalakis et al. '17: PB optimal if values i.i.d $[c, c+1]$ for large c
  - Pavlov '11, Menicucci et al. '15: Other i.i.d distributions
- McAfee and McMillan '88, Manelli and Vincent '06: optimality of deterministic mechanisms

Main Result

PB optimal if high value consumers have high relative utility



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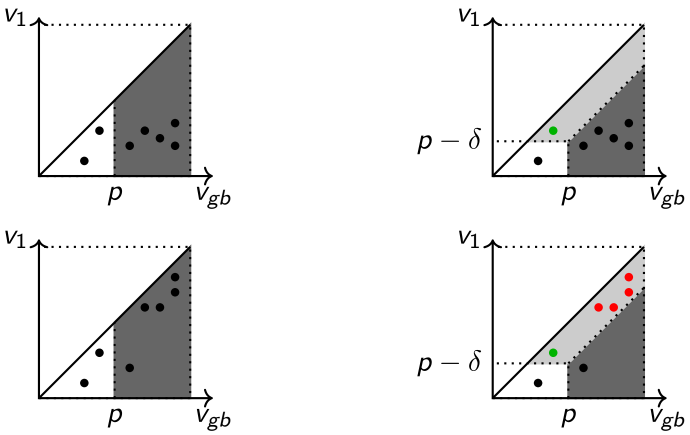

Thanks!