Selling to a Group

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Abstract

A group of agents can collectively purchase a public good that yields heterogeneous benefits to its members. Combining a reduced-form implementation result with a duality argument, we characterize the seller’s profit-maximizing mechanism. Trade outcomes depend solely on a weighted average of the agents’ virtual values, with endogenous voting weights. Heterogeneity in voting weights reflects heterogeneity in agents’ value distributions, where agents with lower value distributions are given more weight in trade decisions. Simple pricing rules are generally not (even approximately) optimal.

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1. Introduction

We study a problem of selling a good to a buyer group, consisting of multiple agents. The problem has two key features. First, the good is public, i.e., conditional on sale, its benefits are enjoyed by all the group members. Second, the purchase is financed from a collective pool of money at the group’s disposal. Examples with these features abound. One can think of a software company that needs to convince a committee of senior managers in an organization to purchase its product, a consultant who sells her proposals to the members of the executive board, or members of a city council deciding on a public project.

Our main goal is to characterize the seller’s optimal (profit maximizing) mechanisms when group members’ benefits from the good are private information, and each group member can individually veto the mechanism. The example below allows us to demonstrate the key forces in our environment, and the qualitative insights that emerge from our characterization.

**Example 1:** Two agents, 1 and 2, have independently drawn values for the good. Agent 1’s value, $\theta_1$, is distributed uniformly over $[0, 2]$; while agent 2’s value, $\theta_2$, is distributed uniformly over $[0, 3]$. The seller can provide at most one unit of the good, and at a zero cost. The seller designs a mechanism—an allocation rule and a transfer rule—that conditions on agents’ reports of their respective types. If a sale occurs and the common transfer is $m$, then each agent’s utility is $\theta_i - m$; if no sale occurs and the transfer is $m$, then each agent’s utility is $-m$. Agents’ outside option from not participating (that is, vetoing the mechanism) is zero.

Let us begin with the simple class of posted price mechanisms. Such mechanisms would be optimal, following classic results, if there were a single agent. Since agent 2’s valuation first-order stochastically dominates that of agent 1, a reasonable starting point is the optimal posted price for agent 2. That is, the seller could post a price of $\frac{3}{2}$ and make a take-it-or-leave-it offer to agent 2. The sale takes place if agent 2 agrees, but not otherwise. However, this mechanism yields a negative (interim) expected payoff to agent 1 when his type is low, and so this agent will want to veto the mechanism. To see this, notice that, say if $\theta_1 = 0$, then the good will be sold at a price of $\frac{3}{2}$ with a positive probability, thereby earning agent 1 a negative utility. It is not difficult to modify the payment rule to make this posted price mechanism individually rational. The seller can offer a subsidy: an additional transfer that she pays the agents regardless of whether a sale occurs. The subsidy must be at least as high as the expected revenue itself to
make the mechanism individually rational, in particular, when $\theta_1 = 0$. Therefore, the seller earns at most zero profit from such a mechanism.

One remedy to the above problem is to require \emph{unanimous} approval of the trade. That is, the good is offered at a price $p$, and is sold if and only if both agents agree to the purchase. We call this a “unanimous posted price” mechanism. This mechanism is individually rational by construction, and generates a profit of $p(1 - \frac{\theta_1}{2})(1 - \frac{\theta_2}{2})$—maximized at $p = \frac{1}{3}(5 - \sqrt{7})$. This maximal profit is approximately 0.35.

Can the seller do better? Her problem is to maximize profit over all (Bayesian) incentive compatible and individually rational mechanisms. If she could use agent-specific transfers, then standard arguments à la Myerson (1981) teach us that a given allocation rule is implementable if and only if its associated interim allocations are nondecreasing. Perhaps surprisingly, even with access only to collective transfers, Lemma 2 shows the same condition characterizes implementability in our setting. However, absent agent-specific transfers, the agents’ average payments must coincide. Hence, the maximal revenue that can be extracted from a given implementable allocation rule is pinned down by the condition that \emph{one} agent’s individual rationality constraint binds (and the others’ are satisfied). Therefore, the profit-maximizing mechanism given an allocation rule $x(\cdot)$ yields a profit of $\min_{i \in N} \mathbb{E}\{x(\theta)[\varphi_i(\theta_i) - c]\}$, where $N$ is the set of agents, $\varphi_i(\cdot)$ is the virtual valuation function of agent $i$, and $c$ is the production cost of the good to the seller.

If we temporarily ignore the above monotonicity condition on allocation rules, we can view the seller’s optimal profit,

$$\max_x \min_{i \in N} \mathbb{E}\{x(\theta)[\varphi_i(\theta_i) - c]\},$$

as the maximin value from a two-player zero-sum game in which one player, the Maximizer, chooses an allocation rule $x(\cdot)$, and the other player, the Minimizer, chooses an agent $i$. We show that this infinite game has a Nash equilibrium, and hence its value is

$$\min_{\omega \in \Delta N} \max_x \mathbb{E}\{x(\theta)[\omega \cdot \varphi(\theta) - c]\},$$

where $\omega$, a distribution over agents, can also be seen as a mixed strategy for the Minimizer. We characterize the equilibria of the above game to establish that the optimal allocation rule is unique and is a weighted voting rule: The good is sold if and only if $\omega \cdot \varphi(\theta) \geq c$, where the optimal weights $\omega$ are characterized by
the simple program, \( \min_{\omega \in \Delta N} E\{[\omega \cdot \varphi(\theta) - c]_+\} \). Moreover, under our regularity assumptions, this allocation rule is monotone and hence solves the seller’s problem. These results are summarized in Theorem 1.

Solving the above problem for Example 1, we obtain that the optimal weights are \( \omega_1 = \sqrt{\frac{2}{7}} \) and \( \omega_2 = 1 - \omega_1 \) and the associated optimal profit is approximately 0.39. Notice how the allocation rule of the optimal mechanism, where the good is sold if and only if \( \omega_1 \theta_1 + \omega_2 \theta_2 \geq 1 + \frac{2}{7} \), is different from the allocation rule of the unanimous posted pricing mechanism, where the good is sold if and only if \( \min(\theta_1, \theta_2) \geq p \). The optimal allocation rule, with its interior \( \omega \), relaxes the stringent requirement of a unanimous agreement.

Consistent with Example 1, generally, the optimal mechanism overweights agents with lower value distributions. In fact, Proposition 2 delivers a quantitative ranking result for the optimal voting weights whenever the optimal mechanism is nontrivial. It says that, if \( \varphi_i \) is smaller than \( \alpha \varphi_j \) in the hazard rate order (a strong form of stochastic dominance that has been previously applied in the auction literature) for some \( \alpha \in (0, 1] \), then the optimal mechanism entails \( \omega_i \geq \alpha \omega_j \). The example exhibits this distributional ranking with \( \alpha = \frac{3}{2} \); and indeed, a direct computation shows \( \omega_1 \geq \frac{3}{2} \omega_2 \). Qualitatively, the seller listens to weaker agents, whose veto constraint binds more tightly, more. The same lesson is reflected in Corollary 2, which shows that the optimal mechanism uses only one agent’s private information if and only if that agent’s value distribution is extremely low relative to others’.

Though unanimous posted prices are not optimal (as we saw above for the example), we do find that they are profit-maximizing within two different natural classes of simple mechanisms. The first is the class of mechanisms for which the price of the good is constant conditional on trade (Proposition 3). The second is the class of deterministic, ex-post individually rational, dominant-strategy incentive compatible mechanisms (Proposition 6). As formalized in Proposition 5, unanimous posted pricing is suboptimal in all interesting instances of our model. In particular, such simple mechanisms are with loss of optimality in such cases. We further quantify this payoff loss in Proposition 7, by considering a many-agent limit case of our model. There, we show that the “price of simplicity”—the limiting ratio of the optimal profit to that of unanimous posted pricing—is unboundedly large except in asymptotically trivial cases.
1.1. Related Work

As the good for sale in our model is public, our work is closely related to the vast literature on designing mechanisms for provision of public goods. The canonical model (e.g., d’Aspremont and Gérard-Varet, 1979) allows for arbitrary monetary transfers between agents. Several papers show, in related contexts, that any mechanism achieving budget balance ex ante can be converted (preserving agents’ incentives) to one that is ex-post budget balanced by choosing ex-post transfers appropriately (Makowski and Mezzetti, 1994; d’Aspremont et al., 2004; Börgers and Norman, 2009). Our construction of ex-post transfer rules that induce a given profile of interim transfer rules is related, especially in the two-agent special case. Rob (1989) shows that with a large number of agents, profit-maximizing mechanisms are inefficient, while Mailath and Postlewaite (1990) extend this inefficiency result to all individually rational and budget-balanced mechanisms. In a setting where agents’ values for a good are symmetric, and each is initially endowed with a share, Cramton et al. (1987) show that efficient and individually rational trading mechanisms exist if and only if agents’ shares are sufficiently symmetric. Guth and Hellwig (1986) identify profit-maximizing mechanisms with incentive compatibility and individual rationality constraints. Hence, our seller’s problem is equivalent to that of Guth and Hellwig (1986), with the added restriction that agent-specific transfers are not available.

Another strand of the literature on public goods studies voting mechanisms without monetary transfers. Starting with Rae (1969), many entries to this literature study mechanisms that maximize utilitarian efficiency. Schmitz and Tröger (2012) and Krishna and Morgan (2015) identify conditions under which (weighted) majority does or does not maximize efficiency. Azrieli and Kim (2014) show that any incentive compatible mechanism must be a weighted majority rule, and characterize the weights that maximize efficiency. Our model is a middle ground between the two aforementioned strands of literature on public goods in the sense that monetary transfers are available in our setting but are restricted to be identical across agents.

Our work is also related to the literature that studies (approximate) optimality of posted pricing mechanisms. Myerson (1981) and Riley and Zeckhauser (1983) show that posted pricing is in fact an optimal strategy for selling a single good to a single agent. Even though posted pricing is no longer optimal in settings with multiple goods or agents, it remains approximately optimal in many such settings (see, for example, Chawla et al., 2010; Chawla et al., 2015; Hart and Nisan, 2017;
In contrast, in our setting, posted pricing mechanisms perform arbitrarily poorly relative to optimal mechanisms as the number of agents grows.

We solve for optimal mechanisms using a simple reduced-form characterization of implementable collective transfer rules. Our work is thus related to the literature on reduced-form implementation in auctions (e.g., Border, 1991; Cai et al., 2012; Alaei et al., 2019). Our implementability result could be repurposed to study interim allocation rules for a real-valued (or nonnegative real-valued) and unbounded public outcome. This explicit, tractable implementability result for transfer rules stands in contrast to the results of Gopalan et al. (2018), who show that if the public outcome is binary-valued (or, equivalently, if it is restricted to some bounded interval), then no computationally tractable characterization of implementable reduced forms of collective transfer rules exists.

Our work is related to the literature on the (in)equivalence of Bayesian and dominant strategy incentive compatibility. Incentive compatibility in our setting with a single good is characterized by standard monotonicity constraints (as in Myerson, 1981). Nonetheless, because individual transfers are not permitted, optimal mechanisms are not dominant-strategy incentive compatible in our setting, except in uninteresting cases. This result stands in contrast to the known results on the equivalence of Bayesian and dominant-strategy incentive compatibility in settings with a single good (Manelli and Vincent, 2010; Gershkov et al., 2013). Such equivalence is known to fail for the sale of multiple goods (Yao, 2017, Manelli and Vincent, 2019).

2. Model

We study the problem of a seller who can sell up to one indivisible good to be shared by a group of agents. We denote the finite nonempty set of agents $N$. The seller incurs a cost $c \geq 0$ if the good is sold. Any monetary transfer paid for the good is borne collectively by the group. Agents are heterogeneous in how they value the good vis-à-vis the group’s money. That is, each agent $i$ has a private type $\theta_i$, which is a random variable taking values in $\Theta_i = [\underline{\theta}_i, \bar{\theta}_i] \subset \mathbb{R}_+$ that is drawn according to a distribution with a CDF $F_i$.\footnote{We use the following standard notation throughout. The set of type profiles is $\Theta := \prod_{i \in N} \Theta_i$, and $\Theta_{-i} := \prod_{j \in N \setminus \{i\}} \Theta_j$ for $i \in N$. We also sometimes use a measure and its CDF interchangeably, and use $F$ and $F_{-i}$ to refer to associated product measures on $\Theta$ and $\Theta_{-i}$, respectively.} We make the following regularity assumption for each $i \in N$: The CDF $F_i$ admits a continuous and strictly positive
density $f_i$ on its support, and the virtual value $\varphi_i(\theta_i) := \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}$ is strictly increasing. It will often be convenient to work directly with an agent’s virtual value $\varphi_i := \varphi_i(\theta_i)$, an atomlessly distributed random variable with convex support.

An outcome of our contracting environment consists of a probability $x \in [0, 1]$ with which the good is sold to the buyer group, and a (signed) transfer $m \in \mathbb{R}$ paid to the seller by the buyer group. The payoff of agent $i$ for this outcome is given by $x\theta_i - m$, while the seller’s payoff is $m - cx$. Let us highlight two distinguishing features of our environment. First, the good is public: Conditional on it being allocated, every agent derives a benefit (equal to his type). Second, the transfers are collective. One could interpret our agents as a group of committee members deciding whether or not to approve a project (purchased from our seller). The transfer paid for the project will come from the common pool of money that the committee can access, while the private benefits each member derives from using the organization’s funds on this particular project may vary.

The seller knows the distribution of types for each agent, but not agents’ realized types. She designs a (direct, without loss) mechanism, which specifies a probability of trade and a total transfer, for every profile of reported types. For most of the paper, we focus on the optimal (i.e., profit-maximizing) mechanism for the seller, among all mechanisms that are incentive compatible and individually rational for the agents. We formally define these standard notions below.

**Definition 1:** A (collective) allocation rule is a measurable function $x : \Theta \to [0, 1]$; let $\mathcal{X}$ denote the set of all allocation rules. A (collective) transfer rule is a bounded measurable function $m : \Theta \to \mathbb{R}$. A (collective selling) mechanism is a pair $(x, m)$ consisting of an allocation rule and a transfer rule.

Say a mechanism $(x, m)$ is incentive compatible (IC) if

$$\theta_i \in \arg\max_{\hat{\theta}_i \in \Theta_i} \mathbb{E}\left[\theta_i x(\hat{\theta}_i, \theta_{-i}) - m(\hat{\theta}_i, \theta_{-i})\right], \ \forall i \in N, \ \forall \theta_i \in \Theta_i;$$

and say it is individually rational (IR) if

$$\mathbb{E}[\theta_i x(\theta_i, \theta_{-i}) - m(\theta_i, \theta_{-i})] \geq 0, \ \forall i \in N, \ \forall \theta_i \in \Theta_i.$$

The profit generated by a mechanism $(x, m)$ is $\Pi(x, m) := \mathbb{E}[m(\theta) - cx(\theta)]$. An optimal mechanism is an IC and IR mechanism that generates a weakly higher profit than any other IC and IR mechanism. Finally, an optimal allocation rule is any allocation rule $x$ such that $(x, m)$ is an optimal mechanism for some
3. Characterizing the Optimal Mechanism

In this section, we fully characterize optimal mechanisms. First, we provide a useful reduced-form implementation result for transfers, characterizing exactly which profiles of interim transfer rules can be implemented with some collective transfer rule. Then, using this characterization, we describe which allocation rules are implementable, and solve for the seller’s optimal profit from implementing such an allocation rule. Next, we establish that a unique optimal allocation rule exists and can be described as a weighted voting rule with weights that we explicitly characterize. Finally, we show that, excepting trivial cases, the voting weights that describe an optimal allocation rule are unique—and hence characterizing them is equivalent to characterizing optimal mechanisms.

We begin by introducing some convenient notation and terminology for standard objects. Just as in the auction setting, the Bayesian incentive properties of our design environment are convenient to discuss in terms of each agent’s interim (that is, conditioning only on his own type) outcomes.

**Definition 2:** Fix any agent $i \in N$. Given an allocation rule $x$, define the \textbf{interim allocation rule} to be $X^x_i : \Theta_i \to \mathbb{R}$ given by $X^x_i(\theta_i) := \mathbb{E}[x(\theta_i, \theta_{-i})]$. Similarly, given a transfer rule $m$, define the \textbf{interim transfer rule} to be $M^m_i : \Theta_i \to [0, 1]$ given by $M^m_i(\theta_i) := \mathbb{E}[m(\theta_i, \theta_{-i})]$.

Now, say an allocation rule $x$ is \textbf{interim-monotone} if $X^x_i$ is weakly increasing for every $i \in N$.

As a first step toward solving our seller’s problem, we provide a simple reduced-form implementation result, which exactly characterizes which profiles of interim transfer rules can be induced when transfers are restricted to be ex-post identical for all agents. It is obviously necessary that the average transfer stipulated by each interim transfer rule should be the same, both being the expected value (by iterated expectations) of a common random variable. The following lemma, which may be of independent interest, shows that this necessary condition is sufficient. Moreover, it shows that, if transfers are restricted to be nonnegative, then pairing that necessary condition with the another obviously necessary condition (that all interim transfers be nonnegative) is again sufficient for implementability.

**Lemma 1 (Reduced-form transfer rules):** Let $M^*_i : \Theta_i \to \mathbb{R}$ be a bounded measur-
able function for each $i \in N$. Then, the following are equivalent:

1. Some transfer rule $m$ exists such that each $i \in N$ has $M^*_i = M^*_i$.
2. The expectations $\{E[M^*_i(\theta_i)]\}_{i \in N}$ all coincide.

Moreover, $m$ can be taken to be nonnegative if and only if each of $\{M^*_i\}_{i \in N}$ is.

The straightforward proof of the above lemma is constructive and resembles previous constructions in the literature (e.g., Makowski and Mezzetti, 1994; d’Aspremont et al., 2004; Börgers and Norman, 2009) that convert ex-ante budget-balanced mechanisms into ex-post budget-balanced mechanisms, while preserving the players’ interim transfer rules. While we apply this result to collective transfers, we imagine future applications could benefit from Lemma 1, treating its nonnegative version as a reduced-form implementability result for (unbounded) public good provision.

Leveraging the above result, the following lemma characterizes allocation rules that are implementable with some transfer rule, as well as the seller’s profit from implementing such an allocation rule.

**Lemma 2 (Implementable allocations):** Let $x$ be some allocation rule.

1. A transfer rule $m$ exists such that $(x, m)$ is IC and IR if and only if $x$ is interim-monotone.
2. If some transfer rule $m$ exists such that mechanism $(x, m)$ is IC and IR, then a maximally profitable such mechanism exists, with resulting profit

$$\min_{\mu \in N} E [x(\mu)(\varphi; - c)].$$

Classic results (Myerson, 1981) would imply that interim-monotonicity fully characterizes implementability, if the seller could freely choose the interim transfer rule that each agent faces. However, our seller is constrained, in that different agents’ interim transfers must be derived from a common ex-post transfer rule. Nevertheless, Lemma 1 tells us the sole constraint that collective transfers place on these interim transfer rules is that they stipulate the same transfer on average. Hence, after modifying the transfer rules by a player-dependent flat subsidy (which does not affect IC), they can be implemented by some ex-post transfer rule. Consequently, interim-monotonicity fully characterizes implementability of an allocation rule, as in the setting with separable payments.

Given that an allocation rule is implementable, the reasoning behind Myerson’s 1981 result determines each agent’s interim transfer rule up to a constant. How-
ever, that each agent must pay the same transfer on average (since they do so ex
post) determines these agent-specific constants up to a single scalar parameter.
Analogous to how an optimal auction would optimize the transfer rule by setting
each agent’s IR constraint to bind, our remaining constant is solved out by impos-
ing that one agent’s IR binds (and the others’ are satisfied). Hence, in contrast
to the implementability question, the seller’s maximum (IC and IR) profit from a
given allocation rule is affected by the fact that interim transfer rules cannot be
separably designed.

With Lemma 2 in hand, our seller’s problem can be cast directly as an opti-
mization over allocation rules, with associated profit of such a rule being pinned
down by revenue equivalence and the principle that IR binds for the worst-off low
type. Formally, the seller’s optimization over allocation rules is:

$$\max_{x \in X} \left\{ \min_{i \in N} \mathbb{E}[x(\theta)(\varphi_i - c)] \right\}$$

s.t. $x$ is interim-monotone.

Our main result is a complete characterization of the solution to the pro-
gram (SP). To this end, we define a class of allocation rules that play a special
role in our analysis and results.

**Definition 3:** Given $\omega \in \Delta N$, the $\omega$-voting rule is the allocation rule $x_\omega := 1_{\omega \cdot \varphi \geq \epsilon}$. Say an allocation rule is a voting rule if it is a $\omega$-voting rule for some
$\omega \in \Delta N$.

We now state our main characterization theorem.

**Theorem 1 (Optimal allocation):** An essentially unique optimal allocation rule
exists and is a voting rule. Namely, the $\omega$-voting rule is optimal for any $\omega \in \Delta N$
that satisfies either of the following two equivalent conditions (and some such $\omega$
exists):

1. $\omega \in \arg\min_{\omega \in \Delta N} \mathbb{E}[(\tilde{\omega} \cdot \varphi - c)_+]$.
2. $\text{supp}(\omega) \subseteq \arg\min_{i \in N} \mathbb{E}[\varphi_i 1_{\omega \cdot \varphi \geq \epsilon}]$.

Moreover, an optimal mechanism exists with nonnegative transfers.

The proof of Theorem 1 studies a relaxed program (RSP) in which the interim-
monotonicity constraint is ignored. In order to solve the relaxed program, we
consider an auxiliary two-player zero-sum game in which the Maximizer chooses
an allocation rule $x$, the Minimizer chooses an agent $i$ whose IC and IR con-
straints must be satisfied, and the objective of the game is $E[x(\theta)(\varphi_i - c)]$—the seller’s highest possible profit from the chosen allocation, subject to the “revenue equivalence” formula and the chosen agent’s IR constraint. Observe that an allocation rule solves (RSP) if and only if it is a cautious optimum for Maximizer in the auxiliary game—that is, a “maximin” strategy. Moreover standard results on zero-sum games imply that a maximin strategy is a Nash equilibrium strategy for the Maximizer, and vice versa. Hence, we turn to characterizing Nash equilibria of the auxiliary game.

We first show that, if the Minimizer is allowed to choose a mixture, then some Nash equilibrium of this auxiliary game exists by the minimax theorem, and every mixed strategy $\omega$ for the Minimizer exhibits a unique (up to almost-everywhere equality) best response $x_\omega$ for the Maximizer. Because the set of Nash equilibria of a two-player zero-sum game exhibits a product structure, it follows that an essentially unique allocation rule can be an optimal strategy for the Maximizer of the auxiliary game. Specifically, this allocation rule is equal to the $\omega$-weighted sum of agents’ virtual values, where $\omega \in \Delta N$ is a Nash equilibrium strategy for the Minimizer. The pair of conditions characterizing such $\omega$ are standard to zero-sum games: The mixed strategy $\omega$ is a cautious optimum for the Minimizer (the first condition) if and only if it is a best response to some Maximizer best response to $\omega$ (the second condition, once Maximizer’s unique best response to $\omega$ is substituted in). Now, observe that the essentially unique Nash equilibrium strategy for the Maximizer is actually interim-monotone: Because virtual values are increasing, a cutoff rule for the $\omega$-weighted virtual value is monotone, hence interim-monotone. The result is a characterization of the unique optimal allocation rule, solving not only (RSP) but also (SP).

Finally, we turn to the form of optimal transfer rules. Having solved for the optimal allocation rule and the expected revenue that the seller garners, each agent’s interim expected transfer rule is fully determined by the classic Myerson (1981) payment formula. Moreover, direct computation shows these interim transfers are always nonnegative; that is, no agent expects (even conditioning on realizing his lowest possible type) to be subsidized on average. Although infinitely many ex-post transfer rules implement these interim transfers, and some will indeed specify a negative payment from the agents, Lemma 1 shows by construction that at least one such transfer rule does not. The theorem follows.

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Given Theorem 1, we can characterize optimal mechanisms by characterizing which voting weights solve the two equivalent conditions listed in the theorem. This goal justifies the following definition.

**Definition 4:** Say \( \omega \in \Delta N \) is an optimal vector of voting weights if it is in \( \text{argmin}_{\omega \in \Delta N} E[(\tilde{\omega} \cdot \varphi - c)_+] \) or, equivalently, has \( E[\varphi_i 1_{\omega \cdot \varphi < c}] \leq E[\varphi_j 1_{\omega \cdot \varphi \geq c}] \) for every \( i, j \in N \) with \( \omega_i > 0 \).

In light of Theorem 1 (together with revenue equivalence), understanding optimal selling mechanisms amounts to understanding which voting weights \( \omega \in \Delta N \) are optimal.\(^3\)

While Theorem 1 delivered the uniqueness of the optimal allocation rule, it is straightforward to see that an optimal \( \omega \) need not be unique in trivial cases. For example, if \( \max_{i \in N} \theta_i \leq c \), it is easy to see that every choice \( \omega \in \Delta N \) of voting weights is optimal, each inducing a mechanism in which trade never occurs. Similarly, optimal weights may not be unique when the optimal mechanism stipulates that trade always occurs. This observation motivates the following definition.

**Definition 5:** The never-trade mechanism is given by \( (x, m) = (0, 0) \). The always-trade mechanism is given by \( (x, m) = (1, \min_{i \in N} \theta_i) \) [resp. \( x = 1 \)]. A mechanism \( (x, m) \) or an allocation rule \( x \) is trivial if \( E[x(\theta)] \in \{0, 1\} \) and non-trivial if \( 0 < E[x(\theta)] < 1 \).

The next result shows that the above multiplicity happens only when it is optimal to either never trade or always trade, and characterizes when this is the case. In all other (and so in all interesting) cases, the optimal \( \omega \) is unique.

**Proposition 1** (Trivial optimal mechanisms and uniqueness of weights):

1. The never-trade mechanism is optimal if and only if \( \min_{i \in N} \theta_i \leq c \).
2. The always-trade mechanism is optimal if and only if some \( i \in \text{argmin}_{j \in N} \theta_j \) exists such that \( \varphi_i(\theta_i) \geq c \).
3. In all other cases, a nontrivial mechanism is optimal, and a unique \( \omega \in \Delta N \) exists such that \( x_\omega \) is optimal.

Because the never-trade and always-trade mechanisms are obviously optimal among mechanisms that never or always allocate, respectively, the real content of the first two statements of the proposition is a characterization of when each of

\(^3\)As a trivial observation, note that an optimal (indirect) mechanism exists in which each bidder submits a vote from a bounded interval, with the good being provided if and only if the weighted sum of votes exceeds a threshold.
the $x = 0$ and $x = 1$ allocation rules is optimal.\footnote{Although Theorem 1 enables a unified simple proof of this and related results, let us note that the characterization of when the never-trade mechanism is optimal is easy to establish directly. If $\bar{\theta}_i \leq c$ for some $i \in N$, then one can show agent $i$’s IR constraint implies $E[m(\theta_i)] \leq c E[x(\theta)]$, precluding positive profit. If $\min_{i \in N} \bar{\theta}_i > c$, then posting a price strictly between these two quantities, and requiring unanimous agreement to buy, generates positive profit.} Given Theorem 1, it suffices to check when an optimal voting weight vector exists that generates each of these allocations—a straightforward computation.

The third statement provides further comfort in directly interpreting $\omega$ as voting weights, as it delivers (in all nontrivial cases) a one-to-one correspondence between the optimal allocation rule and the weights that describe it. The proof is straightforward: We show that distinct voting weights that generate nontrivial allocation rules must generate distinct allocations with positive probability, and then the uniqueness part of Theorem 1 applies.

4. The Role of Heterogeneity

Since optimal mechanisms take the form of a weighted voting rule, it is natural to consider the relative voting weights of different agents. This section asks which agents are assigned a high weight in determining the trade decision.

In a relaxed program in which the seller can observe agents’ types (but is still subject to IR), trade occurs if and only if the lowest-value agent’s type exceeds the production cost. Hence, a reasonable intuition is that agents with the lowest value distribution (in some sense) will be overweighted in the decision. Indeed, examining whether the extreme always-trade and never-trade mechanisms are optimal shows that, when one agent has a stochastically lowest value distribution, then that agent alone determines the (sub)optimality of said mechanism.

**Corollary 1** (Low-value agents and the extensive margin): Suppose $i \in N$ is such that $\theta_i$ is (weakly) first-order stochastically dominated by $\theta_j$ (that is, $F_i \geq F_j$) for each $j \in N$. Then the never-trade [resp. always-trade] mechanism is optimal if and only if $\bar{\theta}_i \leq c$ [resp. $\varphi_i(\theta_i) \geq c$].

The above corollary follows directly from Proposition 1, once one observes that an agent with a (first-order stochastically) lowest value distribution necessarily has a lowest high type, a lowest low type, and a lowest low virtual valuation among those that have the lowest low type.

While Corollary 1 provides a sense in which a weak agent is pivotal to the
nature of the allocation, it is extremely weak. In particular, conditional on one of these trivial mechanisms being used, all agents are treated equally. Hence it is desirable to understand when an agent is important to determining the allocation. A particularly strong notion of pivotality is captured by the following definition.

**Definition 6:** Given an agent \( i \in N \), let \( i \)-dictatorship refer to the vector \( \omega \in \Delta N \) of voting weights in which all agents other than \( i \) are ignored, that is, with \( \omega_j = 1_{i=j} \) for each \( j \in N \).

The following corollary shows exactly when an \( i \)-dictatorship mechanism is optimal for the seller. Such an allocation rule can be implemented by posting a price \( p \in [\bar{\theta}_i, \bar{\theta}_i] \) that ensures trade occurs if and only if \( i \)'s virtual value exceeds the production cost. The next result, a nearly immediate consequence of Theorem 1, shows that such a mechanism is optimal if and only if this price is such that every type of every other agent would happily trade at that price.

**Corollary 2 (Dictatorship):** Given \( i \in N \), the following are equivalent:

1. The \( i \)-dictatorship mechanism is optimal.
2. Either no price \( p \in (c, \bar{\theta}_i) \) exists, or the optimal posted price \( p \) when facing only \( i \) satisfies \( p \leq \theta_j \) for every agent \( j \neq i \).
3. Either \( \bar{\theta}_i \leq c \), or \( \bar{\theta}_i > c \) and \( \mathbb{E}[\varphi_j] \geq \mathbb{E}[\varphi_i | \varphi_i \geq c] \) for every agent \( j \neq i \).

The previous result shows that the case of dictatorship is very special, requiring extreme asymmetry. The next result suggests that the more typical optimal mechanism pays some attention to all agents. Indeed, in the nontrivial case, if the lowest possible value is zero for each agent, then the result says the unique optimal allocation is responsive to every agent’s private information.

**Corollary 3 (Not ignoring the lowest types):** Suppose the optimal allocation rule is nontrivial. If \( i \in N \) has \( \theta_i \leq \theta_j \) for every \( j \in N \), then the unique optimal voting weights \( \omega \in \Delta N \) have \( \omega_i > 0 \). In particular, if \( \theta = \bar{\theta} \), then \( \omega_i > 0 \) for every \( i \in N \).

The proof of the above corollary is nearly immediate from Theorem 1. In the auxiliary zero-sum game that characterized optimal allocation rules, we show that any agent whose lowest type is (weakly) lower than everybody else’s will necessarily be a unique best response for the Minimizer to any allocation rule that ignores his type.

The previous results of this section have all spoken to the choice of which agents will exert *some* influence over the eventual trade decision in the optimal
mechanism, but they have been silent on the degree of such influence. For the remainder of this section, we pursue a quantitative analysis of the optimal voting weights. Specifically, we seek conditions on primitives under which we can rank $\omega_i$ and $\omega_j$ for two agents $i$ and $j$ (and under which we can quantify a wedge between these two weights). To state our main condition, we invest in the following distributional ranking definition.

**Definition 7:** Given two real random variables $v$ and $w$ with respective CDFs given by $G$ and $H$, say $v$ is smaller than $w$ in the hazard rate order if $\sup \supp v \leq \sup \supp w$, and $\frac{1-H}{1-G}$ is weakly increasing on $(-\infty, \sup \supp v])$.  

The above distributional ranking is a useful strengthening of first-order stochastic dominance. Intuitively, the ranking requires that the conditional distributions, when conditioned on lying above any common threshold, are stochastically ranked. This ranking condition has been fruitful in past work in mechanism design. Specifically, in the literature on asymmetric auctions (e.g., Maskin and Riley, 2000; Kirkegaard, 2012), ranking bidders’ value distributions via the hazard rate order has enabled the ranking of equilibrium bidding behavior, which in turn has been used to provide revenue rankings for alternative auction formats. In our setting, as the following proposition shows, a hazard rate order on agents virtual value distributions is of relevance in designing optimal selling mechanisms.

**Proposition 2 (Ranking voting weights):** If $\varphi_i$ is smaller than $\alpha \varphi_j$ in the hazard rate order, where $\alpha \in (0, 1]$, and the optimal allocation rule is nontrivial, then the unique optimal vector of voting weights $\omega$ satisfies $\omega_i \geq \frac{1}{\alpha} \omega_j$.  

We prove the proposition by contradiction, assuming that some optimal voting weights fail to satisfy the desired ranking property. Modifying the weights on agents $i$ and $j$, we construct an alternative vector of voting weights for the seller to use. To show that the modified weight vector is optimal, we consider the loss function that optimal weight vectors minimize, observing that it is an increasing and convex function of the weighted sum of $i$ and $j$’s virtual values. Hence, we can apply known results that translate hazard rate orders on random variables to

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5If $v$ and $w$ both admit continuous densities on their supports, then the monotonicity of the function $\frac{1-H}{1-G}$ can be equivalently expressed (taking a derivative) as requiring that the hazard rate of $v$ is weakly below that of $w$ on the same interval.

6These papers rank bidders’ value distributions according to the reverse hazard rate order—which, by Theorem 1.B.41 of Shaked and Shanthikumar (2007), amounts to a hazard rate ordering of their negative value distributions.

7One can easily show the conclusion also holds for some optimal voting weight vector, in the case that the optimal allocation rule is trivial.
increasing convex orders on their weighted sums, when the weights are rearranged to be assortative with the random variables. The upshot is a distinct voting weight vector that must also be optimal. Although the new optimum we construct may not itself satisfy the $\alpha$-ranking property that the proposition requires, we still derive a contradiction with the uniqueness of optimal voting weights that Proposition 1 guarantees. Hence, the result follows.

We conclude the section with two different classes of examples to which Proposition 2 applies. One is a rigid ranking of value distributions, where the distribution of a given agent’s values can be arbitrary (subject to our standing regularity conditions) but different agents’ distributions are assumed proportional. The other is the full class of uniform distributions. In either case, we note that the agents’ distributions of virtual values inherits the same structure, making the proposition straightforward to apply.

**Example 2** (Proportional value distributions): Suppose $\theta_i / \bar{\theta}_i$ and $\theta_j / \bar{\theta}_j$ are identically distributed—for example, this property would hold if $\theta_i$ and $\theta_j$ were uniformly distributed on $[0, \bar{\theta}_i]$ and $[0, \bar{\theta}_j]$, respectively. Then, a direct computation shows $\varphi_i / \bar{\theta}_i$ and $\varphi_j / \bar{\theta}_j$ are identically distributed too. Proposition 2 then implies, if $\bar{\theta}_i \leq \bar{\theta}_j$, that (in the nontrivial case) the optimal voting weight vector $\omega \in \Delta N$ has $\omega_i \geq (\bar{\theta}_j / \bar{\theta}_i) \omega_j$.

**Example 3** (Uniform value distributions): Suppose $\theta_i$ and $\theta_j$ are both uniformly distributed on their respective supports. Direct computation shows that $\varphi_i$ and $\varphi_j$ are then uniformly distributed on $[2\bar{\theta}_i - \bar{\theta}_i, \bar{\theta}_i]$ and $[2\bar{\theta}_j - \bar{\theta}_j, \bar{\theta}_j]$, respectively. Hence, within this parametric class, the distributional ranking of $\varphi_i$ and $\alpha \varphi_j$ reduces to $\bar{\theta}_i \leq \alpha \bar{\theta}_j$ and $2\bar{\theta}_i - \bar{\theta}_i \leq \alpha (2\bar{\theta}_j - \bar{\theta}_j)$. Applying Proposition 2 tells us that (in the nontrivial case) the optimal voting weight vector $\omega \in \Delta N$ has either $\omega_i = \omega_j = 0$ or $\omega_i > \omega_j$, if $2(\bar{\theta}_j - \bar{\theta}_i) > \bar{\theta}_j - \bar{\theta}_i > 0$.

5. **Simple Mechanisms**

In this section, we study simple mechanisms. We first formulate a permissive class of posted price mechanisms in which the allocation rule is potentially flexible but the price is fixed, and show that a unanimous posted price mechanism is maximally profitable within this class. The unanimous posted price mechanism is strategically simple, namely, it is dominant-strategy incentive compatible and ex-post individually rational. We in fact show that the unanimous posted pricing
is maximally profitable within a subclass of mechanisms satisfying these two properties; our argument rests on a characterization of all mechanisms that implement deterministic outcomes via dominant strategies. Using the same characterization, we show that, in all interesting cases of our model, no optimal mechanism is dominant-strategy incentive compatible. Finally, we compare optimal mechanisms to simple ones as the number of agents grows, showing that unanimous posted pricing profit can perform arbitrarily poorly relative to optimal profit.

5.1. Posted price mechanisms

An influential result in the mechanism design literature is that a take-it-or-leave-it posted price is the optimal mechanism for selling a single indivisible good to a single agent (Myerson, 1981; Riley and Zeckhauser, 1983). This type of mechanism is ubiquitous and simple, and enjoys appealing computational properties. Moreover, beyond the single-agent setting, environments have been identified in which such pricing mechanisms remain approximately optimal (Chawla et al., 2010; Chawla et al., 2015; Hart and Nisan, 2017; Babaioff et al., 2020). A natural question, then, is whether posted price mechanisms remain optimal for our seller. Having focused on characterizing optimal implementable allocation rules, with relatively little attention paid to the exact implementing transfers, our analysis to this point has left this question unaddressed.

Logically prior to the above question about optimal mechanisms is the question of how one should define a posted price mechanism. In the one-agent setting, the IC direct mechanisms that correspond to a posted price are exactly those satisfying two properties. First, the transfer is directly proportional to the allocation probability. And second, the allocation probability is 1 for types above the price and 0 for those below it. The first condition—which we can interpret as a restriction that money never changes hands if the good is not sold and that the price at which trade occurs is constant when it does—generalizes immediately. But the second condition—which we can interpret as the agent freely deciding whether or not to execute trade—is less immediate to generalize to the multi-agent setting. Who decides whether trade occurs? Once the seller announces a price for the good, a complex negotiation process could ensue between the agents in deciding whether to buy or not. Might eventual trade outcomes arise from a mixed-strategy equilibrium of the resulting bargaining game between the agents? Can the seller intervene and aid the bargaining process?

In light of these difficulties, we define a collective posted price rather permis-
sively, only incorporating the first of the two conditions mentioned in the previous paragraph. We also introduce a specific, interpretable pricing mechanism, that will be important for our results.

**Definition 8:** A mechanism \((x, m)\) is a **collective posted price** mechanism if some \(p \in \mathbb{R}\) exists such that \(m = px\). Say it is a **unanimous posted price** mechanism if it is a collective posted price mechanism in which \(x(\theta) = 1_{\min_{i \in N} \theta, \geq p}\) for every \(\theta \in \Theta\).

One can envision several examples of collective posted price mechanisms. For example, the seller could set a price \(p\) and execute a sale if and only if all agents agree to the purchase—defined above as a unanimous posted price. Alternatively, the principal could post a price, and select an agent, or even a subset of agents, perhaps randomly, and sell the good if all the agents in this chosen subset agree to the purchase. Another mechanism would post the price and execute trade if and only if at least one agent, or perhaps some majority of agents, champions the sale.

While the space of all collective posted price mechanisms is rather rich, the next result shows that (perhaps) the simplest example is optimal among them.

**Proposition 3 (Optimal price is unanimous):** A unanimous posted price mechanism, with price \(p\) that solves

\[
\max_{p \in [c, \infty)} \left\{(p - c) \prod_{j \in N} [1 - F_j(p)]\right\},
\]

gerates the highest profit among all IC and IR collective posted price mechanisms.

Toward establishing this result, we first show that the collection of IC and IR collective posted price mechanisms is quite limited. Indeed, an agent will want to minimize [resp. maximize] the probability of trade when his type is below [resp. above] the price, so that his interim allocation rule must be constant below [resp. above] the price. But then, IR implies that this interim probability is actually zero when the agent’s type is below the price. Therefore, trade is a zero-probability event conditional on any agent having a realized valuation below the price. From this observation, it is easy to see that any collective posted price mechanism is outperformed by some unanimous one. Indeed, if the price were weakly below the production cost \(c\), then profit would be nonpositive, and a unanimous posted price of \(c\) would be at least as good; and if the price were \(p > c\), then a unanimous
posted price of $p$ would generate profitable trade with a higher probability.

Having characterized the optimal form of collective posted price mechanism, we are poised to answer the question that motivated this subsection: When are collective posted price mechanisms optimal? The result below establishes that they never are, restricted to interesting instances of our model. Specifically, whenever it is optimal to factor in multiple agents’ information at all, it is optimal to use their reports to fine-tune the price of trade.

**Proposition 4 (Posted prices are suboptimal):** The following are equivalent.

1. Some optimal mechanism $(x, m)$ is a collective posted price.
2. An $i$-dictatorship is optimal for some $i \in N$.

Because the optimal $i$-dictatorship mechanism is a posted price (and this price is below other agents’ lowest types by Corollary 2), we need only show that a non-dictatorship optimal mechanism is not a collected posted price. Because the never-trade and always-trade mechanisms are special instances of dictatorship mechanisms, we need only focus on nontrivial mechanisms. Moreover, in light of Proposition 3, it suffices to show it is not a unanimous posted price. So consider an agent who receives weight $\omega_i \in (0, 1)$ in the optimal allocation rule. His interim allocation probability is nonconstant and varies smoothly with his type, and so cannot be a step function. But a unanimous posted price would make this interim allocation rule a step function, so that the two cannot coincide.

5.2. Dominant strategies

The notion of incentive compatibility we have used so far in our analysis is Bayesian incentive compatibility (BIC, which we have denoted IC earlier in the paper), which requires only that agents’ reports be best responses in expectation, given their own realized types. Similarly, as a participation constraint, we have required that each agent (knowing his own type) prefers in expectation to interact with the mechanism rather than walk away. Here, we explore our environment under more demanding incentive and participation constraints, which we formalize through direct mechanisms below (in light of the revelation principle).

**Definition 9:** Say a mechanism $(x, m)$ is dominant-strategy incentive compatible (DIC) if

$$\theta_i \in \arg \max_{\hat{\theta}_i \in \Theta_i} \left\{ \theta_i x(\hat{\theta}_i, \theta_{-i}) - m(\hat{\theta}_i, \theta_{-i}) \right\}, \forall i \in N, \forall \theta \in \Theta;$$

(DIC)
and say it is ex-post individually rational (epIR) if

$$\theta, x(\theta) - m(\theta) \geq 0, \forall i \in N, \forall \theta \in \Theta.$$  \hspace{1cm} \text{(epIR)}

A mechanism \((x, m)\) or an allocation rule \(x\) is bang-bang if \(x(\theta) \in \{0, 1\}\) almost surely.

A mechanism is DIC if an agent finds truthful reporting dominant in the direct revelation game—that is, he would willingly report truthfully even if he knew others’ reported types. Likewise, the mechanism is epIR if an agent (knowing his own type) would rather interact with the mechanism than take an outside option of zero, even if he knew others’ reported types. Finally, it will be useful for our analysis to consider mechanisms which are bang-bang, in that the allocation rule is deterministic conditional on the reported profile of types.

We showed in Lemma 2 that, for a given allocation rule, interim monontonicity was equivalent to BIC implementability. Said differently, we showed that being able to BIC-implement an allocation rule with agent-specific transfers was equivalent to being able to do so with only collective transfers. Moreover, Theorem 1 explicitly characterized the allocation rule from optimal BIC and IR mechanisms, showing it stipulated trade if and only if an player-weighted virtual value exceeded the cost of production. Notice, though, that this allocation rule is monotone in the agents’ profile of types. If our seller could engage in agent-specific transfers, such monotonicity would render the same allocation rule DIC-implementable too. A natural conjecture, then, is that (like in optimal auction settings) DIC can be attained by our seller at no additional cost.

We will show that, in the interesting case of our model, the restriction to DIC mechanisms is with loss of optimality for the seller. Toward showing this fact, we first provide a structural lemma that characterizes the full class of DIC bang-bang mechanisms, as summarized in two properties. The first property concerns the transfer: Any DIC bang-bang mechanism is, up to a flat upfront transfer, equivalent to a collective posted price mechanism. To be precise, the transfer rule can be decomposed into one quantity \((-s)\) that will appear whether or not trade occurs, and another transfer \((p)\) that will additionally be made if and only if trade occurs. The second property gives a representation of the allocation rule. It says that trade is determined by the price and \(J\), a collection of subsets of \(N\) such that the good is sold if and only if, for some \(J \in J\), every agent in \(J\) agrees to the purchase.
**Lemma 3 (DIC mechanisms):** Suppose \((x,m)\) is a DIC bang-bang mechanism. Then, some \(p \in \mathbb{R}_+\), \(s \in \mathbb{R}\) and \(\mathcal{J} \subseteq 2^N\) exist such that, almost surely:

1. \(m(\theta) = px(\theta) - s\);
2. \(x(\theta) = 1_{\cup_{j \in \mathcal{J} \cap \{i \mid \theta_j > p\}}}

Moreover, we may assume without loss that no two members of \(\mathcal{J}\) are nested, and that \(\bar{\theta}_j < p < \bar{\theta}_j\) for each \(j \in \bigcup \mathcal{J}\).

The above class of mechanisms is somewhat flexible. As examples, setting \(\mathcal{J} = \{N\}\) corresponds to a unanimous posted price; setting \(\mathcal{J} = \{\{i\}\}\) for \(i \in N\) means posting a price and letting \(i\) decide; setting \(\mathcal{J} = \{\{j\} : j \in N\}\) means posting a price at which trade will occur if *at least* one agent wants it; and setting \(\mathcal{J} = \{|J| : J \subseteq N, |J| > \frac{1}{2}|N|\}\) means posting a price and having a majority vote on whether or not to purchase.

The proof of the above structural lemma proceeds in two steps. First, we show that the transfer rule is constant among type profiles leading to certain trade, and constant among type profiles leading to non-trade. To prove this property, consider any two type profiles \(\theta\) and \(\theta'\) such that \(x(\theta) = x(\theta')\); say this trade probability is equal to 1, the alternative case being analogous. Letting \(\theta^*\) be a type profile that is coordinatewise higher than both \(\theta\) and \(\theta'\), it is easy to construct a finite sequence of type profiles with the following property: The first type profile in the sequence is \(\theta\) and the last is \(\theta^*\), the type profiles get coordinatewise higher as the sequence progresses, and consecutive entries in the sequence differ in only one agent’s type. But then, because DIC (for the agent whose type is raised in a given increment of the sequence) implies \(x\) must be monotone, it follows that every type profile in the sequence generates probability 1 of trade. Hence, DIC (again, for the agent whose type is incremented) implies that consecutive sequence profiles yield an identical transfer. A symmetric argument applies to \(\theta'\), so that \(m(\theta') = m(\theta^*) = m(\theta)\).

The second property that the proof establishes is the structure on the allocation rule. Given that the mechanism is incentive-equivalent to a collective posted price of \(p\), DIC implies that (given a realization of others’ types) the trade decision must be identical for all types of agent \(i\) below \(p\) and for all types of agent \(i\) above \(p\). Hence, the allocation rule is essentially an increasing \(\{0,1\}\)-valued transformation of the set-valued function \(\theta \mapsto \{j \in N : \theta_j \geq c\}\). The second property amounts to a more explicit description of such increasing functions.

Using the above characterization of DIC bang-bang mechanisms, we can address
the main question of this subsection. Analogous to Proposition 4, the following result says that the optimal allocation rule can be made DIC if and only if it is a dictatorship.

**Proposition 5 (Dominance binds):** The following are equivalent:

1. An optimal IC, IR mechanism exists that is also DIC.
2. An $i$-dictatorship mechanism is optimal for some $i \in N$.

Because a dictatorship conditions only on one agent’s private information, it is immediate that implementing transfers can be chosen (e.g. with a posted price) to ensure that DIC is satisfied. To prove the converse, suppose that the optimal allocation rule $x$ characterized in Theorem 1 is DIC; we aim to show it is a dictatorship. Because this allocation rule is bang-bang, we can apply Lemma 3 to it. So let a $(p, s, J)$ be the parameters delivered by that lemma (and suppose, in line with the last sentence in the lemma, that $J$ has no redundancy in it). In particular, letting $Z := \prod_{j \in N} [\varphi_j(\theta_j), \varphi_j(\bar{\theta}_j)]$ be the set of virtual value profiles, the set of profiles at which trade occurs is essentially $Z^* := \bigcup_{J \in \mathcal{J}} \bigcap_{j \in J} \{z \in Z : z_j \geq \varphi_j(p)\}$. Because $x$ is equivalent to $x_\omega$ for some weight vector $\omega \in \Delta N$, it follows that $Z^*$ and $Z \backslash Z^*$ are both convex. However, looking at averages of virtual value profiles near the boundary of $Z^*$ reveals that $Z^*$ cannot be convex if $J$ contains two different sufficient coalitions, and $Z \backslash Z^*$ cannot be convex if some minimal coalition in $J$ contains at least two agents. It follows that the mechanism is either the never-trade mechanism ($J = \emptyset$), the always-trade mechanism ($J = \{\emptyset\}$), or a nontrivial $i$-dictatorship ($J = \{i\}$) for some $i \in N$. The result follows.

Given that a DIC constraint is typically with loss of optimality, and that DIC constitutes a desirable robustness property for a mechanism, it is natural to wonder what the seller’s profit-maximizing mechanism is in the presence of this constraint. Moreover, the same robustness concern that makes DIC appealing as an incentive constraint also makes epIR the natural participation constraint. Hence we would like to know the seller’s profit-maximizing mechanism among all DIC and epIR mechanisms. As of now, the structure of such a mechanism remains an open question. However, given Lemma 3, we can provide a characterization for the restricted (ad hoc) class of mechanisms in which the allocation is deterministic for each realized type profile.

**Proposition 6 (Best DIC mechanism):** The unanimous posted price of Proposition 3 maximizes the seller’s profit among all DIC epIR bang-bang mechanisms.

To prove the result, we start with an arbitrary DIC epIR bang-bang mecha-
nism, which Lemma 3 says we can represent via parameters \((p, s, J)\). The epIR constraint tells us that the subsidy \(s\) is always nonnegative, and that the price \(p\) is weakly higher than the subsidy. We then show that lowering the price and subsidy by the same amount, and keeping fixed the set \(J\) of sufficient buying coalitions, can only increase the profitability of the mechanism. The payoff gains come from two sources. First, the subsidy is paid to agents always, while the price is collected only sometimes; and hence reducing both would be profitable if trade decisions were not affected. Second, the probability of trade is increased as the price is lowered; this increase benefits the seller because the price is higher than the production cost (if the original mechanism was generating positive profit). It follows from this argument that our arbitrary (DIC epIR bang-bang) can be improved upon by replacing it with one that uses no subsidy. But then, the original mechanism is less profitable than some collective posted price, and so the optimality of unanimous posted prices follows directly from Proposition 3.

As mentioned above, our analysis in this subsection restricts attention to bang-bang mechanisms for the sake of tractability. While this restriction is sufficient to show that DIC is with loss of optimality (Proposition 5), a natural question is how the optimization over DIC epIR mechanisms (Proposition 6) extends beyond the bang-bang case. If our seller could engage in agent-specific transfers, so that DIC-implementability would only correspond to an ex-post monotonicity constraint on the allocation rule, then the extreme points of the space of DIC-implementable allocation mechanisms would be exactly the bang-bang ones.\(^8\) Hence, with agent-specific transfers, a standard argument would tell us that profit-maximizing DIC epIR allocation rules could be found among mixtures of few bang-bang such mechanisms.\(^9\) But we have already observed that monotonicity is not sufficient for DIC-implementability when transfers are collective (e.g., Proposition 5 typically shows our optimal allocation rule serves as an example), so that the above argument does not apply. It is an interesting open question whether or not the DIC bang-bang mechanisms of Lemma 3 are sufficient to “generate” (by averaging or in some other useful sense) the universe of DIC-implementable allocations.

---

\(^8\)The proof of Lemma 2.7 from Börgers (2015) extends immediately to show any non-bang-bang monotone allocation rule is non-extreme among the monotone allocation rules. Moreover, that the space of allocation rules is convex and (in some appropriate topology) compact implies, given Choquet’s theorem, that any monotone allocation rule is an average of bang-bang monotone allocation rules.

\(^9\)The relevant program for the seller would amount to an linear optimization over monotone allocation rules, subject to \(|N|\) linear constraints, and so an optimum could be found among mixtures of \(|N| + 1\) or fewer extreme points (Dubins, 1962).
5.3. Many agents and suboptimality of posted pricing

Our model, with each agent needing to approve of the mechanism (captured by our IR constraint) is perhaps best suited to model bargaining with a small-to-moderate group of agents. Nevertheless, because our seller’s optimum typically combines the private information of multiple agents in a detailed way, it is natural to study the behavior of our model in the limiting case where the number of agents is large. Doing so is the goal of the present subsection.

Suppose all agents are identical, with \( \theta_i = \bar{\theta}, \bar{\theta}_{i} = \bar{\theta}, \) and \( F_i = F_1 \) for every \( i \in N \). Holding fixed these parameters and the cost of production, we can think of our model as being parameterized by the number of agents, \( |N| \in \mathbb{N} \).

The result below characterizes the limiting profit that our seller derives from employing an optimal mechanism, in the many-agent limit. Also, because simple mechanisms (either formalized by a constant price condition as in Proposition 3 or robust incentive conditions as in Proposition 6) seem especially appealing when there are many agents collectively deciding, we record the asymptotic profit of using a simple mechanism.

**Proposition 7 (Asymptotically optimal mechanisms):** Suppose all agents are identical, with types drawn via \( F_1 \). Then, as \( |N| \to \infty \):

1. If \( c \not= \bar{\theta}_1 \), then the optimal profit converges to \( (\bar{\theta}_1 - c)_+ \), and the highest profit among collective posted price mechanisms converges to the same.

2. If \( c = \bar{\theta}_1 \), then the optimal profit converges to \( \max_{p \in \mathbb{R}} (p - c)[1 - F_1(p)] > 0 \), and the highest profit from a collective posted price converges to zero.

To prove the proposition, first note that the set of optimal voting weights is convex and invariant to coordinate permutations, and hence the uniform weight vector is optimal. But then, trade occurs in the optimal allocation rule if and only if the (unweighted) average virtual value exceeds the cost. As agents’ virtual values are independent (and bounded), the distribution of their average converges to a degenerate distribution on their expectation, \( \mathbb{E}[\varphi_1] = \bar{\theta}_1 \). From here, the proposition’s first point is straightforward. The probability of trade converges to 1 if \( \bar{\theta}_1 > c \) and to 0 if \( \bar{\theta}_1 < c \). Hence, in these two special cases, the always-trade mechanism or never-trade mechanisms are asymptotically optimal. But both of these asymptotically optimal mechanisms are specific instances of unanimous posted price mechanisms, and so this restricted class of mechanisms has the same asymptotic payoff.
All that remains is the specific case in which $c = \theta_1$. This case is in some sense canonical, capturing the case of revenue maximization with no a priori lower bound on the potential gains from trade. First, consider posted price mechanisms; by Proposition 3, we can focus on unanimous posted prices. Let $p$ be any asymptotically profit-maximizing posted price, that is, any limit point of profit-maximizing unanimous posted prices as $|N| \to \infty$. If $p \leq c$, then the asymptotic profit per trade is nonpositive; and if $p > c = \theta_1$, then the asymptotic probability of trade is $\lim_{n \to \infty} [1 - F_1(p)]^n = 0$. In either case, positive profit is not attainable in the limit with a posted price.

Now, in the case that $c = \theta_1$, let us consider the optimal (IC and IR) mechanism. Our limit characterization relies on the fact that a representative agent (say agent 1) faces a very predictable average of other agents’ virtual values. Therefore, the centered random variable

$$\epsilon := c - \left(1 - \frac{1}{|N|}\right) \sum_{j \neq 1} \frac{1}{|N|-1} \varphi_j$$

is independent of agent 1’s private information, and it is very concentrated around zero. Because this concentration is strong enough (as quantified by a Hoeffding bound) and trade occurs if and only if $\frac{1}{|N|-1} (\varphi_1 - c) \geq \epsilon$, it follows that the allocation probability conditional on $\theta_1$ is very close to $\mathbb{1}_{\varphi_1 \geq c}$. Therefore, the profit of the optimal mechanism approaches the profit from a one-agent problem with a virtual value cutoff of $c$—exactly the optimal profit of the monopolist problem with one buyer whose value has a CDF of $F_1$. The proposition follows.

It is instructive to specialize Proposition 7 to the case of revenue maximization with no a priori lower bound on agents’ valuations ($c = \theta_1 = 0$). In this canonical case, unanimous posted pricing yields a limit expected revenue of zero as the number of agents grows, while the optimal mechanism yields a strictly positive limit profit. Hence, in the many-agent limit, a restriction to simple mechanisms (either to collective posted prices or to DIC, epIR, bang-bang mechanisms), is unboundedly costly to the seller.

References


A. Proofs

A.1. Proofs for Section 3

Proof of Lemma 1. If transfer rule \( m \) has \( M_i^m = M_i^* \) for each \( i \in N \), then iterated expectations implies

\[
E[M_i^*(\theta_i)] = E[m(\theta)] = E[M_j^*(\theta_j)] \forall i, j \in N.
\]

Conversely, suppose \( \bar{m} \in \mathbb{R} \) is such that \( E[M_i^*(\theta_i)] = \bar{m} \) for every \( i \in N \). It then follows immediately from independence of \( \{\theta_i\}_{i \in N} \) that the following transfer rule generates the desired interim versions for any constant \( m_0 \in \mathbb{R} \backslash \{\bar{m}\} \):

\[
m : \Theta \rightarrow \mathbb{R} \quad \theta \mapsto m_0 + \frac{1}{(m_0 - \bar{m})|N|-1} \prod_{i \in N} [M_i^*(\theta_i) - m_0].
\]

Now, we turn to the final assertion. If nonnegative transfer rule \( m \) has \( M_i^m = M_i^* \) for \( i \in N \), then monotonicity of integration implies \( M_i^*(\theta_i) \geq 0 \) for every \( \theta_i \in \Theta_i \). Conversely, suppose each of \( \{M_i^*\}_{i \in N} \) is nonnegative, and their expectations are all equal to \( \bar{m} \in \mathbb{R} \). Monotonicity of integration then implies \( \bar{m} \geq 0 \). If \( \bar{m} > 0 \) [resp. \( \bar{m} = 0 \)], then the above-constructed transfer rule is nonnegative when for \( m_0 = 0 \) [resp. \( m_0 = -1 \)].

\[10\]

Proof of Lemma 2. Let \( X_i := X_i^* \) for each \( i \in N \). Given a transfer rule \( m \), standard arguments (Myerson, 1981) show that \( (x, m) \) is IC if and only if each \( i \in N \)

\[10\]We should note that our proof does not use any of our regularity assumptions on the type spaces and their distributions: types can be independently drawn from any probability spaces whatsoever. Moreover, boundedness can be replaced with integrability, in which case the implementing \( m \) need not be bounded.
has \( X_i \) weakly increasing and

\[
M_i^m(\theta_i) = X_i(\theta_i)\theta_i - \int_{\theta_i}^{\theta} X_i(\tilde{\theta}_i) \, d\tilde{\theta}_i - U_i, \quad \forall \theta_i \in \Theta_i \tag{1}
\]

for some constant \( U_i \in \mathbb{R} \); that such a mechanism is IR if and only if \( U_i \geq 0 \) for each \( i \in N \); and that any \( M_i : \Theta_i \rightarrow \mathbb{R} \) satisfying equation (1) has \( \mathbb{E} [M_i(\theta_i)] = \mathbb{E} [X_i(\theta_i)\varphi_i] - U_i \). The latter expression implies that, by iterated expectations, any transfer rule \( m \) such that \( (x, m) \) is IC has

\[
\mathbb{E} [m(\theta)] = \mathbb{E} [x(\theta)\varphi_i] - U_i \quad \forall i \in N. \tag{2}
\]

Let us now observe how the two parts of the lemma follow from the above standard observations. For the necessity of interim-monotonicity in the first part, nothing remains to show. To see that the payoff expression in the second part is an upper bound on attainable profits, note that every IR and IC mechanism \( (x, m) \) generates, by equation (2), a profit of

\[
\Pi(x, m) = \min_{i \in N} \{ \mathbb{E} [x(\theta)(\varphi_i - c)] - U_i \} \leq \min_{i \in N} \mathbb{E} [x(\theta)(\varphi_i - c)].
\]

So the lemma will follow if we can construct, given an arbitrary allocation rule \( x \) whose induced interim allocation rules \( \{X_i\}_{i \in N} \) are all weakly increasing, a transfer rule \( m \) such that \( (x, m) \) is IC and IR with \( \Pi(x, m) = \min_{i \in N} \mathbb{E} [x(\theta)(\varphi_i - c)] \). To that end, fix some \( i_* \in \arg\min_{i \in N} \mathbb{E} [x(\theta)\varphi_i] \). For each \( i \in N \), define the hypothetical payoff lower bound \( U_i := \mathbb{E} [x(\theta)(\varphi_i - \varphi_{i_*})] \) and hypothetical interim transfer rule

\[
M_i^* : \Theta_i \rightarrow \mathbb{R}, \quad \theta_i \mapsto X_i(\theta_i)\theta_i - \int_{\theta_i}^{\theta} X_i(\tilde{\theta}_i) \, d\tilde{\theta}_i - U_i.
\]

If we had a transfer rule \( m \) whose interim transfer rules satisfied \( M_i^m = M_i^* \forall i \in N \), then \( (x, m) \) would be as desired. Indeed, that each of \( \{M_i^m\}_{i \in N} \) satisfies equation (1) would imply IC, that \( \{U_i\}_{i \in N} \) are all nonnegative would imply IR, and that \( U_{i_*} = 0 \) would imply \( \Pi(x, m) = \mathbb{E} [x(\theta)(\varphi_{i_*} - c)] = \min_{i \in N} \mathbb{E} [x(\theta)(\varphi_i - c)] \).

Hence, the lemma follows if some transfer rule \( m \) exists with interim transfer rules equal to \( \{M_i^*\}_{i \in N} \). Because, for every \( i \in N \),

\[
\mathbb{E} [M_i^*(\theta_i)] = \mathbb{E} [x_i(\theta_i)\varphi_i] - U_i = \mathbb{E} [x_{i_*}(\theta_{i_*})\varphi_{i_*}] = \mathbb{E} [M_{i_*}^*(\theta_{i_*})],
\]

such a transfer rule exists by Lemma 1. \( \square \)
Proof of Theorem 1. Let \( \tilde{X} \) denote the set \( X \), modulo the \( F \)-almost everywhere equivalence relation. One can view \( \tilde{X} \) as subset of \( L^\infty(\Theta, F) \), and the Banach-Alaoglu theorem then implies \( \tilde{X} \) is weak*-compact. \(^{11}\) Consider the optimization problem,

\[
\max_{x \in \tilde{X}} \left\{ \min_{i \in N} \mathbb{E}[x(\theta)(\varphi_i - c)] \right\}, \tag{RSP}
\]

which is our seller’s problem without the monotonicity constraint. In what follows, we will pursue a solution to this relaxed problem. As we will show, this program is solved by a unique \( x^* \in \tilde{X} \), and this \( x^* \) happens to exhibit monotone interim allocation probabilities. Hence, it will follow that \( x^* \) is the unique solution to our seller’s problem.

Toward solving (RSP), consider a two-player zero-sum game where the maximizer (Max) chooses \( x \in \tilde{X} \) and the minimizer (Min) chooses \( \omega \in \Delta N \). The objective (that is, the payoff to Max) is \( \mathbb{E}[x(\theta)(\omega \cdot \varphi - c)] \). Because \( \tilde{X} \) is weak*-compact and convex (the space \( \Delta N \) obviously is as well), and the objective as weak*-continuous in the strategy profile, it follows from Sion’s minimax theorem that

\[
\max_{x \in \tilde{X}} \min_{\omega \in \Delta N} \mathbb{E}[x(\theta)(\omega \cdot \varphi - c)] = \min_{\omega \in \Delta N} \max_{x \in \tilde{X}} \mathbb{E}[x(\theta)(\omega \cdot \varphi - c)],
\]

where all maxima/minima in the equation are attained by Berge’s theorem.

Because the auxiliary game is zero-sum (Proposition 22.2, Osborne and Rubinstein, 1994), the Nash equilibria are exactly the pairs \((x^*, \omega^*) \in \tilde{X} \times \Delta N\) for which

\[
x^* \in \arg\max_{x \in \tilde{X}} \min_{\omega \in \Delta N} \mathbb{E}[x(\theta)(\omega \cdot \varphi - c)] \quad \text{and} \quad \omega^* \in \arg\min_{\omega \in \Delta N} \max_{x \in \tilde{X}} \mathbb{E}[x(\theta)(\omega \cdot \varphi - c)].
\]

Observe, though, that \( \min_{\omega \in \Delta N} \mathbb{E}[x(\theta)(\omega \cdot \varphi - c)] = \min_{i \in N} \mathbb{E}[x(\theta)(\varphi_i - c)] \) for each \( x \in \tilde{X} \). Hence, \( x^* \) maximizes this quantity if and only if \( x^* \) solves (RSP). Moreover, \( \max_{x \in \tilde{X}} \mathbb{E}[x(\theta)(\omega \cdot \varphi - c)] = \max_{x \in \tilde{X}} \mathbb{E}[(\omega \cdot \varphi - c)_+] \) for each \( \omega \in \Delta N \), so that maximizing the two expressions is equivalent. Finally, because these maxima/minima are obtained, some Nash equilibrium exists.

To summarize our progress so far, we know that a Nash equilibrium exists for the zero-sum game, and Nash equilibria are exactly the pairs \((x^*, \omega^*) \in \tilde{X} \times \Delta N\) for which \( x^* \) solves (RSP) and \( \omega^* \) solves \( \min_{\omega \in \Delta N} \max_{x \in \tilde{X}} \mathbb{E}[(\omega \cdot \varphi - c)_+] \).

\(^{11}\)As is standard, \( L^\infty(\Theta, F) \) is isometrically isomorphic to the dual of \( L^1(\Theta, F) \).
Now, for an arbitrary \( \omega \in \Delta N \). Because \( \{\theta_i\}_{i \in N} \) are atomless and independent and \( \{\varphi_i\}_{i \in N} \) are all strictly increasing, it follows that \( \mathbb{P}\{\omega \cdot \varphi\} = 0 \), so that the \( \omega \)-voting rule \( x_\omega \) is Minimizer’s unique best response to \( \omega \). From the product structure of the set of Nash equilibria, then, it follows that Maximizer has a unique Nash equilibrium strategy \( x^* \), which is then the unique solution to (RSP). Moreover, because a voting rule is obviously interim-monotone (given independent types and increasing virtual values), it follows that the unique solution to (RSP) is also the unique solution to the seller’s problem (SP).

All that remains for our characterization of optimal allocation rules is to show the equivalence of the two conditions in the theorem’s statement for a given \( \omega \in \Delta N \), and that these conditions imply \( x_\omega \) is optimal. We have argued above that the first condition is equivalent to \( \omega \) being a Nash equilibrium strategy for Minimizer. Meanwhile, because we have argued \( x_\omega = 1_{\omega \cdot \varphi \geq c} \) is a unique Maximizer best response to \( \omega \), it follows readily that the second condition is equivalent to \( \omega \) being a Nash equilibrium strategy for Minimizer. Hence the first and second conditions are equivalent. Moreover, we have argued that, if \( \omega \) is a Nash equilibrium strategy for Minimizer, then the \( \omega \)-voting rule is an optimal allocation rule. Therefore, if \( \omega \) satisfies (1) or (2), then \( x_\omega \) is an optimal allocation rule.

We now address the theorem’s final statement—that transfers are without loss taken to be nonnegative. If the never-trade mechanism is optimal, there is nothing to show. So focus on the complementary case in which an optimal mechanism generates strictly positive revenue.

We have proved that the optimal allocation rule is \( x = x_\omega \) for some optimal \( \omega \in \Delta N \). Let \( \{M_i^*\}_{i \in N} \) denote interim transfer rules, constructed in the proof of Lemma 2, that implement \( x \) at maximum possible profit. By Lemma 1, it suffices to show \( M_i^* \) is nonnegative for each \( i \in N \). To see this feature, note that (given the functional form of its construction) \( M_i^* \) is always weakly increasing and is constant if \( X_i^* \) is constant. With this observation, we can establish that \( M_i^* \geq 0 \) in two exhaustive cases. First, if \( \omega_i = 0 \), then \( X_i^* \) is constant and so \( M_i^* \) is constant, hence equal to \( \mathbb{E}[m(\theta)] \geq c\mathbb{E}[x(\theta)] \geq 0 \), where the first inequality holds because an optimal mechanism is weakly better than the never-trade mechanism. Second, if \( \omega_i > 0 \), optimality of \( \omega \) implies \( i \in \arg\min_{j \in N} \mathbb{E}[\varphi_j x(\theta)] \). But, in this case the constructed transfer rules satisfy

\[
0 = \varphi_i X_i^*(\theta_i) - M_i^*(\theta_i) \geq -M_i^*(\theta_i),
\]

so that \( M_i^* \geq 0 \).

**Proof of Proposition 1.** First, it is immediate that the never-trade [resp. always-trade] mechanism is optimal among all IC and IR mechanisms using allocation
the support of $\phi$ such that $\forall i \in N$, then choosing $\omega \in \Delta N$ with $\omega_j = 1_{i=j}$ for each $j \in N$ satisfies the first condition in Theorem 1, so that the zero allocation rule $x_\omega$ is optimal. Conversely, suppose $\tilde{\theta}_i > c$ for every $i \in N$. Theorem 1 says some optimal $\omega \in \Delta N$ exists, and the $\omega$-voting rule is a uniquely optimal allocation rule—but observe this rule entails a positive trade probability.

Next, we characterize when the always-trade allocation rule is optimal. By Theorem 1, this allocation rule is optimal if and only if some $\omega \in \Delta N$ exists such that $x_\omega$ is the always-trade allocation rule if and only if some $\omega \in \Delta \left( \arg\min_{j \in N} \theta_j \right)$ exists such that $\omega \cdot \varphi(\theta) \geq c$. Because taking $\omega$ to be the right degenerate voting weights (subject to a given support constraint) maximizes $\omega \cdot \varphi(\theta)$, this property is equivalent to some $i \in \arg\min_{j \in N} \theta_j$ existing such that $\varphi_i(\theta) \geq c$.

Finally, we turn to uniqueness. Suppose $\omega, \tilde{\omega} \in \Delta N$ are such that $x_\omega$ and $x_{\tilde{\omega}}$ are both optimal. Theorem 1 shows that $x_{\tilde{\omega}}(\theta) = x_\omega(\theta)$ almost surely. Our aim is to show, assuming these allocation rules are nontrivial, that $\omega = \tilde{\omega}$. Toward establishing this equality, define $G := \prod_{i \in N} \left( \varphi_i(\theta) - c, \tilde{\theta}_i - c \right)$, the interior of the support of $\varphi - c1_N$. Now, define the linear map $L : \mathbb{R}^N \rightarrow \mathbb{R}^2$ by letting $L(z) := (\omega \cdot z, \tilde{\omega} \cdot z)$ for each $z \in \mathbb{R}^N$.

Let us now observe some properties of $G$ and $L$. First, that $x_\omega$ and $x_{\tilde{\omega}}$ are nontrivial implies $L(G)$ is not a subset of $\mathbb{R}_+ \times \mathbb{R}$, of $\mathbb{R}_- \times \mathbb{R}$, of $\mathbb{R} \times \mathbb{R}_+$, or of $\mathbb{R} \times \mathbb{R}_-$. Second, that $\mathbb{P} \{ x_{\tilde{\omega}}(\theta) = x_\omega(\theta) \} = 1$ implies $L(G)$ is a subset of $\mathbb{R}^2_+ \cup \mathbb{R}^2_-$. Third, because $L$ is linear and $G$ is convex, the set $L(G)$ is convex. Combining these three observations tells us that $L(G)$ is contained in a single line through the origin. Because $G$ is open and $L$ is linear, then, $L(\mathbb{R}^N)$ is contained the same line. Said differently, the rank of the linear map $L$ is 1, so that vectors $\omega, \tilde{\omega} \in \mathbb{R}^N$ are proportional. Because $||\omega||_1 = 1 = ||\tilde{\omega}||_1$, it follows that $\omega = \tilde{\omega}$. \qed
A.2. Proofs for Section 4

Proof of Corollary 1. The stochastic dominance hypothesis implies that \( \bar{\theta}_i \leq \bar{\theta}_j \) and \( \bar{\theta}_i \leq \bar{\theta}_j \) for each \( j \in N \). Moreover, it implies that any \( j \in N \) with \( \bar{\theta}_j = \bar{\theta}_i \) has \( f_i(\bar{\theta}_i) \geq f_j(\bar{\theta}_j) \), so that \( \varphi_i(\bar{\theta}_i) \geq \varphi_i(\bar{\theta}_j) \). The result then follows directly from Proposition 1.

Proof of Corollary 2. The second and third conditions are equivalent, because the profit maximizing price for agent \( i \) induces allocation rule \( \theta \mapsto 1_{\varphi_i(\theta) \geq c} \), and \( \mathbb{E}[\varphi_j] = \bar{\theta}_j \) for each \( j \in N \). We now turn to showing the first and third conditions are equivalent.

By Theorem 1, this \( \omega \) is optimal if and only if \( \min_i \mathbb{E}[\varphi_j] \mathbb{1}_{\varphi \geq c} \) over all \( j \in N \). But observe each \( j \in N \) has \( \mathbb{E}[\varphi_j] \mathbb{1}_{\varphi \geq c} = \mathbb{E} [\varphi_j] \mathbb{P} \{ \varphi_i \geq c \} \) if \( j \neq i \). Hence \( \omega \) is optimal if and only if each \( j \in N \{ i \} \) has \( \mathbb{E}[\varphi_j] \mathbb{P} \{ \varphi_i \geq c \} = \mathbb{E}[\varphi_j] \mathbb{P} \{ \varphi_i \geq c \} \).

Consider now two exhaustive cases. First, if \( \bar{\theta}_i \leq c \), then \( \mathbb{P} \{ \varphi_i \geq c \} = \mathbb{E}[\varphi_i \mathbb{1}_{\varphi \geq c}] = 0 \), and so the inequalities are trivially satisfied. Second, if \( \bar{\theta}_i > c \), then dividing the inequalities by \( \mathbb{P} \{ \varphi_i \geq c \} > 0 \) tells us \( \omega \) is optimal if and only if each \( j \in N \{ i \} \) has \( \mathbb{E}[\varphi_j] \geq c \).

Proof of Corollary 3. Consider any \( \omega \in \Delta N \) such that \( x_\omega \) is nontrivial and has \( \omega_i = 0 \). It suffices to show that \( \omega \) cannot be optimal.

Fixing some \( j \in \text{supp}(\omega) \), observe that

\[
\mathbb{E} [\varphi_i \mathbb{1}_{\varphi \geq c}] = \mathbb{E}[\varphi_i] = \bar{\theta}_i \leq \bar{\theta}_j = \mathbb{E}[\varphi_j]
\]

\[
\leq \mathbb{E} \left[ \varphi_j \mid \varphi_j \geq \frac{1}{\omega_j} \left( c - \sum_{k \in N \setminus \{j\}} \omega_k \varphi_k \right) \right]
\]

\[
= \mathbb{E} [\varphi_j] \mathbb{1}_{\varphi \geq c},
\]

where the first equality follows from \( \{ \varphi_k \}_{k \in N} \) being independent, and the strict inequality follows from the same and from the conditioning event having interior probability. Hence, \( \omega \) is not optimal.

Proof of Proposition 2. Assume for a contradiction that the unique (by Proposition 1) optimal vector of voting weights \( \omega \) satisfies \( \omega_i < \frac{1}{\alpha} \omega_j \). Let

\[
\beta := \frac{\alpha (\omega_i + \omega_j)}{\alpha^2 \omega_i + \omega_j} \in (0, 1],
\]

and define \( \tilde{\omega} \in \mathbb{R}^N \) by letting \( \tilde{\omega}_i := \frac{\beta}{\alpha} \omega_j \) and \( \tilde{\omega}_j := \beta \omega_i \), and letting \( \tilde{\omega}_k = \omega_k \) for
every other \( k \in N \). Notice that \( \tilde{\omega} \in \Delta N \) since \( \tilde{\omega}_i, \tilde{\omega}_j \geq 0 \) and

\[
\tilde{\omega}_i + \tilde{\omega}_j = \beta \left( \frac{\omega_i}{\alpha} + \alpha \omega_i \right) = \frac{\alpha (\omega_i + \alpha)}{\alpha \omega_i + \omega_j} \left( \frac{\alpha^2 \omega_i + \omega_j}{\alpha} \right) = \omega_i + \omega_j.
\]

Also, note that \( \hat{\omega} \neq \omega \). Indeed, if \( \omega_i = 0 \) then this fact follows from \( \hat{\omega}_i = \frac{\beta}{\alpha} \omega_j > 0 \); and otherwise, it follows from \( \hat{\omega}_i = \frac{\alpha^2 \omega_i}{\alpha \omega_i + \omega_j} < \frac{\omega_i}{\omega_i} \). Observe now that it suffices to show \( E[(\omega \cdot \varphi - c)_+] \geq E[(\hat{\omega} \cdot \varphi - c)_+] \). Indeed, if we could show this ranking, then \( \hat{\omega} \) would be optimal too—\( \beta \)-in contradiction to the unique optimality of \( \omega \).

So let us turn to showing that \( E[(\omega \cdot \varphi - c)_+] \geq E[(\hat{\omega} \cdot \varphi - c)_+] \). To prove the

result, we invoke results from Shaked and Shanthikumar (2007). First, as \( \varphi_i \) is smaller than \( \alpha \varphi_j \) in the hazard rate order, it follows from Theorem 1.B.41 that \( -\alpha \varphi_j \) is smaller than \( -\varphi_i \) in the reverse hazard order. Hence, Theorem 4.A.37 implies that \( \omega_i(-\varphi_i) + \frac{1}{\alpha} \omega_j(-\alpha \varphi_j) = -(\omega_j \varphi_i + \omega_i \varphi_j) \) is smaller than \( \frac{1}{\alpha} \omega_j(-\varphi_i) + \omega_i(-\alpha \varphi_j) \) in the increasing and concave order. Therefore, by 4.A.1, \( \omega_i \varphi_i + \omega_j \varphi_j \) is larger than \( \frac{1}{\alpha} \omega_j \varphi_i + \alpha \omega_i \varphi_j = \frac{1}{\alpha}(\omega_i \varphi_i + \omega_j \varphi_j) \) in the increasing and convex order. Moreover, \( \frac{1}{\beta} (\omega_i \varphi_i + \omega_j \varphi_j) \) is larger than \( \omega_i \varphi_i + \omega_j \varphi_j \) in the increasing and convex order since \( \beta \leq 1 \).\(^{12}\)

Therefore, \( E[\eta(\omega \varphi_i + \omega_j \varphi_j)] \) \( \geq \) \( E[\eta(\omega_i \varphi_i + \omega_j \varphi_j)] \) for every (weakly) increasing and convex \( \eta : \mathbb{R} \rightarrow \mathbb{R} \). The desired inequality then follows from applying this ranking to \( \eta \) given by

\[
\eta(y) := E \left[ \left( y - c + \sum_{k \in N \setminus \{i,j\}} \omega_k \varphi_k \right)_+ \right],
\]

which is convex and increasing because \((\cdot)_+\) is.

\( \square \)

A.3. Proofs for Section 5

A.3.1. Proofs for Section 5.1

**Lemma 4:** In any mechanism with a collective posted price \( p > 0 \), for \( i \in N \), the associated interim allocation rule \( X_i \) is zero for types strictly below \( p \) and constant on the interval of types strictly above \( p \).

**Proof.** First, consider \( \theta_i \in \Theta_i \) with \( \theta_i < p \). Letting \( M_i \) denote the mechanisms, interim transfer rule for agent \( i \), we have \( pX_i(\theta_i) = M_i(\theta_i) \) since the mechanism is a posted price mechanism. IR requires that \( (\theta_i - p)X_i(\theta_i) = \theta_iX_i(\theta_i) - M_i(\theta_i) \geq 0 \),

\(^{12}\)As the random variable \( v := \frac{1}{\beta}(\tilde{\omega}_i \varphi_i + \tilde{\omega}_j \varphi_j) \) has nonnegative mean, any convex increasing \( \eta \) has \( E\eta(v) = (1 - \beta)E\eta(v) + \beta E\eta(v) \geq (1 - \beta)E\eta(Ev) + |E\eta(\beta v) - (1 - \beta)\eta(0)| \geq E\eta(\beta v) \).
so $X_i(\theta_i) = 0$. Now, supposing $p < \tilde{\theta}_i$, for each $\theta_i \in \Theta_i$ with $\theta_i > p$, we have

$$
(\theta_i - p)X_i(\theta_i) = \theta_iX_i(\theta_i) - M_i(\theta_i) \\
= \max_{\tilde{\theta}_i \in \Theta_i} \left\{ \theta_iX_i(\tilde{\theta}_i) - M_i(\tilde{\theta}_i) \right\} \text{ (by IC)} \\
= (\theta_i - p)X_i(\tilde{\theta}_i) \\
\implies X_i(\theta_i) = X_i(\tilde{\theta}_i).
$$

\[ \square \]

Proof of Proposition 3. Consider an arbitrary collective posted price mechanism $(x, m)$ with price $p \geq 0$. We will show a unanimous posted price (with price $p \geq c$) performs weakly better (and strictly better if it produces different trade outcomes).

If $p < c$, then the profit associated with the mechanism is always non-positive, and so a unanimous posted price with price $c$ is weakly better—strictly so if it induces different trade outcomes.

Now, we suppose that $p > c$. By Lemma 4, if an agent $i \in N$ has a type strictly below $p$, then his interim allocation is zero—and so the allocation probability must be zero for all but a null set of others’ realized types. This observation yields an upper bound on $x_i(p)$. In particular, the allocation probability may (almost surely) be positive only in the event that $\theta_j \geq p$ for all agents $j \in N$. That is, $x(\theta) \leq x^U(\theta)$ almost surely, where allocation rule $x^U$ is given by $x^U(\theta) := 1_{\theta_j \geq p \forall j \in N}$. Hence, the profit from the original mechanism is

$$(p - c)\mathbb{E}[x(\theta)] \leq (p - c)\mathbb{E}[x^U(\theta)],$$

where the inequality is strict unless $x(\theta) = x^U(\theta)$ almost surely. Hence, a unanimous posted price of $p$, generating allocation rule $x^U$ and trade probability $(p - c) \prod_{j \in N}[1 - F_j(p)]$, is more profitable for the seller. \[ \square \]

Proof of Proposition 4. The equivalence is trivial in the case that always-trade or never-trade is optimal. Hence, we restrict attention to the case that the optimal allocation rule is nontrivial; let $\omega$ denote the unique (by Proposition 1) optimal voting weights, and $x := x_\omega$.

First, if $\omega$ is a dictatorship, then the optimal mechanism is trivially a posted price mechanism. We now suppose $\omega$ is not a dictatorship, so that some $i \in N$ has $0 < \omega_i < 1$. For each $\theta_i \in \Theta_i$, then, $i$’s interim probability of trade is given by

$$
X_i^e(\theta_i) = \mathbb{P} \left\{ \sum_{j \in N \setminus \{i\}} \omega_j\varphi_j(\theta_j) \geq c - \omega_i\varphi_i(\theta_i) \right\}.
$$
Recall that $\theta$ admits a density, $\omega_i$ and $\omega_{\bar{i}}$ are both nonzero, and $\{\varphi_j\}_{j \in N}$ are all continuous. It follows that the random variable on the left side of the above inequality is atomlessly distributed, while the number on the right side varies continuously with $\theta_i$. Hence, the interim allocation rule $X_i^x$ is continuous. Moreover, this function is not constant, because the allocation rule is nontrivial while the random variable on the left side of the inequality has convex support. Therefore, $X_i^x$ cannot be a step function. Hence, by Lemma 4, no optimal mechanism is a collective posted price.

\[ \tag*{□} \]

A.3.2. Proofs for Section 5.2

**Lemma 5.** Suppose that $(x, m)$ is a DIC mechanism and $\theta, \theta' \in \Theta$ have $x(\theta) = x(\theta') \in \{0, 1\}$. Then $m(\theta) = m(\theta')$.

**Proof.** Define $\theta^* := \theta \lor \theta'$ if $x(\theta) = x(\theta') = 1$, and $\theta^* := \theta \land \theta'$ if $x(\theta) = x(\theta') = 0$. We will observe that $m(\theta) = m(\theta^*) = m(\theta')$; by symmetry, it suffices to show $m(\theta) = m(\theta^*)$. To show it, define the type profile

$$
\theta^\ell := (\theta^*_i 1_{i \leq \ell} + \theta_j 1_{j > \ell})_{i \in N} \in \Theta \text{ for each } \ell \in \{0, \ldots, N\} = N \cup \{0\}.
$$

Observe, either $\theta^0 \leq \cdots \leq \theta^N$ and $x(\theta^0) = 1$, or $\theta^0 \geq \cdots \geq \theta^N$ and $x(\theta^0) = 0$. In either case, because $x$ is weakly increasing (due to DIC) and can only take values in $[0, 1]$, it follows by induction that $x(\theta^0) = \cdots = x(\theta^N)$. For each $i \in N$, because $\theta^i$ and $\theta^{i-1}$ differ only in the $i$ coordinate and $x(\theta^{i-1}) = x(\theta^i)$, it follows from DIC (for agent $i$) that $m(\theta^{i-1}) = m(\theta^i)$. Thus, $m(\theta) = m(\theta^0) = \cdots = m(\theta^N) = m(\theta^*)$, as desired.

**Proof of Lemma 3.** Fix a DIC mechanism $(x, m)$ such that $x(\theta)$ almost surely in $\{0, 1\}$. By Lemma 5, some constants $m^L, m^H \in \mathbb{R}$ exists such that $m(\theta) = m^L$ [resp. $m^H$] for every $\theta \in \Theta$ with $x(\theta) = 0$ [resp. 1]. Moreover, DIC implies $m^L \leq m^H$ if there exist type profiles leading to both allocation probabilities; and we may without loss take $m^L \leq m^H$ in the complementary case. So, defining $p := m^H - m^L \geq 0$ and letting $s := -m^L$, we have $m(\theta) = px(\theta) - s$ whenever $x(\theta) \in \{0, 1\}$, an almost sure event.

Now, modifying $x$ on a measure-zero subset of its domain, and similarly modifying the transfer rule to maintain $m = px - s$, we may assume without loss that $x$ is (statewise) $\{0, 1\}$-valued. Indeed, if $x(\theta) = 0$ almost surely, we can replace the allocation rule with the zero allocation rule; and in the complementary case, we can replace the allocation rule with $\theta \rightarrow 1_{x(\theta) \geq 0}$. It is easy to see that DIC of the modified mechanism follows from DIC of the original one.

Now, we show $x$ has the desired structure. Given an agent $i \in N$ and type realization $\theta_i \in \Theta_i$, his payoff from a reported type profile of $\hat{\theta}$ is $(\theta_i - p)x(\hat{\theta}) - s$,
which is strictly increasing [resp. decreasing] in $x(\hat{\theta})$ if $\theta_i > p$ [resp. $\theta_i < p$]. Hence, given $\theta_i \in \Theta_{-i}$ DIC implies that one the following three possibilities holds: $x(\cdot, \theta_{-i}) = 1$, $x(\cdot, \theta_{-i}) = 0$, or $x(\theta_i, \theta_{-i}) = 1$ [resp. $x(\theta_i, \theta_{-i}) = 0$] for each $\theta_i \in \Theta_i$ with $\theta_i > p$ [resp. $\theta_i < p$]. Hence, letting $\tilde{\Theta} := \prod_{i \in N} [\Theta_i \setminus \{p\}]$, some $y :\{0,1\}^N \rightarrow \{0,1\}$ exists such that every $\theta \in \tilde{\Theta}$ has $x(\theta) = y((1_{\theta_i \geq p})_{i \in N})$. Moreover, we may assume without loss that $y$ is constant in its $i$ coordinate if $p \leq \bar{\theta}_i$ or $p \geq \bar{\theta}_i$ for $i \in N$. Then, monotonicity of $x$ implies $y$ is monotone too. If we let $\tilde{\mathcal{J}} := \{J \subseteq N : y(1_J) = 1\}$, then, $x(\theta) = 1_{\bigcup_{J \in \tilde{\mathcal{J}}} \bigcap_{j \in J} \Theta_j \geq p}$ almost surely.

Now, let $\tilde{\mathcal{J}} := \{j \in J : \theta_j < p\} : \tilde{\mathcal{J}} \in \tilde{\mathcal{J}}$ with $\tilde{\theta}_j > p \forall j \in \tilde{\mathcal{J}}$. Then, $x(\theta) = 1_{\bigcup_{J \in \tilde{\mathcal{J}}} \bigcap_{j \in J} \Theta_j \geq p}$ almost surely, and $\tilde{\theta}_j < p < \tilde{\theta}_j$ for each $j \in \bigcup \tilde{\mathcal{J}}$. Finally, let $\mathcal{J} := \{J \in \tilde{\mathcal{J}} : \exists \tilde{j} \in \tilde{\mathcal{J}} \text{ with } J \subseteq \tilde{j}\}$. Then, $x(\theta) = 1_{\bigcup_{J \in \mathcal{J}} \bigcap_{j \in J} \Theta_j \geq p}$ almost surely, $\tilde{\theta}_j < p < \tilde{\theta}_j$ for each $j \in \bigcup \mathcal{J}$, and no two members of $\mathcal{J}$ are nested. Thus, $(p, s, \mathcal{J})$ are as required. \hfill \Box

**Proof of Proposition 5.** That the second condition implies the first is trivial: A dictatorship mechanism is trivially DIC because no agent both affects the outcome and has some co-player who affects the outcome. To show that the second condition implies the first, suppose allocation rule $x$ is both optimal and DIC. Theorem 1 implies $x(\theta) = x_\omega(\theta)$ almost surely, for some $\omega \in \Delta N$. Lemma 3 implies $x(\theta) = 1_{\bigcup_{J \in \mathcal{J}} \bigcap_{j \in J} \Theta_j \geq p}$ almost surely, for some $p \in \mathbb{R}$ and $\mathcal{J} \subseteq 2^N$ such that $\mathcal{J}$ are pairwise non-nested and every $j \in N^* := \bigcup \mathcal{J}$ has $\tilde{\theta}_j < p < \tilde{\theta}_j$.

If we can establish that the set $\mathcal{J}$ is equal to $\emptyset$, to $\{\emptyset\}$, or to $\{i\}$ for some $i \in N$, then the proposition will follow because an $i$-dictatorship mechanism is optimal in each of these three cases. To that end, define $Z := \prod_{i \in N} [\varphi_i(\bar{\theta}_i), \tilde{\theta}_i]$, and let $Z^* \subseteq Z$ denote the support of the measure on $Z$ which assigns mass $\mathbb{E}[x(\theta)1_{\varphi \in Z}]$ to every Borel $\tilde{Z} \subseteq Z$. That $x(\theta) = x_\omega(\theta)$ almost surely implies both $Z^*$ and $Z \setminus Z^*$ are convex. Meanwhile, that $x(\theta) = 1_{\bigcup_{J \in \mathcal{J}} \bigcap_{j \in J} \Theta_j \geq p}$ almost surely implies

$$Z^* = \bigcup_{j \in \mathcal{J}} \{z \in Z : z_j ≥ \varphi_j(p) \forall j \in J\}.$$

Using this characterization, we can show that $\mathcal{J}$ is one of the aforementioned sets.

First, let us see that $|\mathcal{J}| \leq 1$. Assume for contradiction that $J, J' \in \mathcal{J}$ have $J \neq J'$. Define now the elements $z, z' \in Z$ via $z := (\varphi_i(\bar{\theta}_i)1_{i \notin J} + \varphi_i(p)1_{i \in J})_{i \in N}$ and $z' := (\varphi_i(\bar{\theta}_i)1_{i \notin J'} + \varphi_i(p)1_{i \in J'})_{i \in N}$. Observe that $z$ and $z'$ are in $Z^*$, but their midpoint is is not—contradicting the convexity of $Z^*$.

Hence, if $\mathcal{J}$ is nonempty, then $\mathcal{J} = \{J\}$ for some $J \subseteq N$. Now, let us see that $|J| \leq 1$ in this case. Assume for a contradiction that $j, j' \in J$ have $j \neq j'$. Given $\epsilon \in (0, \min_{i \in J} [\varphi_i(\bar{\theta}_i) - \varphi_i(p)])$, define $z_\epsilon, z'_\epsilon \in Z$ by letting $z_\epsilon$ :=
\[
([\varphi_i(p) - \epsilon] 1_{i=j'} + \varphi_i(\hat{\theta}_i) 1_{i \neq j'})_{i \in N} \text{ and } z'_\epsilon := ([\varphi_i(p) - \epsilon] 1_{i=j} + \varphi_i(\hat{\theta}_i) 1_{i \neq j})_{i \in N}. \]

Observe that \(z_\epsilon\) and \(z'_\epsilon\) are outside of \(Z^*\), but their midpoint is in \(Z^*\) when \(\epsilon\) is sufficiently small—contradicting the convexity of \(Z^*\).

Thus, \(|J| \leq 1\), and \(|J| \leq 1\) for any \(J \in \mathcal{J}\). It follows that \(\mathcal{J}\) is equal to \(\emptyset\), \(\{\emptyset\}\), or \(\{\{i\}\}\) for some \(i \in N\), as desired.

**Proof of Proposition 6.** Given \(p \in \mathbb{R}_+, s \in \mathbb{R}\), and \(\mathcal{J} \subseteq 2^N\), we can define the \((p, s, \mathcal{J})\)-mechanism as the mechanism \((x_{p,s,\mathcal{J}}, m_{p,s,\mathcal{J}})\) by letting
\[
x_{p,s,\mathcal{J}}(\theta) := 1_{\bigcup_{j \in \mathcal{J} \cap \theta \ni \{j\}} \theta \ni p} \text{ for every } \theta \in \Theta, \text{ and letting } m_{p,s,\mathcal{J}} := px_{p,s,\mathcal{J}} - s.
\]

Clearly, every mechanism as described in the above paragraph is DIC, and Lemma 3 says every DIC bang-bang mechanism is almost surely equivalent to some mechanism of that form. Moreover, a mechanism almost surely equivalent to the \((p, s, \mathcal{J})\)-mechanism is epIR if the \((p, s, \mathcal{J})\)-mechanism is itself epIR. Given these observations, it suffices to optimize over mechanisms of this special form. Observe, the seller’s profit from mechanism \((x_{p,s,\mathcal{J}}, m_{p,s,\mathcal{J}})\) is
\[
\tilde{\Pi}(p, s, \mathcal{J}) := (p - c) \mathbb{P}\left(\bigcup_{j \in \mathcal{J} \cap \theta \ni \{j\}} \theta \ni p\right) - s.
\]

Now, given an arbitrary epIR and DIC mechanism, let us establish that the \((p, s, \mathcal{J})\)-mechanism is also epIR (and DIC) and generates a weakly higher profit—strictly so if it is not almost surely identical—for some \(p^* \geq 0\) and \(\mathcal{J} \subseteq 2^N\). Given the argument in the above paragraph, we may without loss assume the original mechanism is the \((p, s, \mathcal{J})\)-mechanism for some \(p \in \mathbb{R}_+, s \in \mathbb{R}, \text{ and } \mathcal{J} \subseteq 2^N\). If trade occurs with probability 1 in this mechanism, we may replace \((p, s)\) with \((p + s, 0)\) for an outcome-equivalent mechanism. We now focus on the complementary case that trade occurs with probability strictly less than 1 in the \((p, s, \mathcal{J})\)-mechanism. Now, epIR implies \(s \geq 0\). Moreover, if trade occurs with positive probability, then epIR implies \(p - s \geq c\); and otherwise, we may without loss raise \(p\) to ensure this is the case. Consider now, for any \(\epsilon \in [0, s]\), the \((p - \epsilon, s - \epsilon, \mathcal{J})\)-mechanism. It remains DIC, and is epIR because the original mechanism is epIR and \(s - \epsilon \geq 0\). Moreover, defining the quantity
\[
\xi(\epsilon) := \mathbb{P}\left(\bigcup_{j \in \mathcal{J} \cap \theta \ni \{j\}} \theta \ni p - \epsilon\right),
\]

the profit this mechanism delivers satisfies
\[\tilde{\Pi}(p - \epsilon, s - \epsilon, \mathcal{J}) = (p - \epsilon - c)\xi(\epsilon) - (s - \epsilon) \quad \Rightarrow \quad \frac{\partial}{\partial \epsilon}\tilde{\Pi}(p - \epsilon, s - \epsilon, \mathcal{J}) = [1 - \xi(\epsilon)] + (p - \epsilon - s)\xi'(\epsilon).\]

As \(\xi\) is weakly increasing, and [0, 1]-valued, it follows that this profit is weakly increasing in \(\epsilon\). Moreover, because \(\xi(0) < 1\), the profit is not constant in \(\epsilon\); hence, if \(s > 0\), then taking \(\epsilon = s\) yields a strictly higher profit to the seller, as desired.

It remains to see, given an epIR \((p^*, 0, \mathcal{J})\)-mechanism for \(p^* \geq 0\) and \(\mathcal{J} \subseteq 2^N\), that a unanimous posted price mechanism generates a weakly higher profit to the seller, strictly so if it is not essentially outcome-equivalent. But, as the \((p^*, 0, \mathcal{J})\)-mechanism is a collective posted price mechanism, this result follows immediately from Proposition 3.

\[\Box\]

### A.4. Proofs for Section 5.3

**Proof of Proposition 7.** In a mild abuse of notation, let \(N \in \mathbb{N}\) denote the number of agents, and say \(N = \{1, \ldots, N\}\). Because the set of optimal voting weight vectors is clearly convex and invariant to permutations, it follows that the uniform voting weight vector \(\omega^N := \frac{1}{N}1_N \in \Delta N\) is optimal. Hence, defining \(x^N := \frac{1}{N}\sum_{i \in N} \varphi_i \geq c\), the seller’s optimal profit is \(E[x^N(\varphi_1 - c)]\). Note that \(E[\varphi_1] = \bar{\theta}_1\), and let \(\zeta := \bar{\theta}_1 - \varphi_1(\bar{\theta}_1)\), a uniform upper bound on \(|\varphi_1 - c|\).

To start, let us establish the result for the (easier) case in which \(\bar{\theta}_1 \neq c\). Note that a unanimous posted price of \(\bar{\theta}_1\) [resp. \(\bar{\theta}_1\)] attains a profit of 0 [resp. \(\bar{\theta}_1 - c\)]. Hence our the limiting value from an optimal collective posted price will necessarily coincide with the seller’s limiting profit from an optimal mechanism, if we show that the latter attains the specified limit. Observe now that the law of large numbers implies \(E[x^N]\) converges to 0 [resp. 1] as \(N \to \infty\) if \(\bar{\theta}_1 < c\) [resp. \(\bar{\theta}_1 > c\)]. Using this fact, let us show that the seller’s optimal value converges to the desired quantity in each of these two cases. First, if \(\bar{\theta}_1 < c\), then the distance between the seller’s optimal profit and zero is

\[|E[x^N(\varphi_1 - c)]| \leq \zeta E[x^N] \to 0 \text{ as } N \to \infty.\]

Next, if \(\bar{\theta}_1 < c\), then the distance between the seller’s optimal profit and \(\bar{\theta}_1 - c\) is

\[|(\bar{\theta}_1 - c) - E[x^N(\varphi_1 - c)]| = |E[(1 - x^N)(\varphi_1 - c)]| \leq \zeta E[1 - x^N] \to 0 \text{ as } N \to \infty.\]

\[\text{Here, } \xi' \text{ denotes the right-hand-side derivative of } \xi, \text{ which exists because each of } \{F_i\}_{i \in N} \text{ is differentiable.}\]
We now turn to the complementary case that \( c = \bar{\theta}_1 \). As this equality implies \( \varphi_1(\bar{\theta}_1) < c < \varphi_1(\hat{\theta}_1) \), let \( \bar{\theta}_1 \) := \( \varphi_1^{-1}(c) \), the unique profit-maximizing posted price for a single agent with value distribution distributed according to CDF \( F_1 \). Let 
\( \pi := \max_{p \in \mathbb{R}_+} (p - c)[1 - F_1(p)] = (\bar{\theta}_1 - c)[1 - F_1(\bar{\theta}_1)] \), which is easily seen to be an upper bound on the seller’s profit.\(^{14}\) Let us show, given an arbitrary \( \epsilon > 0 \), that our seller’s optimal limit optimal profit, as \( N \to \infty \), is at least \( \pi - \epsilon \). Because \( F_1 \) is continuous, some \( \theta_L, \theta_H \in \Theta_1 \) exist such that \( \theta_L < \hat{\theta}_1 < \theta_H \) and \( F_1(\theta_H) - F_1(\theta_L) \leq \frac{\epsilon}{N} \).

Given \( N \in \mathbb{N} \) with \( N > 1 \), let \( X_1^N \) denote the interim allocation rule for agent 1 in the optimal mechanism. Observe now that

\[
X_1^N(\theta_H) - X_1^N(\theta_L) = \mathbb{P} \left[ -\frac{1}{N} \varphi_1(\theta_H) - c \leq -\frac{1}{N} \varphi_1(\theta_L) \right] \\
\geq \mathbb{P} \left[ \frac{-\delta}{N-1} \leq \left( \frac{1}{N-1} \sum_{i=2}^{N} \varphi_i \right) - \frac{\delta}{N-1} \right] \\
\geq 1 - \gamma_N,
\]

for \( \gamma_N := 2e^{\frac{2\delta^2}{(N-1)^2}} \), where the last inequality follows from Hoeffding’s inequality. Interim monotonicity then tells us any \( \bar{\theta}_1 \in \Theta_1 \) has \( X_1^N(\bar{\theta}_1) \leq \gamma_N \) if \( \bar{\theta}_1 \leq \theta_L \) and \( X_1^N(\bar{\theta}_1) \geq 1 - \gamma_N \) if \( \bar{\theta}_1 \geq \theta_H \). Therefore, the difference between \( \pi \) and the optimal profit is

\[
\pi - \mathbb{E} \left[ X_1^N(\theta_1)(\varphi_1 - c) \right] = \mathbb{E} \left\{ \left[ 1_{\theta_1 = \bar{\theta}_1} - X_1^N(\theta_1) \right] (\varphi_1 - c) \right\} \\
\leq \mathbb{E} \left\{ \left[ \gamma_N + 1_{\theta_1 \in \theta_L, \theta_H} \right] (\varphi_1 - c) \right\} \\
\leq \mathbb{E} \left[ \gamma_N + 1_{\theta_1 \in \theta_L, \theta_H} \right] \frac{\pi}{N} \\
= \left[ \gamma_N + F_1(\theta_H) - F_1(\theta_L) \right] \frac{\pi}{N} \\
\leq \frac{\pi}{N} + \epsilon \\
\to \epsilon \text{ as } N \to \infty.
\]

As \( \epsilon > 0 \) was arbitrary, it follows that the optimal profit approaches \( \pi \) as \( N \to \infty \).

Now, let us see that the highest profit \( \max_{p \geq 0} (p - c)[1 - F_1(p)]^N \) attainable with a collective posted price mechanism goes to 0 as \( N \to \infty \). First, by Proposition 3, we can express this quantity as \( \pi_N := \max_{p \geq 0} (p - c)[1 - F_1(p)]^N \). Now, let \( p_N \in [\bar{\theta}_1, \tilde{\theta}_1] \) be a maximizer of \( p \to (p - c)[1 - F(p)]^N \), which exists by compactness and continuity. Because \([\bar{\theta}_1, \tilde{\theta}_1]\) is compact, we can decompose the sequence \( (p_N)_N \) into a collection of convergent subsequences \( (p_{N_\ell})_\ell \). Consider an arbitrary such

\(^{14}\)Consider a relaxed program in which IC and IR are required only for one agent. As is well-known, the optimal profit in the relaxed program, a single-agent monopolist problem, is \( \pi \).
subsequence converging to $p_\ast$. The sequences $(p_{N_\ell} - c)_\ell$ and $([1 - F_1(p_{N_\ell})]^{N_\ell})_\ell$ are both bounded, and one of them converges to zero—the former if $p_\ast = \theta_1$, the latter if $p_\ast > \theta_1$. Hence the product converges to zero. \qed