Sequential Mechanisms With ex post Individual Rationality*

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Abstract

We study optimal screening mechanisms for selling multiple products to a buyer who learns her value for each product at a different time period. A mechanism may screen types over time or be static, i.e., screen types only in the last period. A mechanism must provide the buyer a non-negative ex post utility. We show that there exists an optimal mechanism that determines the allocation of each product as soon as the buyer learns her value for that product. This observation allows us to solve for optimal mechanisms recursively, and to provide several structural properties of optimal mechanisms. We show that static mechanisms are sub-optimal if the buyer first learns her values for products that are ex ante less valuable. Under this condition, the ability to bundle products is less profitable than the ability to screen types dynamically.

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1 Introduction

A buyer who visits a multi-product seller's website may be uncertain about her willingness to pay for different products. She resolves this uncertainty by viewing products one by one and reading their descriptions and reviews. The website can track the buyer's behavior and customize offers to her based on her observed behavior. For instance, if the buyer expresses interest in a book by adding it to her wish list, the website may offer a discount for buying a bundle of books which includes that book. What strategy should the seller use to maximize profit, taking into account the sequential arrival of information to the buyer?

The seller chooses a strategy from a rich class of screening mechanisms. In our model, the buyer learns her value for one product in each period and then sends a message to the mechanism. At the end of the last period, the mechanism specifies the final allocation of products and the payment as a function of all these messages. A special case is the class of static mechanisms in which the buyer sends a message only at the last period. That is, after the buyer learns all her values, she can choose form a menu that offers different bundles of products at different prices. A static mechanism does not utilize the possibility of screening the values of the buyer over time. Another special case is the class of mechanisms in which the allocation of each product is specified immediately once the buyer learns that product's value. For example, each product is offered at a price that may depend on the messages sent in the previous periods. Such a mechanism screens the buyer over time, but cannot arbitrarily bundle products in the sense that the allocation of a product does not depend on what the buyer learns in the future periods.

We restrict attention to mechanisms in which the buyer has an ex post non-negative utility. That is, once all the uncertainty is resolved, the buyer's utility for the allocation and prices specified by the mechanism must be non-negative. This restriction excludes mechanisms in which the seller "sells the store in advance" to the buyer. In such a mechanism, before any uncertainty is resolved, the seller requests an advance payment (equal to expected surplus of an efficient allocation) from the buyer in exchange for the allocation of all products. For a static mechanism, our ex post non-negative utility constraint is equivalent to the standard notion of individual rationality. Thus, our class of mechanisms includes well-studied static multi-product screening mechanisms (going back to Stigler, 1963; Adams and Yellen, 1976; McAfee et al., 1989). The constraint that ex post utility must be non-negative allows us to compare dynamic and static mechanisms on an equal footing by isolating the value of inter-temporal incentives from the ability to give negative utility to the agent ex post.

Our first result is that, in order to maximize profit, it is without loss of generality to restrict attention to a class of *separable* mechanisms. A separable mechanism has two features. First, the allocation of each product is specified immediately once the value for that product is revealed to the buyer (as in "perishable products" mechanisms discussed above). Second, in each period, the buyer only reports the value learned in that period, instead of her interim type (the profile of values learned so far).

A separable mechanism is handicapped. It does not have the ability to "bundle" the products by arbitrarily tying the allocation of one product to the value for another. To see this, consider selling two products, and assume that the value for each product is either 1 or 2. A static mechanism can bundle the products. For example, it can offer the two products only as a bundle at a take it or leave it price of 3. If the buyer's value for either one of the products is 2, she buys the bundle, and otherwise she buys nothing. Since the allocation of each product depends on the value for the other product, no separable mechanism can implement this allocation.

Bundling is a strong instrument to screen types in static settings (McAfee et al., 1989). Since a separable mechanism cannot use such an instrument, and given that the ex post utility constraint restricts the use of advance payments, it is a priori not clear that a separable mechanism can be optimal. Indeed, we show with an example that if values are correlated, then there exists a static mechanism that outperforms all separable mechanisms, and thus separable mechanisms are sub-optimal. Nevertheless, in our setting with independently drawn values, we show that any mechanism can be converted to a separable one with the same profit. The result provides two insights. First, timing is a more powerful screening instrument than bundling. Second, perishability of products does not harm the seller's profits. Note that optimality of separable mechanisms does not imply sub-optimality of static mechanisms. Indeed, it is possible that for some distributions, a static mechanism is optimal. We revisit optimality of static mechanisms later.

The fact that separable mechanisms are optimal significantly simplifies the problem since optimal separable mechanisms can be characterized via standard recursive methods (Green, 1987; Spear and Srivastava, 1987; Thomas and Worrall, 1990). For a given period, consider possible continuation mechanisms that specify the allocation of products in future periods. Since values are independent, different interim types of the buyer assign the same expected utility to a given continuation mechanism. Thus, any continuation mechanism of an optimal mechanism must be optimal over all continuation mechanisms with the same expected utility. Otherwise, the continuation mechanism can be replaced with one that yields a higher profit without affecting incentive constraints. As a result, to ensure incentive compatibility, the mechanism needs only to maintain a single variable, the *promised utility*, which is the buyer's continuation utility. The promised utility is the information rent that must be paid to the buyer to respect incentive constraints. For any given period and any promised utility, the optimal continuation mechanism can be characterized via backward induction. In particular, in each period, the optimal mechanism solves a single-product *continuation profit problem*. The continuation profit problem maximizes the stage profit plus the continuation profit from future periods (calculated recursively), subject to delivering the utility that is promised.

We further study the properties of the continuation profit problem. As will be discussed, we use these properties to solve for optimal mechanisms for selling two products and to study the performance of static mechanisms. The solution to the last period problem is particularly simple. In the last period, the promised utility cannot be deferred and must be immediately fulfilled with money. Thus the problem is the standard screening problem of selling a single product, but subject to a novel promise keeping constraint. The optimal mechanism randomizes over at most two prices. We provide a geometric representation of the problem that relies on the concavification of an appropriately constructed revenue function.

We apply our analysis of the continuation profit problem to solve for optimal mechanisms with two products and two values. We identify five mechanisms, and show that at least one of them is optimal. All five mechanisms are deterministic. Four of these five mechanisms are static.[1] The fifth mechanism is not static: In the first period, the buyer reports her value, gets the first product regardless of her report, and pays her reported value. The second product is sold via a take it or

[1]In particular, once we specify optimal profit by identifying optimal separable mechanisms, we ask whether there exists a static mechanism that can achieve the same profit, and is thus optimal. (Recall that optimality of separable mechanisms does not necessarily imply that static mechanisms are sub-optimal.) In four out of five cases, a static mechanism is indeed optimal. However, whenever the fifth mechanism is optimal, all static mechanisms are sub-optimal.

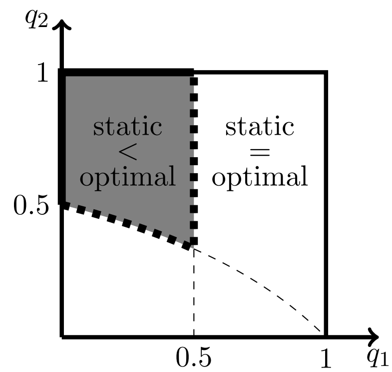

Figure 1: The dark-shaded region is the set of (q_1, q_2) for which static mechanisms are sub-optimal. The right and the bottom boundaries are not in the set.

leave it price that depends on the first period report. If the report is low, the price in the second period equals the high value. If the report is high, the price in the second period is p, to be identified shortly. The mechanism pays no information rent in the first period and defers rent to the second period. The low type in the first period receives no information rent in either period. To ensure incentive compatibility, the high type in the first period must receive an information rent equal to the difference between the values in the first period. The price p is set to deliver such information rent.

We identify conditions under which static mechanisms are sub-optimal. With two types and two values, static mechanisms are sub-optimal if the fifth mechanism described above (which is not static) is optimal. To specify the distributions for which the fifth mechanism is optimal, assume that the value for each product is either 1 or 2. Let q_1 be the probability that that the first product's value equals 2, and q_2 be the probability that the second product's value equals 2. The set of possible pairs (q_1, q_2) for which static mechanisms are sub-optimal is specified in Figure 1. We provide a closed form characterization for the set. Roughly speaking, static mechanisms are sub-optimal if and only if q_1 is low and q_2 is high, that is, the first product is ex ante less valuable than the second product.

Notice that if a static mechanism is optimal among all mechanisms, then it must also be optimal among only static mechanisms. Thus a corollary of the above result is identifying conditions under which the notoriously difficult problem of selling multiple products with static mechanisms can be solved. Beyond this example, our recursive formulation provides a tractable upper bound on the revenue of all static mechanisms. It may thus be useful in identifying approximately optimal static mechanisms.

Beyond the case of two products and two values, we provide a sufficient condition for sub-optimality of static mechanisms. Static mechanisms are sub-optimal if all optimal monopoly prices for the first product are lower than all optimal monopoly prices for the second product.[2] This result implies that a seller who could choose the order by which the buyer learns her values should first reveal the value for the product with the lower optimal monopoly price. The result generalizes the intuition that static mechanisms are sub-optimal if the first product is ex ante less valuable than the second product. To see the connection between the two results, consider again two products with values 1 and 2. All optimal monopoly prices for the first product are lower than all optimal monopoly prices for the second product if and only if $q_1 < 0.5 < q_2$. Indeed, if $q_1 < 0.5 < q_2$, the optimal monopoly price for the first product is 1, and the optimal monopoly price for the second product is 2. Note in Figure 1 that if $q_1 < 0.5 < q_2$, then static mechanisms are sub-optimal. Thus the result for any number of products and values partially generalizes the result for the case of two values and two products.

Related Work. Closest to our work are the papers that consider selling multiple products with ex post participation constraints. These papers assume that products perish. As a result, unlike ours, these papers are not concerned with the performance of static bundling mechanisms. Papadimitriou et al. (2016) show that when the buyer's values are correlated, finding optimal mechanisms is computationally hard. Mirrokni et al. (2016) characterize approximately optimal mechanisms with multiple buyers recursively. Mirrokni et al. (2020) consider the design of approximately optimal mechanisms when buyers have different expectations of future distributions. Balseiro et al. (2017) imposes a martingale constraint on the buyer's utility, and show that the seller's profit approaches first best (full surplus extraction) as the number of products grow.

The ex post utility constraint is related to limited liability constraints in dynamic principal agent models. Krishna et al. (2013), Krähmer and Strausz (2017), and Grillo and Ortner (2018) study contracts in which the agent's stage utility is non-negative. In our setting, a mechanism with non-negative stage utilities also has non-negative ex post utilities. Nonetheless, the two constraints are equivalent when solving for optimal mechanisms, since a separable mechanism satisfies the non-

[2]An optimal monopoly price for a product is an optimal take it or leave it price for selling that product.

negative stage utility constraint. Relatedly, Sappington (1983); Clementi and Hopenhayn (2006); DeMarzo and Sannikov (2006) assume that the agent cannot make monetary transfers to the principal.

Ex post participation constraints have also been studied for selling a single product. Krähmer and Strausz (2015) consider a problem where the seller has a single item to sell and the buyer sequentially receives signals about her valuation. They show that assuming a monotone hazard rate condition, static mechanisms are optimal. Bergemann et al. (2017) consider the same setting and provide necessary and sufficient conditions for optimality of static mechanisms. Krähmer and Strausz (2016) consider a multi-unit extension of the problem. When the buyer's utility is linear in quantity but seller's costs are nonlinear, static mechanisms are sub-optimal. The main question studied in these papers, namely optimality of static mechanisms, is similar to ours. Nevertheless, the settings and results are different.

More broadly, our work relates to two well-studied branches of literature on mechanism design, namely multi-product bundling and dynamic mechanism design.

The literature on multi-product bundling goes back to Stigler (1963) and Adams and Yellen (1976). This literature considers static mechanisms. That is, the buyer walks into the store knowing her values (alternatively, the buyer learns no new information about her values) McAfee et al. (1989) and Manelli and Vincent (2007) show that optimal screening mechanisms typically involve mixed bundling, i.e., offering a menu of bundles and prices. More generally, the literature shows that optimal mechanisms are complex. The optimal menu may include unboundedly many randomized bundles (Manelli and Vincent, 2007). As such, characterizations of optimal mechanisms are rare. Exceptions exist, such as Rochet and Chone (1998) and Daskalakis et al. (2017). Rochet and Chone (1998) characterize optimal mechanisms via a sweeping procedure that generalizes ironing. Daskalakis et al. (2017) characterize optimal mechanisms via a dual measure that satisfies certain stochastic dominance conditions. To apply either characterization, one must be able to identify sweeping procedure or the dual measure, for which no general construction is known. In contrast, optimal mechanisms can be characterized recursively in our dynamic setting.

The literature on dynamic mechanism design is similarly broad. The main thrusts in this literature study dynamic arrivals and departures of agents such as in Pai and Vohra (2008) and Gershkov and Moldovanu (2009, 2010), and agents whose private information evolves such as in

Courty and Li (2000); Eső and Szentes (2007); Kakade et al. (2013); Pavan et al. (2014); Bergemann and Välimäki (2010); Boleslavsky and Said (2012). Garrett (2016) combines these two branches by considering a setting with dynamic arrival and evolving values. A main difference with our paper lies in the ex post non-negative utility constraint. Prior literature, other than the papers we discuss below, considers weaker notions of individual rationality requiring that, in the beginning of each period, the expected utility from all future periods must be non-negative.

2 The Model

A seller has k products to sell to a single buyer. The cost of production is normalized to zero. The buyer's value for product $i \in \{1, \ldots, k\}$ is $v_i \in V_i \subseteq \mathbb{R}^+$. Assume that V_i is finite (this assumption is for simplicity and can be relaxed). Each value v_i is drawn independently from all other values with probability $f_i(v_i) > 0$. The distributions f_1 to f_k are commonly known to the seller and the buyer. We refer to $v = (v_1, \ldots, v_k)$ as the ex post type of the buyer. The utility of an ex post type v for receiving a set of products $S \subseteq \{1, \ldots, k\}$ and transferring $t \in \mathbb{R}$ units of money to the seller is $(\sum_{i \in S} v_i) - t$. The buyer is risk neutral, that is, the utility of receiving each product i with probability a_i and a random monetary transfer with expectation t to the seller is $v \cdot a - t = (\sum_i v_i a_i) - t$.

The buyer privately learns her values over time. In particular, in each period i from 1 to k, v_i is privately revealed to the buyer. Thus, in period i, the buyer knows values v_1 to v_i. Define $\Theta^i = \prod_{j=1}^{i} V_j$. We refer to $\theta^i \in \Theta^i$ as an interim type of the buyer in period i. For any ex post type v, let $v^{1:i} = (v_1, \ldots, v_i) \in \Theta^i$. If the buyer's ex post type is v, her interim type in period i is $v^{1:i}$.

By the revelation principle, we focus on direct incentive compatible mechanisms. A (direct) mechanism (a, t) consists of an allocation rule $a_i : \Theta^1 \times \ldots \times \Theta^k \to [0, 1]$ for each $i \in \{1, \ldots, k\}$, and a transfer rule $t : \Theta^1 \times \ldots \times \Theta^k \to \mathbb{R}$ (by risk neutrality we can assume that the transfer rule is deterministic). The interpretation is that in each period i, upon realizing each value v_i, the buyer reports an interim type $\theta^i \in \Theta^i$ to the mechanism. At the end of the last period k the buyer receives each product i with probability $a_i(\theta^1, \ldots, \theta^k)$ and transfers $t(\theta^1, \ldots, \theta^k)$ units of money to the mechanism. Note that in each period the buyer reports *all the values* she has observed so far. Such a formulation allows us to focus on "one-shot" incentive constraints in which the buyer reports her interim type truthfully regardless of past reports, as formalized below. One-shot incentive constraints lend themselves to a recursive solution of the problem. An alternative formulation of the revelation principle is to focus on mechanisms in which the buyer only reports v_i in period i, but in which incentive constraints may be multi-shot and thus less tractable for our purposes (see Appendix A for details). One of our results in the subsequent sections is that to find *optimal* mechanisms, however, it is indeed without loss of generality to focus not only on certain mechanisms in which the buyer reports v_i in period i, but also on one-shot deviations.

A mechanism (a, t) is periodic incentive compatible (PIC) if for all i, $v^{1:i}$, and $\theta^1, \ldots, \theta^{i-1}$,

$$v^{1:i} \in \arg\max_{\theta^i} \mathbf{E}\left[v \cdot a(\theta^1, \ldots, \theta^i, v^{1:i+1}, \ldots, v^{1:k}) - t(\theta^1, \ldots, \theta^i, v^{1:i+1}, \ldots, v^{1:k})\right], \tag{1}$$

where the expectation is taken over $v_{i+1}, \ldots, v_k$. That is, in period i, following a history of reports $\theta^1, \ldots, \theta^{i-1}$, the buyer with interim type $v^{1:i}$ maximizes her expected utility over all possible reports θ^i by reporting truthfully, $\theta^i = v^{1:i}$. Note that (1) concerns only one-shot deviations since the reports after period i are assumed to be truthful. Nevertheless, by backward induction, reporting truthfully in periods $i+1$ to k is indeed the optimal strategy of the buyer, regardless of the report she makes in period i.

A mechanism is ex post individually rational (ex post IR) if it guarantees non-negative utility for the buyer. Let us abuse notation and denote by $a(v)$ and $t(v)$ the outcome of the mechanism if the buyer reports $v^{1:i}$ in each period i, that is, $a(v) = a(v^{1:1}, v^{1:2}, \ldots, v^{1:k})$, and similarly for t. A mechanism (a, t) is ex post IR if at the end of period k, given the buyer's optimal strategy (reporting truthfully), the expected utility of the buyer is non-negative,

$$v \cdot a(v) - t(v) \geq 0 \tag{2}$$

for all ex post types v. Note that a_i denotes the probability of allocation. Thus the ex post individual rationality states that the utility of the buyer is non-negative for all ex post types v, but *in expectation* over the random choices of the mechanism. Even though (2) is written in expectation, it is possible to guarantee non-negative utility for all random choices of the mechanism by appropriately correlating transfers with allocation. We defer the argument to Appendix B. Following that argument, we abuse terminology and refer to the constraint (2) as the ex post IR constraint even though it is written in expectation over the randomization of the mechanism. In addition, we refer to $a(v), t(v)$, and $v \cdot a(v) - t(v)$ as buyer's ex post allocation, transfer, and utility.

The problem is to find a mechanism (a, t) that maximizes the expected revenue

$$\mathbf{E}_{v_1,\ldots,v_k}\left[t(v)\right], \tag{3}$$

subject to the periodic incentive compatibility (1) and ex post IR (2) constraints.

A special class of mechanisms is the class of *static* mechanisms. A static mechanism is a mechanism where the outcome depends only on the report in the last period k. Formally, a mechanism (a, t) is static if $(a, t)(\theta^1, \ldots, \theta^k) = (a, t)(\hat{\theta}^1, \ldots, \hat{\theta}^k)$ whenever $\theta^k = \hat{\theta}^k$. We can therefore represent such a mechanism more succinctly by its allocation rule $a^{ST} : \Theta^k \to X$ and transfer rule $t^{ST} : \Theta^k \to \mathbb{R}$ in the last period. The interpretation is that in the last period k, having learned all her values, the buyer makes a report v to the mechanism. The buyer then receives each product i with probability $a_i^{ST}(v)$ and transfers $t^{ST}(v)$ to the mechanism. Since the reports in periods before k are irrelevant, a static mechanism trivially satisfies all periodic incentive compatibility constraints before the last period k. Therefore a static mechanism is PIC if it satisfies the last period incentive compatibility condition,

$$v \cdot a^{ST}(v) - t^{ST}(v) \geq v \cdot a^{ST}(\hat{v}) - t^{ST}(\hat{v}).$$

for all $v, \hat{v} \in \Theta^k$. Similarly, a static mechanism (a^{ST}, t^{ST}) is ex post IR if

$$v \cdot a^{ST}(v) - t^{ST}(v) \geq 0.$$

This formulation is used in the multi-product mechanism design literature, e.g., in Manelli and Vincent (2007); Daskalakis et al. (2014). Thus our model nests the optimal mechanism design problem for selling k products with static mechanisms as a special case. Note further that our model also nests "semi-static" mechanisms where the products $1, \ldots, k$ are partitioned into intervals, and the allocation of products in each interval does not depend on the reports in other periods. For example, if the partitioning is $\{1, 2\}$, $\{3, 4, 5\}$, etc, then the allocation of products 1 and 2 is decided at period 2, allocation of products 3 to 5 is decided at period 5 depending only the reports about the value of products 3 to 5, and so on. Furthermore, our set of mechanisms includes "perishable products" mechanisms in which the allocation of any product i depends only on the reports made up to and including that period. That is, $a_i(\theta^1, \ldots, \theta^k) = a_i(\hat{\theta}^1, \ldots, \hat{\theta}^k)$ if $\theta^j = \hat{\theta}^j$ for all $j \leq i$.

The optimal revenue among all mechanisms is at least as high as the revenue from any static mechanism. This observation immediately follows from the fact that static mechanisms are a subclass of all mechanisms. A question we ask is whether the optimal revenue is strictly higher than that from static mechanisms. To this end, we first identify optimal revenue, and then ask whether it can be achieved by a static mechanism.

3 Recursion, Separability, and Promised Utility

The periodic incentive compatibility constraints are complex. In each period, the buyer may misreport different dimensions of her interim type. Even for the special case of static mechanisms where all incentive constraints before the last period are trivially satisfied, the incentive constraints are complex. Nevertheless, we show that the optimization problem can be solved by making two observations. First, to maximize revenue, it is sufficient to focus on a simple class of *separable* mechanisms. Second, it is possible to optimize over separable mechanisms recursively.

A separable mechanism satisfies two properties. First, no re-reporting is required. That is, in each period i, the buyer only reports her value v_i for product i (instead of her interim type). Second, the allocation of product i is based on the reports made up to (and including) period i, but does not depend on reports made in periods $i + 1$ to k. Formally,

Definition 1. A mechanism (a, t) is *separable* if for all $\theta^1, \ldots, \theta^k$ and $\hat{\theta}^1, \ldots, \hat{\theta}^k$,

1. $t(\theta^1, \ldots, \theta^k) = t(\hat{\theta}^1, \ldots, \hat{\theta}^k)$ if $\theta^i_i = \hat{\theta}^i_i$ for all i, and

2. for all i, $a_i(\theta^1, \ldots, \theta^k) = a_i(\hat{\theta}^1, \ldots, \hat{\theta}^k)$ if $\theta^j_j = \hat{\theta}^j_j$ for all $j \leq i$.

The first property states that the payment rule depends on the report θ^i in each period i only through the value learned in that period θ^i_i. The second property states that the allocation of product i depends on the report θ^j in period $j \leq i$ only through the value learned in that period θ^j_j, and does not depends on the report $\theta^{j'}$ in period $j' > i$. We will henceforth represent a separable mechanism more succinctly with functions $a_i^{SP} : \Theta^i \to [0,1]$ and $t^{SP} : \Theta^k \to \mathbb{R}$ (as opposed to $a_i : \Theta^1 \times \ldots \times \Theta^k \to [0,1]$ and $t : \Theta^1 \times \ldots \times \Theta^k \to \mathbb{R}$ for a general mechanism). The interpretation is that the buyer reports v_i in each period i. Given reports $(v_1, \ldots, v_k)$, product i is allocated with probability $a_i^{SP}(v_1, \ldots, v_i)$, and the transfer is $t^{SP}(v_1, \ldots, v_k)$.

We now show that to maximize revenue, it is without loss of generality to restrict attention to separable mechanisms. Notice that a separable mechanism is handicapped. It does not have the ability to bundle the products together since it cannot tie the allocation of a product to the allocation of future products (and the buyer's reports about those values). In contrast, a static mechanism does have the ability to bundle (we later return to this comparison and provide examples). As a result, it is a priori not clear that the optimal separable mechanism obtains at least as much revenue as all static mechanisms, let alone all mechanisms (that include separable and static mechanisms as special cases).

To argue that restricting to separable mechanisms is without loss of generality for maximizing revenue, we convert any mechanism to a separable mechanism with the same revenue (but with a different allocation rule). In particular, given a mechanism (a, t), define its *induced separable mechanism* (a^{ISP}, t^{ISP}) as follows. The allocation probability a_i^{ISP} is the expectation of the allocation probability a_i assuming truthful reporting in all future periods. That is, for any $v_1, \ldots, v_i$,

$$a_i^{ISP}(v_1, \ldots, v_i) := \mathbf{E}_{v_{i+1}, \ldots, v_k}\left[a_i(v)\right]. \tag{4}$$

(Recall that $a_i(v)$ is the shorthand for the allocation when the buyer reports $(v_1, \ldots, v_j)$ in each period j.) For $v = (v_1, \ldots, v_k)$, define the transfer as follows

$$t^{ISP}(v) := t(v) - v \cdot a(v) + v \cdot a^{ISP}(v). \tag{5}$$

(Recall similarly that $t(v)$ is the shorthand for the transfer when the buyer reports $(v_1, \ldots, v_j)$ in each period j.)

Let us verify properties of the above construction. Consider the separable mechanism represented by (a^{ISP}, t^{ISP}) defined in Equations (4) and (5). First, if a mechanism is ex post IR, then so is its induced separable mechanism. This is because the transfer rule of the induced separable mechanism is defined such that the two mechanisms have the same ex post utility. That is, rearranging Equation (5) we have

$$v \cdot a^{ISP}(v) - t^{ISP}(v) = v \cdot a(v) - t(v) \tag{6}$$

for all v. Second, the two mechanisms have the same expected revenue. The reason is that the two mechanisms have the same ex post utility and also create the same surplus. More precisely, take the expectation of Equation (5),

$$\begin{aligned}
\mathbf{E}_v\left[t^{ISP}(v)\right] &= \mathbf{E}_v\left[t(v)\right] + \mathbf{E}_v\left[v \cdot a^{ISP}(v) - v \cdot a(v)\right] \\
&= \mathbf{E}_v\left[t(v)\right] + \sum_i \mathbf{E}_v\left[v_i a_i^{ISP}(v_1, \ldots, v_i) - v_i a_i(v)\right] \\
&= \mathbf{E}_v\left[t(v)\right] + \sum_i \mathbf{E}_{v_1,\ldots,v_i}\left[v_i\Big(a_i^{ISP}(v_1, \ldots, v_i) - \mathbf{E}_{v_{i+1},\ldots,v_k}\left[a_i(v)\right]\Big)\right] \\
&= \mathbf{E}_v\left[t(v)\right],
\end{aligned} \tag{7}$$

where the last equality follows from Equation (4). It only remains to verify that these adjustments do not violate the PIC constraints.

To see that the construction above preserves incentive compatibility, let us first verity incentive compatibility on path, that is, following a history of truthful reports. Consider the utility of a buyer with value v_i from reporting $\hat{v}_i$. By (6), the expected utility of the buyer in the induced separable mechanism is equivalent to the utility it would get in mechanism (a, t) if she reports $\hat{v}_i$ instead of v_i in *every period* i to k (recall that the buyer reports her full interim type in each period, and $a(v)$ and $t(v)$ stand for the outcome if the buyer reports $v_1, \ldots, v_i$ in each period i). However, by incentive compatibility of (a, t), the buyer is better off if it reports v_i instead of $\hat{v}_i$ in every period. Thus the incentive constraint is satisfied on path.

The equivalence discussed above no longer holds off path. To establish incentive compatibility off path, we notice that a separable mechanism is PIC if it is PIC on path. Indeed, in a separable mechanism, the report in period i does not affect the allocation of products 1 to $i-1$. Thus, the incentive constraint at period i for an interim type $(v_1, \ldots, v_i)$ following a history of reports $\hat{v}_1, \ldots, \hat{v}_{i-1}$ is identical, up to a constant, to the incentive constraint for an interim type $(\hat{v}_1, \ldots, \hat{v}_{i-1}, v_i)$ following a history of truthful reports $(\hat{v}_1, \ldots, \hat{v}_{i-1})$. Thus, if a separable mechanism is incentive compatible for all histories of truthful reports, then it is incentive compatible for all histories. Formally, we have the following lemma.

Lemma 1. *A separable mechanism (a^{SP}, t^{SP}) is PIC if it is PIC on path, that is, for all i and $v_1, \ldots, v_{i-1}$,*

$$v_i \in \arg\max_{\hat{v}_i} \mathbf{E}\left[v \cdot a^{SP}(v_1, \ldots, v_{i-1}, \hat{v}_i, v_{i+1}, \ldots, v_k) - t^{SP}(v_1, \ldots, v_{i-1}, \hat{v}_i, v_{i+1}, \ldots, v_k)\right]$$

where the expectation is taken over $v_{i+1}, \ldots, v_k$.

Proof. The incentive constraint in period i is that for an interim type $v_1, \ldots, v_{i-1}$, and following a history of reports $\hat{v}_1, \ldots, \hat{v}_{i-1}$, the expected utility of the buyer

$$\mathbf{E}\left[v \cdot a^{SP}(\hat{v}_1, \ldots, \hat{v}_{i-1}, \hat{v}_i, v_{i+1}, \ldots, v_k) - t^{SP}(\hat{v}_1, \ldots, \hat{v}_{i-1}, \hat{v}_i, v_{i+1}, \ldots, v_k)\right]$$

is maximized over all reports $\hat{v}_i$ by setting $\hat{v}_i = v_i$. Separability implies that the utility of the buyer from the allocation of products 1 to $i-1$, $\sum_{j<i} v_j a_j^{SP}(\hat{v}_1, \ldots, \hat{v}_j)$, does not depend on the report in period i. Therefore, the solution to the maximization problem does not change if $\sum_{j<i} v_j a_j^{SP}(\hat{v}_1, \ldots, \hat{v}_j)$ is replaced by $\sum_{j<i} \hat{v}_j a_j^{SP}(\hat{v}_1, \ldots, \hat{v}_j)$, which also does not depend on $\hat{v}_i$. As a result, the incentive constraint holds if

$$\mathbf{E}\left[(\hat{v}_1, \ldots, \hat{v}_{i-1}, v_i, \ldots, v_k) \cdot a^{SP}(\hat{v}_1, \ldots, \hat{v}_{i-1}, \hat{v}_i, v_{i+1}, \ldots, v_k) - t^{SP}(\hat{v}_1, \ldots, \hat{v}_{i-1}, \hat{v}_i, v_{i+1}, \ldots, v_k)\right]$$

is maximized over all $\hat{v}_i$ by setting $\hat{v}_i = v_i$. This constraint is the PIC constraint of interim type $(\hat{v}_1, \ldots, \hat{v}_{i-1}, v_i)$ following a truthful history of reports $\hat{v}_1, \ldots, \hat{v}_{i-1}$. □

The following proposition summarizes the arguments made so far.

Proposition 1. *The expected revenue of any mechanism is equal to the expected revenue of its induced separable mechanisms. If a mechanism is PIC and ex post IR, then so is its induced separable mechanism.*

3.1 Examples: Separability, Bundling, and Correlation

Before we proceed with characterizing optimal separable mechanisms, let us provide some examples.

The first example shows that while static mechanisms can bundle the products, separable mechanism do not have this ability.

Example 1. There are two products, and the value for each product is either 1 or 2. Consider a static mechanism that only offers the bundle of both products for a price of 3. The allocation probabilities and transfers are shown in the table below for all types.

v_1	v_2	a_1	a_2	t
1	1	0	0	0
1	2	1	1	3
2	1	1	1	3
2	2	1	1	3

Notice that the allocation of product 1 depends on the value for product 2, $a_1(1,2) \neq a_1(1,1)$, and vice versa for product 2. Thus, no separable mechanism can implement this allocation.

The following example shows that Proposition 1 does not hold if the buyer's values for products are correlated.

Example 2. There are two products, and the value for product is either 1 or 2. The probabilities of profiles $(1,2)$ and $(2,1)$ are 0.5 each, and the probabilities of profiles $(1,1)$ and $(2,2)$ are 0 each. Similar examples can be constructed wherein the probabilities of profiles $(1,1)$ and $(2,2)$ are non-zero but small. The optimal mechanism is static. It extracts the full surplus by offering the bundle for a price of 3, thus obtaining an expected revenue of 3. However, no separable mechanism has revenue 3 as we show below.

First consider a naive generalization of the construction in Section 3 where the expected allocation probabilities and the ex post utilities of the separable mechanism are equal to those of the static mechanism. Conditioned on $v_1 = 1$, v_2 is equal to 2 with probability one. Thus, $a_1(1) = 1$. Similarly we have $a_1(2) = 1$. By definition, the allocation probabilities of product 2 are equal to those of the static mechanism. The separable mechanism is shown below.

v_1	v_2	a_1	a_2	t
1	1	1	0	1
1	2	1	1	3
2	1	1	1	3
2	2	1	1	3

Notice that the expected revenue of the separable mechanism is indeed 3. However, the separable mechanism is not PIC. Indeed, the expected utility of an interim type $v_1 = 2$ from truth-telling is zero since conditioned on $v_1 = 2$, $v_2 = 1$ with probability one. On the other hand, the expected utility from reporting $v_1' = 1$ is 1 since by doing so, the buyer receives product 1 and pays 1.

We now argue that indeed no separable mechanism can obtain an expected revenue of 3. By ex post IR, for the revenue to be 3, both types $(1, 2)$ and $(2, 1)$ must receive both products and pay 3. Thus in a separable mechanism, $a_1(v_1) = 1$ for all v_1. Further, the incentive compatibility constraint in the second period requires that the probability of allocation of product 2 for type $(2, 2)$ must be no lower than that for type $(2, 1)$. Thus, $a_2(2, 2) = 1$ and $t(2, 2) = 3$. We summarize our discussion in the table below, in which the only free parameters are $a_2(1, 1)$ and $t(1, 1)$.

v_1	v_2	a_1	a_2	t
1	1	1	$a_2(1,1)$	$t(1,1)$
1	2	1	1	3
2	1	1	1	3
2	2	1	1	3

The ex post IR constraint for type $(1, 1)$ is

$$1 + a_2(1, 1) - t(1, 1) \geq 0.$$

Now consider the PIC constraint in period 2 for ex post type $(1, 2)$ following a history of truthful report $v_1' = 1$. The constraint is

$$0 \geq 1 + 2a_2(1, 1) - t(1, 1).$$

Given the two constraints above, we must have $a_2(1, 1) = 0$ and $t(1, 1) = 1$. Thus the mechanism is equal to the induced separable mechanism of the static mechanism that sells the bundle at price 3. As we argued above, the separable mechanism is not incentive compatible.

3.2 Recursive Optimization: Separability and Promised Utility

We now turn to the problem of maximizing revenue. Equipped with Proposition 1, we study optimization over separable mechanisms. We will henceforth represent a separable mechanism simply by (a, t). The PIC constraint for a separable mechanism is simpler than the PIC constraint for a general mechanism. Consider the incentive compatibility constraint at a period i. By Lemma 1, we need to only consider the PIC constraints on path. In a separable mechanism, the allocation of products 1 to $i-1$ does not depend on the report at period i. Therefore, to choose her report in period i, the buyer only takes into account the allocations of products i to k and the transfer. For reports $(v_1, \ldots, v_i)$, define the *continuation utility* CU_i of the buyer to be the expected utility from the allocation of products $i+1$ to k and the transfer, assuming truthful reporting in future periods,

$$CU_i(v_1, \ldots, v_i) = \mathbf{E}_{v_{i+1},\ldots,v_k}\left[\left(\sum\nolimits_{j>i} v_j a_j(v^{1:j})\right) - t(v)\right].$$

Note also that the continuation utility does not depend on the buyer's interim type in period i, and instead is only a function of the reports that the buyer makes. The PIC constraint on path at every period i is that for all $v_1, \ldots, v_{i-1}$ and v_i, the buyer maximizes the sum of her stage utility in period i plus her continuation utility from the future periods by reporting her value truthfully,

$$v_i \in \arg\max_{\hat{v}_i} v_i a_i(v_1, \ldots, v_{i-1}, \hat{v}_i) + CU_i(v_1, \ldots, v_{i-1}, \hat{v}_i). \tag{8}$$

We now recursively characterize optimal separable mechanisms. The main observation is that every history can be summarized by the continuation utility as a state variable. In particular, consider reports $(v_1, \ldots, v_i)$. If we modify the mechanism in periods $i+1$ to k following the history $(v_1, \ldots, v_i)$, while leaving the continuation utility unchanged, then the incentive constraints in period i will remain satisfied. Thus, the continuation mechanism following the history $(v_1, \ldots, v_i)$ must be optimal over all continuation mechanisms with promised utility equal to $u_i(v_1, \ldots, v_i)$

(otherwise the continuation mechanism can be modified to one with higher revenue). We can thus maintain the "promised utility" $\mathsf{PU} = CU_i(v_1, \ldots, v_i)$ as a scalar state variable that summarizes the history and characterize optimal mechanisms recursively, as discussed below.

In order to describe the optimal mechanism recursively, we first define the seller's continuation revenue, i.e., the optimal revenue it can achieve for any given promised utility. To do so, it will be simpler to consider mechanisms in which the buyer pays the surplus $v_i a_i$ in each period $i < k$. Notice that this transformation is simply an accounting technique and is without loss of generality. That is, all transfers in such a mechanism can be deferred to the last period, resulting in a mechanism where transfers are only made in the last period, without affecting incentive and participations constraints and revenue. Nevertheless, this transformation simplifies the analysis since the stage utility of the buyer in each period $i < k$ becomes zero, and thus we can verify the ex post IR constraint by simply checking that the state utility is non-negative in the last period. Given this transformation, in a period i and for a given promised utility, the mechanism optimizes the revenue that can be extracted in period i, which is equal to the surplus in period i, plus the continuation revenue that can be extracted in the future periods given the updated promised utility in period $i + 1$.

The following definition recursively formulates optimal revenue for a given promised utility at every period PU. The optimal continuation revenue in period i for a given promised utility PU is $CR_i(\mathsf{PU})$. For a PU and report v_i in period i, the buyer receives product i with probability $\mathcal{A}_i^{\mathsf{PU}}(v_i)$ and is given a promised utility of $\mathcal{U}_i^{\mathsf{PU}}(v_i)$ for future periods.

Definition 2 (The Continuation Revenue Problem)**.** Define the *seller's continuation revenue* functions $CR_k, \ldots, CR_0$ recursively as follows. Let $CR_k(\mathsf{PU}) = -\mathsf{PU}$ if $\mathsf{PU} \geq 0$, and $CR_k(\mathsf{PU}) = -\infty$ if $\mathsf{PU} < 0$. For all $i \leq k$ and PU, define $CR_{i-1}(\mathsf{PU})$ to be equal to the maximum of

$$\mathbf{E}_{v_i}\left[v_i a_i^{OP}(v_i) + CR_i(u_i^{OP}(v_i))\right],$$

$$\text{subject to: } u_i^{OP}(v_i) \geq (v_i - \hat{v}_i)a_i^{OP}(\hat{v}_i) + u_i^{OP}(\hat{v}_i); \forall v_i, \hat{v}_i, \tag{9}$$

$$\mathbf{E}_{v_i}\left[u_i^{OP}(v_i)\right] = \mathsf{PU}, \tag{10}$$

over all one-product allocation functions $a_i^{OP} : V_i \to [0, 1]$ and continuation utility functions u_i^{OP} :

$V_i \to \mathbb{R}$. Define $(\mathcal{A}_i^{\mathsf{PU}}, \mathcal{U}_i^{\mathsf{PU}})$ to be the optimal solution to the above problem.

The functions $\mathcal{A}_i^{\mathsf{PU}}$ and $\mathcal{U}_i^{\mathsf{PU}}$ are the optimal allocation and continuation utility functions at period i, given a promised utility PU. As discussed above, the optimal mechanism maximizes the sum of the surplus in period i and the revenue from the future periods. The constraint (9) is the incentive constraint at period i. By reporting truthfully, the buyer receives zero stage utility in period i and a continuation utility of $u_i^{OP}(v_i)$ in periods $i+1$ to k. On the other hand, by reporting $\hat{v}_i$, the buyer receives product i with probability $a_i^{OP}(\hat{v}_i)$, pays $\hat{v}_i a_i^{OP}(\hat{v}_i)$, and gets a continuation utility of $u_i^{OP}(\hat{v}_i)$. The constraint (10) is the "promise-keeping" constraint. Since the buyer receives no stage utility in period i, the promised utility is equal to the expectation of the buyer's continuation utility. We will revisit Definition 2 later and show that it can be transformed to a more familiar one-product static mechanism design problem via a change of variables.

We now characterize optimal mechanisms recursively, formalizing the discussion above. The optimal mechanism maintains a promised utility PU, initialized appropriately, as a state variable. In each period i, the allocation rule is given by $\mathcal{A}_i^{\mathsf{PU}}$, and the state variable is updated given $\mathcal{U}_i^{\mathsf{PU}}$ defined in Definition 2.

Definition 3 (The Promised Utility Mechanism)**.** Define the promised utility mechanism as follows. Let $(\mathcal{A}, \mathcal{U})$ be solutions to the continuation revenue problem (Definition 2). Set the initial promised utility PU equal to the maximizer of $CR_0(\mathsf{PU})$. At each period i, given the current promised utility PU and report v_i,

- the buyer gets product i with probability $\mathcal{A}_i^{\mathsf{PU}}(v_i)$,
- the buyer pays $v_i \mathcal{A}_i^{\mathsf{PU}}(v_i)$ if $i < k$ or $v_i \mathcal{A}_i^{\mathsf{PU}}(v_i) - \mathcal{U}_i^{\mathsf{PU}}(v_i)$ if $i = k$, and
- the promised utility is updated by setting $\mathsf{PU} := \mathcal{U}_i^{\mathsf{PU}}(v_i)$.

The lemma below shows that the promised utility mechanism defined above is optimal.

Lemma 2. *The promised utility mechanism in Definition 3 is optimal.*

4 Structure of Optimal Mechanisms

We now explore the structure of optimal mechanisms by studying the continuation revenue problem (Definition 2). To simplify notation, we drop the index i whenever it is clear from the context. That is, we use v, V, f, and CR instead of v_i, V_i, f_i, and CR_i. We also write a, t, u instead of a^{OP}, t^{OP}, u^{OP}, keeping in mind that they refer to one-product mechanisms. We use the structural properties of the solutions to the continuation revenue problem to solve for optimal mechanisms with two products and two values in this section, and to provide a revenue comparison of dynamic and static mechanisms in the next section.

The continuation revenue problem can be converted into a more familiar form via a change of variables. In particular, let $t(v) = va(v) - u(v)$ and write Definition 2 as the problem of maximizing

$$\mathbf{E}_v\left[va(v) + CR\Big(va(v) - t(v)\Big)\right], \tag{11}$$

$$\text{subject to:}\quad va(v) - t(v) \geq va(v') - t(v'); \forall v, v', \tag{12}$$

$$\mathbf{E}_v\left[va(v) - t(v)\right] = \mathsf{PU}. \tag{13}$$

In the above formulation, the objective is to maximize the expected surplus plus the expected continuation revenue, over (one-product) mechanisms with allocation rule a and transfer rule t. The incentive constraint (12) is the standard static incentive constraint for selling a single product. The promise-keeping constraint (13) requires the expected utility of the mechanism to be equal to PU.

We use a characterization of incentive compatible mechanisms that consists of two parts, formalized in the lemma below. The first part is standard. Namely, for the incentive constraint (12) to be satisfied, the allocation rule a must be monotone non-decreasing. Second, in an optimal mechanism, the transfer rule minimizes the utility to the buyer, which is the area under a. See Figure 2. Even though this part parallels the standard characterization, it involves a subtlety. Without the promise-keeping constraint, and if the goal were to maximize the expected transfer t for a given allocation rule, it is clearly optimal to minimize the utility to the buyer. However, in the continuation revenue problem, simply lowering the buyer's utility may cause two complications. First, the promise-keeping constraint may be violated. Second, the continuation revenue may decrease (the

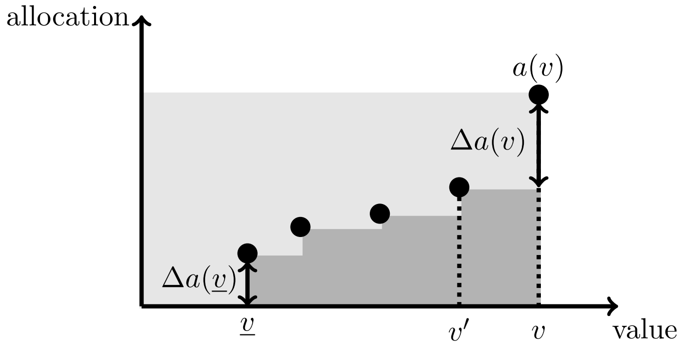

Figure 2: The transfer rule of an optimal mechanism. The area shaded light is the transfer of type v. The area shaded dark is the utility of type v. $\Delta a(v)$ is the difference between $a(v)$ and $a(v')$, where v' is the highest value in V lower than v.

continuation revenue function CR is not necessarily monotone). Nevertheless, we show that if the transfer rule t does not minimize the utility of the buyer given an allocation rule a, there exists *another* mechanism that is also feasible and obtains higher revenue than (a, t).

The lemma requires some notation. For $v \in V$, let $\Delta a(v) = a(v) - a(\max_{v'<v} v')$ denote the change in a at v, where $\Delta a(\underline{v}) = a(\underline{v})$ for the smallest value $\underline{v}$ in V (see Figure 2).

Lemma 3. *There exists t such that the mechanism (a, t) satisfies incentive constraints (12) if only if a is monotone non-decreasing. If a mechanism (a, t) is an optimal solution to the continuation revenue problem (Definition 2), then $t(v) = va(v) - \left(\sum_{v' \leq v}(v - v')\Delta a(v')\right) - u(\underline{v})$.*

Equipped with Lemma 3, to solve the continuation revenue problem we assume $t(v) = va(v) - \left(\sum_{v' \leq v}(v - v')\Delta a(v')\right) - u(\underline{v})$. Even though the equation need not hold for all feasible mechanisms, it can be imposed without loss of generality to solve for optimal mechanisms.

In the rest of this section we consider two special cases of the continuation revenue problem, that is, the last period and the first period. We then combine the two cases and solve some examples with two periods. The analyses of the two cases are also helpful in the next section where we compare the performance of dynamic and static mechanisms.

4.1 The Last Period

Consider the last period. Recall that $CR_k(\mathsf{PU}) = -\infty$ if $\mathsf{PU} < 0$. Thus, the utility in the last period must be non-negative, $u \geq 0$. Additionally, $CR_k(\mathsf{PU}) = -\mathsf{PU}$ for $\mathsf{PU} \geq 0$, that is, any promised utility must be paid back to the buyer immediately with money. The problem becomes to maximize the expected revenue

$$\mathbf{E}_v\left[va(v) - u(v)\right] = \mathbf{E}_v\left[t(v)\right],$$

subject to the incentive compatibility constraint (12), the promise keeping constraint (13), and non-negative utility constraint,

$$va(v) - t(v) \geq 0.$$

Thus the problem is the standard problem of maximizing revenue of selling a single product, but with an additional promise-keeping constraint.

We now characterize the structure of optimal mechanisms in the last period. We show that it is optimal to choose one of at most two prices at random, and sell the product at that price as a take it or leave it offer. To do so, we construct a revenue function that maps the expected utility of any price to the expected revenue of that price. We then show that the optimal revenue can be found by "concavifying" that revenue function. To be formal, we define some notation.

Define a revenue function RU that for each $p \in V$, maps the expected utility (to the buyer) of posting a price p for the product to its expected revenue. In particular, let $U = \{\mathbf{E}[\max(v - p, 0)] \mid p \in V\}$ be the set of possible expected utilities from posting prices in V, and for each $u \in U$, let $p(u) \in V$ be the price that induces that expected utility (note that such a price is unique since different prices induce different utilities). Notice that the smallest utility in U is zero, and let $\bar{u}$ denote the largest utility. Now define the revenue function $RU : U \to \mathbb{R}$, where $RU(u)$ is the expected revenue from posing a price that gives the buyer expected utility u,

$$RU(u) = p(u) \sum_{v \geq p(u)} f(v). \tag{14}$$

Define $\hat{RU} : \mathbb{R}^+ \to \mathbb{R}$ as follows. On the interval $[0, \bar{u}]$, $\hat{RU}$ is the concavification of RU, that is, the smallest concave function that is pointwise at least as large as RU. Above $\bar{u}$, the function $\hat{RU}$ continues linearly with slope -1. The lemma below shows that $\hat{RU}$ is the continuation revenue.

To identify the prices in an optimal mechanism, define $\ell(u)$ to be the largest $u' \leq u$ at which

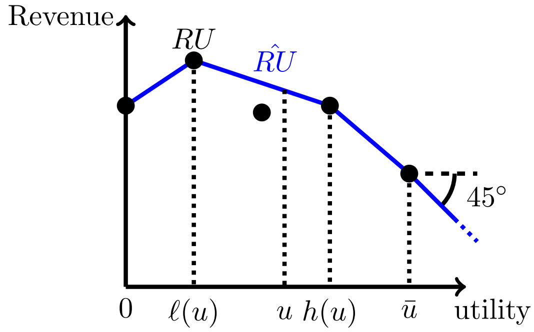

Figure 3: The function RU maps, for each possible p in V, the utility of posting price p to the revenue of posting that price. The function $\hat{RU}$ is its concave hull, and continues with slope -1 above $\bar{u}$. For each u, $\ell(u)$ is the largest utility less than u where the two functions are equal. Similarly, $h(u)$ is the smallest utility larger than u where the two functions are equal.

the two functions are equal $RU(u') = \hat{RU}(u')$, and similarly $h(u)$ to be the smallest $u' \geq u$ at which $RU(u') = \hat{RU}(u')$. See Figure 3.

We now define two mechanisms that are optimal depending on the value of PU. The first mechanism is optimal when PU is large. The mechanism gives the product to all types and pays them $\mathsf{PU} - \bar{u}$,

$$a(v) = 1, t(v) = \bar{u} - \mathsf{PU}, \forall v. \tag{15}$$

The second mechanism is optimal when PU is small. The mechanism randomizes over two prices that induce expected utilities $\ell(\mathsf{PU})$ and $h(\mathsf{PU})$ with probabilities set such that the promise-keeping constraint is satisfied. That is, the probabilities of $p(\ell(\mathsf{PU}))$ and $p(h(\mathsf{PU}))$ are $1-\alpha$ and α, respectively, where

$$\alpha = \frac{\mathsf{PU} - \ell(\mathsf{PU})}{h(\mathsf{PU}) - \ell(\mathsf{PU})}.$$

Formally, the allocation and transfer rules are defined as follows (notice that $p(h(\mathsf{PU}))$ is weakly lower than $p(\ell(\mathsf{PU}))$). See Figure 4.

$$(a,t)(v) = \begin{cases} (0,0) & \text{if} \quad v < p(h(\mathsf{PU})), \\ (\alpha, \alpha p(h(\mathsf{PU}))) & \text{if} \quad p(h(\mathsf{PU})) \leq v < p(\ell(\mathsf{PU})), \\ (1, (1-\alpha)p(\ell(\mathsf{PU})) + \alpha p(h(\mathsf{PU}))) & \text{if} \quad p(\ell(\mathsf{PU})) \leq v. \end{cases} \tag{16}$$

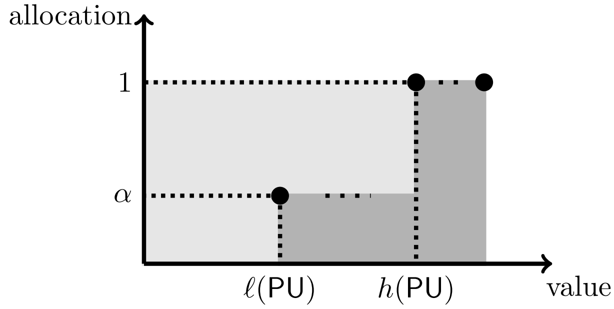

Figure 4: The solution to the continuation revenue problem in the last period for small PU.

Lemma 4. *The solution to the continuation revenue problem (Definition 2) in the last period is as follows. If* $\mathsf{PU} \geq \bar{u}$*, then the mechanism defined in Equation* (15) *is optimal. Otherwise, the mechanism defined in Equation* (16) *is optimal. In addition,* $CU_{k-1}(\mathsf{PU}) = \hat{RU}(\mathsf{PU})$*.*

We state the following corollary of Lemma 4 for future reference. Increasing the promised utility changes the optimal allocation rule, unless the allocation probability of all types is 1. This is clear from (16) and Figure 3. As long as $\mathsf{PU} \leq \bar{u}$, increasing the promised utility either changes the prices $\ell(\mathsf{PU})$ and $h(\mathsf{PU})$ or their probabilities. If the allocation rule does not change in PU, then any additional promised utility must be fulfilled with monetary transfers to the buyer. Thus the interpretation of the corollary is that it is never optimal to fulfill promises by paying money, if it is possible to do so by increasing the allocation.

Corollary 1. *Consider the solution to the continuation revenue problem (Definition 2) in the last period. If* $\mathcal{A}^{\mathsf{PU}} = \mathcal{A}^{\mathsf{PU}'}$ *for* $\mathsf{PU} \neq \mathsf{PU}'$*, then* $\mathcal{A}^{\mathsf{PU}}(v) = \mathcal{A}^{\mathsf{PU}'}(v) = 1$ *for all* v*.*

4.2 The First Period

In the first period, the promise-keeping constraint can be relaxed. This is because in Definition 3, the initial promised utility PU can be arbitrarily chosen.

The following lemma shows that the optimal allocation probabilities are at least as high as what they would be if the goal were to maximize only the stage revenue and to ignore the continuation revenue. In particular, let P be the set of *optimal monopoly prices.* That is, $P = \arg\max_p p \sum_{v \geq p} f(v)$ is the set of prices that maximize revenue of selling only the one product with value distribution

f. Let $\underline{p}$ and $\bar{p}$ be the lowest and highest such price. A mechanism is optimal for selling only one product with distribution f if and only if it posts a price that is randomly chosen from P. This observation has two implications. First, in every optimal mechanism, any $v \geq \bar{p}$ must receive the product with probability one. Second, there exists an optimal mechanism (namely the mechanism that posts a price $\underline{p}$) in which any $v \geq \underline{p}$ receives the product with probability one. The lemma below shows that the same must be true for the continuation revenue problem.

Lemma 5. *Consider the solution to the continuation revenue problem (Definition 2) in the first period. There exists an optimal mechanism where $a(v) = 1$ for all $v \geq \underline{p}$, where $\underline{p}$ is the lowest optimal monopoly price. Additionally, in every optimal mechanism, $a(v) = 1$ for all $v \geq \bar{p}$, where $\bar{p}$ is the highest optimal monopoly price.*

4.3 Two Periods and Two Values

We now apply the analyses of Section 4.1 and Section 4.2 to solve for optimal mechanisms for selling two products, where each product has two possible values, that is, $V_1 = \{\underline{v}_1, \bar{v}_1\}$ and $V_2 = \{\underline{v}_2, \bar{v}_2\}$. To simplify notation, let $q_1 = f_1(\bar{v}_1)$ and $q_2 = f_2(\bar{v}_2)$.

In order to calculate the continuation revenue function in the first period, we start our backward induction with the second period. The continuation revenue function CR_1 can be calculated using the analysis of Section 4.1. The first step is to define the revenue function RU that maps the utility of posting a price to its revenue. For a take it or leave it price $\underline{v}_2$, the buyer's utility is $(\bar{v}_2 - \underline{v}_2)q_2$ and the seller's revenue is $\underline{v}_2$. Thus $RU((\bar{v}_2 - \underline{v}_2)q_2) = \underline{v}_2$. For a take it or leave it price $\bar{v}_2$, the buyer's utility is 0 and the seller's revenue is $\bar{v}_2 q_2$. Thus $RU(0) = \bar{v}_2 q_2$. Given RU, we can define the continuation revenue $CR_1 = \hat{RU}$. The continuation revenue CR_1 consists of two linear pieces. The first piece connects the two points in the graph of RU. The slope may be positive or negative. The second piece has slope of negative one. See Figure 5.

We now write the problem in the first period. Recall that P_1 is set of optimal monopoly prices that maximize the stage revenue for selling the first product. There are three possibilities, $P_1 = \{\underline{v}_1\}, P_1 = \{\bar{v}_1\}$, or $P_1 = \{\underline{v}_1, \bar{v}_1\}$. In either case, $\bar{v}_1 \geq \underline{p}_1$, where $\underline{p}_1$ is the lowest monopoly price. Thus, by Lemma 5, the allocation probability of $\bar{v}_1$ in the first period is equal to one, $a_1(\bar{v}_1) = 1$. Thus, to specify the mechanism in the first period, we need to only specify the

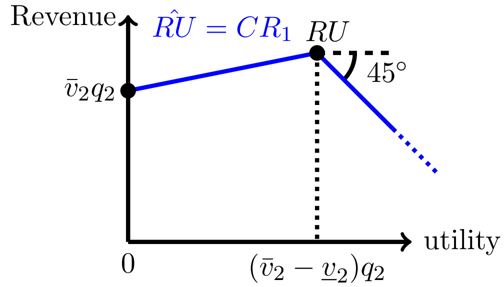

Figure 5: The functions RU and $\hat{RU}$ for selling a product with two possible values $\underline{v}$ and $\bar{v}$, and distribution f_2. The slope of $\hat{RU}$ from 0 to $(\bar{v}_2 - \underline{v}_2)q_2$ may be positive or negative.

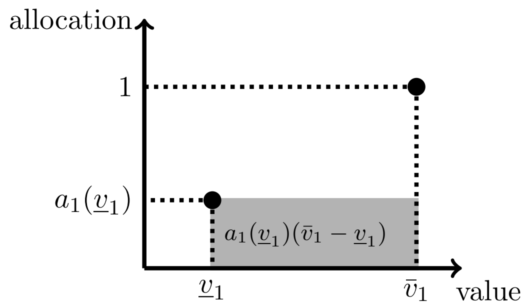

Figure 6: The allocation probabilities in the first period. The shaded region is the difference in the utilities of the two values.

allocation probability of the low value, $a_1(\underline{v}_1)$, and the utility of the low value. The utilities are $u_1(\underline{v}_1)$ and $u_1(\bar{v}_1) = u_1(\underline{v}_1) + a_1(\underline{v}_1)(\bar{v}_1 - \underline{v}_1)$. See Figure 6. Substituting $a = a_1(\underline{v}_1)$ and $u = u_1(\underline{v}_1)$ to simplify notation, the problem is

$$\max_{a,u}(1-q_1)\Big(a\underline{v}_1 + CR_1(u)\Big) + q_1\Big(\bar{v}_1 + CR_1(u + a(\bar{v}_1 - \underline{v}_1))\Big) \tag{17}$$

subject to $0 \leq a \leq 1$. Solving the problem above involves considering several cases, and is rather tedious. In order to simplify the analysis, we further assume that the supports are identical, $V_1 = V_2 = \{\underline{v}, \bar{v}\}$.

The solution to (17) identifies the optimal separable mechanism, which by Proposition 1 is optimal among all mechanisms. However, it is possible that there exist other mechanisms, and in particular static mechanisms, that are optimal as well. To study this possibility, once the solution to (17) is obtained, we verify whether there exists a static mechanism that obtains the same revenue. If so, the static mechanism is optimal as well. If no static mechanism has revenue equal to that of the solution (17), then static mechanisms are sub-optimal.

The following proposition identifies optimal mechanisms. There are five cases. In four cases, a static mechanism is optimal. The four static mechanisms are simple. Three of them sell the products separately. That is, each product has a price, and the buyer can buy each product by paying its price. The fourth static mechanism is a bundling mechanism that only offers the two products as a bundle. The fifth mechanism identified by the proposition is separable. We will show later that whenever the fifth mechanism is optimal, all static mechanisms are sub-optimal. Thus for future reference, the proposition identifies conditions under which the fifth mechanism is optimal. The optimal separable mechanism sells the second product via a take it or leave it price that depends on the reported value in the first period.

To state the proposition, define the price $p^* = \underline{v} - (1 - q_2)(\bar{v} - \underline{v})$. The price p^* is constructed such that the expected utility of the buyer from being offered a take it or leave it price p^* for the second product is equal to $\bar{v} - \underline{v}$.

Proposition 2. *Assume that $k = 2$ and $V_1 = V_2 = \{\underline{v}, \bar{v}\}$. At least one of the following five mechanisms is optimal.*

1. *Sell each product separately at price $\underline{v}$.*

2. *Sell each product separately at price $\bar{v}$.*

3. *Sell each product separately, at price $\bar{v}$ for the first product and $\underline{v}$ for the second product.*

4. *Sell only the grand bundle at price $\underline{v} + \bar{v}$.*

5. *In the first period, the buyer reports v_1, receives the first product, and transfers v_1. The second product is sold via a take it or leave it price $\bar{v}$ if $v_1 = \underline{v}$, and p^* if $v_1 = \bar{v}$.*

In addition, the fifth mechanism is the unique optimal separable mechanism if and only if $q_1 < \underline{v}/\bar{v}$ and $q_2 > \underline{v}(1 - q_1)/(\bar{v} - q_1\underline{v})$.

Thus, to identify optimal mechanisms, one needs only to compare the revenue of the above five mechanisms. The revenue of each mechanism can be written in closed form. For the first four, revenue is simply the prices times the probability of purchase. For the fifth mechanism, revenue is given by (17), substituting $a = 1$ and $u = 0$.

It is illuminating to compare the optimal mechanism Proposition 2 to optimal static mechanisms. Optimal static mechanisms also take five possible forms. The first four are identical to one identified in Proposition 2. The fifth mechanism is to sell the products separately, at price $\underline{v}$ for the first product and $\bar{v}$ for the second product.[3]

5 Optimality of Static Mechanisms with Two Products

We now apply our analysis to the simple case of selling two products to study whether static mechanisms can be optimal. First, we provide necessary and sufficient conditions for optimality of static mechanisms with two products and two values. Second, we provide sufficient conditions for sub-optimality of static mechanisms with two products and any number of values. We obtain the sufficient conditions using the structural properties of the optimal mechanisms developed in Section 4.1 and Section 4.2.

5.1 Tight Conditions for Two Products and Two Values

We first provide necessary and sufficient conditions for optimality of static mechanisms with two products and when $V_1 = V_2 = \{\underline{v}, \bar{v}\}$. The proposition below specifies three conditions that are together necessary and sufficient for sub-optimality of static mechanisms. The argument follows the analysis of Section 4.3. Recall that four out of five optimal mechanisms in Proposition 2 are static. Thus, if static mechanisms are sub-optimal, the fifth mechanism identified in Proposition 2 must be optimal. Optimality of the fifth mechanism provides the two conditions in the proposition below. As in Section 4.3, let $q_1 = f_1(\bar{v})$ and $q_2 = f_2(\bar{v})$.

Proposition 3. *Assume that $k = 2$ and $V_1 = V_2 = \{\underline{v}, \bar{v}\}$. Any static mechanism is sub-optimal if and only if $q_1 < \underline{v}/\bar{v}$ and $q_2 > \underline{v}(1 - q_1)/(\bar{v} - q_1\underline{v})$.*

There are two ways to interpret this result. First, as discussed throughout the paper, it provides conditions under which dynamic screening is valuable after taking away the seller's ability to charge advance payments. Second, it can be used to identify optimal static screening mechanisms. In

[3] These five cases cover all the possible cases of selling the two products separately or as a bundle. There are four possibilities for selling the products separately, where the prices are either $\underline{v}$ or $\bar{v}$. There are three possibilities for selling the two products as a bundle, with prices equal to $2\underline{v}$, $\underline{v} + \bar{v}$, and $2\bar{v}$. However, selling the bundle at price $2\underline{v}$ or $2\bar{v}$ is equivalent to selling each product separately at price $\underline{v}$ or $\bar{v}$, respectively.

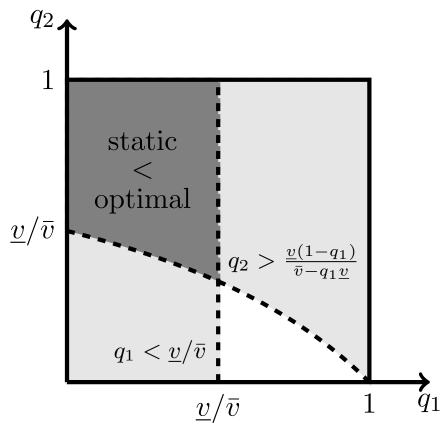

Figure 7: The region shaded dark is the set of (q_1, q_2) for which static mechanisms are sub-optimal.

particular, if $q_1 \geq \underline{v}/\bar{v}$ or $q_2 \leq \underline{v}(1-q_1)/(\bar{v}-q_1\underline{v})$, then one of the four static mechanisms identified in Proposition 2 is optimal among all static mechanisms. This suggests that our approach may be more generally useful for solving, either exactly or approximately, the notoriously difficult problem of selling multiple products using static mechanisms. In principle, the case of two products and two values can be solved in a static setting via case analysis, but such analyses are typically tedious. For instance, Armstrong and Rochet (1999) solve a screening problem with four types. They consider all possible ways to relax subsets of the incentive constraints, and identify conditions under which the solution to each relaxation satisfies all the constraints, and therefore is optimal. In comparison, our recursive formulation allows the incentive constraints in the two periods to be separated and solved using standard tools.

The set of parameters q_1 and q_2 for which static mechanisms are sub-optimal is drawn in Figure 7. Notice that the constraint $q_1 < \underline{v}/\bar{v}$, or equivalently $\bar{v}q_1 < \underline{v}$, states that the unique optimal monopoly price for the first product is $\underline{v}$. The constraint $q_2 > \underline{v}(1-q_1)/(\bar{v}-q_1\underline{v})$ is more complex. Nevertheless, since $\underline{v}(1-q_1)/(\bar{v}-q_1\underline{v})$ is decreasing in q_1, it is sufficient that $q_2 > \underline{v}/\bar{v}$, or equivalently $\bar{v}q_2 > \underline{v}$. That is, the unique optimal monopoly price for the second product is $\bar{v}$. To summarize, static mechanisms are sub-optimal if (but not only if) the optimal monopoly price for product 1 is strictly less than the optimal monopoly price for product 2. In the next section, we show that this statement generalizes to any number of values for the two products.

5.2 Sufficient Conditions for Two Products and Any Number of Values

In this section we identify sufficient conditions for sub-optimality of static mechanisms with two products with identical supports $V_1 = V_2$, but we allow any number of values. In particular, we show that static mechanisms are sub-optimal if the first product has lower monopoly prices than the second product, partially generalizing Proposition 3 to any number of values. To do so we use the structural properties of optimal mechanisms developed in Section 4.

We start with defining the condition. Recall that for each i, P_i is the set of optimal monopoly prices for selling product i, and that $\bar{p}_i$ and $\underline{p}_i$ are largest and smallest such prices. We say that product 1 has *lower monopoly prices* than product 2 if $\underline{p}_1 < \bar{p}_2$. If the optimal monopoly prices are unique, the condition simply demands that the monopoly price for product 1 is lower than the monopoly price for product 2. Notice that if $V_1 = V_2 = \{\underline{v}, \bar{v}\}$, then product 1 has lower monopoly prices than product j if and only if $P_1 = \{\underline{v}\}$ and $P_2 = \{\bar{v}\}$. Thus, the condition is weaker than the conditions of Proposition 3, but allows for a generalization to any number of products and values.

To explain the proof, suppose for simplicity that the optimal monopoly prices p_1 and p_2 are unique. Assume that $p_1 < p_2$. Assume for contradiction that a static mechanism (a, t) is optimal. By Proposition 1, its induced separable mechanism (a^{ISP}, t^{ISP}) must also be optimal. Lemma 5 demands that all interim types weakly above p_1 receive product 1 with probability one in the optimal separable mechanism. Since the probability of allocation of product 1 is positive for all such interim types, the promised utility of these types must be strictly increasing. Recall that Corollary 1 states that if the optimal mechanism in period 2 does not change as the promised utility increases, then the possibility of fulfilling utility promises by increasing allocation must be exhausted. In particular, type $(p_1, \underline{v})$ must be getting both products with probability one. Individual rationality of the static mechanism then demands that the price of the bundle of products must be at most $p_1 + \underline{v}$. Therefore, the utility of type $(\underline{v}, p_2)$ must be strictly positive. However, we show that following a history of a report $v_1 = \underline{v}$, the product in period 2 is sold at prices that are no less than p_2. Therefore, the utility of type $(\underline{v}, p_2)$ must be zero, which contradicts the earlier conclusion that the utility must be positive.

Proposition 4. *Assume that $k = 2$ and $V_1 = V_2$. Any static mechanism is sub-optimal if product 1 has lower monopoly prices than product 2.*

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Electronic Companion

A The Revelation Principle

In this section we formally establish the revelation principle and discuss alternative formulations.

We start with defining the class of all mechanisms. A mechanism is identified by a set of messages M_i that the buyer can send in each period i, and a pair of functions $a : M_1 \times \ldots \times M_k \to [0,1]^k$ and $t : M_1 \times \ldots \times M_k \to \mathbb{R}$ that specify the probability of allocating each product and the transfer as a function of the messages sent (it is possible to further allow for multiple rounds of messages in each period, but we exclude that possibility to keep the notation simple). A strategy σ of the buyer consists of a function $\sigma_i : \Theta^i \times M_1 \times \ldots M_{i-1} \to M_i$ for each i that maps the history at period i to a message. The history includes the interim type of the buyer and the reports that were made in the previous periods. Abusing notation let $a(\sigma(v))$ and $t(\sigma(v))$ denote the probabilities of allocation and the expected transfer when the buyer follows strategy σ. A strategy σ is optimal for the buyer in a mechanism if it maximizes the ex ante expected utility of the buyer

$$\mathbf{E}\left[v \cdot a(\sigma(v)) - t(\sigma(v))\right] \tag{18}$$

over all strategies. The expectation is taken over v since $a(\sigma(v))$ and $t(\sigma(v))$ already incorporate the randomness in buyer's strategy.

A direct mechanism defined in Section 2 is a mechanism in which for each i, the set of messages M_i is equal to the set of interim types Θ^i. The notion of periodic incentive compatibility defined in (1) is equivalent to requiring that the truth-telling strategy is optimal for the buyer.

The revelation principle states that for any mechanism (M, a, t) with optimal strategy σ, there exists a direct mechanism (a', t') in which truth-telling is optimal (equivalently, (a', t') is PIC) and implements the same outcome as the original mechanism, that is, $a(\sigma(v)) = a'(v^{1:1}, \ldots, v^{1:k})$ and $t(\sigma(v)) = t'(v^{1:1}, \ldots, v^{1:k})$. This can be achieved by simply defining

$$a'_i(\theta^1, \ldots, \theta^i) = \mathbf{E}\left[a_i\Big(\sigma_1(\theta^1), \ldots, \sigma_i(\theta^i, m_1, \ldots, m_{i-1})\Big)\right],$$

where the expectation is taken over the random messages $m_1, \ldots, m_{i-1}$ created by strategy σ. The transfer rule t' is defined similarly.

It is possible to formulate the revelation principle differently so that the buyer reports only her value v_i in each period i. Even though such mechanisms appear simpler since they do not require re-reporting, the incentive constraints for such a mechanism must account for multi-shot deviations. This is in contrast with the formulation in Section 2 where the PIC constraints only concern one-shot deviations. Formally, let us define "type-2" direct mechanisms as follows. Any such mechanism is defined by allocation rules $a_i : \Theta^k \to [0, 1]$ for all i and a transfer rule $t : \Theta^k \to \mathbb{R}$. The interpretation is that in each period i the buyer reports her value v_i and at the end of the last period receives each product i with probability $a_i(v_1, \ldots, v_k)$ and transfers $t(v_1, \ldots, v_k)$ to the mechanism. The revelation principle holds. Namely, for any (M, a, t) with optimal strategy σ, there exists a type-2 direct mechanism (a', t') in which truth-telling is optimal and implements the same outcome as the original mechanism, that is, $a(\sigma(v)) = a'(v)$ and $t(\sigma(v)) = t'(v)$. Nevertheless, the incentive compatibility constraint for a type-2 direct mechanism does not simplify much beyond its definition in (18). Namely, in a type-2 direct mechanism (a, t), a strategy σ in which the buyer reports her value truthfully in each period i regardless of her interim type and past reports satisfies

$$\mathbf{E}\left[v \cdot a(\sigma(v)) - t(\sigma(v))\right] \geq \mathbf{E}\left[v \cdot a(\sigma'(v)) - t(\sigma'(v))\right] \tag{19}$$

for any strategy σ'. It is possible to restrict to one-shot deviations on path by requiring the buyer to report her value in period i truthfully if she also reported truthfully in past periods as well, but following a non-truth-telling history, the buyer may find it optimal to be not truthful in period i.

Given the recursive nature of the PIC constraint, we allow for direct mechanisms with re-reporting instead of type-2 direct mechanisms. An additional advantage is that our formulation makes the fact that static mechanisms are a special case transparent: static mechanisms are ones in which the allocation and transfer only depend on the last period reports.

B An Equivalence of Ex post IR Notions

We describe mechanisms in Section 2 by identifying the marginal probability a_i of allocation of products and the expected transfer t, without discussing how the mechanism possibly correlates these decisions. Since the buyer maximizes her expected utility, the correlation is irrelevant for incentive constraints. However, to make sure that the ex post utility of the buyer is non-negative even after the random choices of the mechanism, we here explicitly discuss a way to correlate the allocation of the products and the transfer, assuming that the mechanism satisfies (2). Since the correlation does not affect incentive constraints, we here fix an ex post type v.

Thus fix v and let $u = v \cdot a - t \geq 0$ be the expected utility (over the random choices of the mechanism). Let n be the number of products i with positive probability of allocation $a_i > 0$. Assume $n \geq 1$, since otherwise if $n = 0$ the allocation is zero and no randomization is required. Allocate each product i independently with probability a_i. For each product i with $a_i > 0$, the buyer transfers $v_i - u/(a_i n)$ if it receives the product, and zero otherwise. Note that the expected total transfer is indeed t, since $\sum_{i:a_i>0}(v_i - u/(a_i n))a_i = v \cdot a - u = t$. Note also that the ex post utility (after the randomization of the mechanism) is non-negative, since for each product i that the buyer receives, she pays $v_i - u/(a_i n)$ which is lower than the value of the product.

C Proofs from Section 4

C.1 Proof of Lemma 3

Proof. Necessity of monotonicity of a and the fact that (a, t) for t defined in the lemma satisfies all incentive constraints is standard. We only verify the optimality of t.

In our model with a finite set of values, the allocation probabilities of types in V does not pin down the transfers. That is, for a given allocation rule a, there may be multiple transfer rules t such that the mechanism (a, t) is incentive compatible. Nevertheless, the transfer rule is pinned down (up to a constant which is the utility of the lowest type) once the mechanism is extended to specify the allocation and transfer of all values in $[\underline{v}, \bar{v}]$. For instance, if $V = \{\underline{v}, \bar{v}\}$, then the allocation rule a where $a(\underline{v}) = 0$ and $a(\bar{v}) = 1$ can be implemented by offering the product at any price p between $\underline{v}$ and $\bar{v}$. The extended allocation rule is $a(v) = 0$ if $v < p$ and $a(v) = 1$ if $v \geq p$. The unique transfer rule that makes the extended allocation rule incentive compatible is $t(v) = 0$ if $v < p$ and $t(v) = p$ if $v \geq p$.

Thus assume without loss of generality that (a, t) is defined over $[\underline{v}, \bar{v}]$. For such a mechanism,

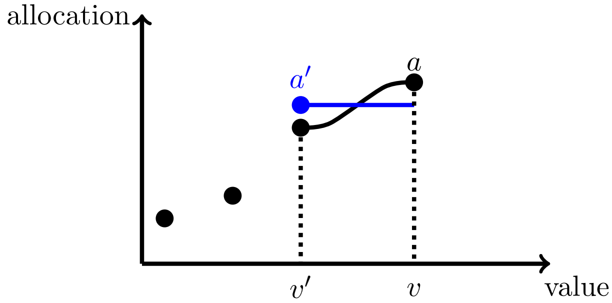

Figure 8: The construction of a' in the proof of Lemma 3.

the transfer rule is defined from the allocation rule as follows,

$$t(v) = va(v) - (\int_{\underline{v}}^{v} a(z)dz) - u(\underline{v}).$$

We show that for (a, t) to be optimal, a must be constant on $[v', v)$ for any adjacent pair of values $v' < v$. Assume for contradiction that this is not the case. See Figure 8.

Construct a' to be constant between v' and v, and such that the areas between v and v' is the same under a and a'. See Figure 8. Define t' as follows

$$t'(v) = va'(v) - (\int_{\underline{v}}^{v} a'(z)dz) - u(\underline{v}).$$

Note that the mechanism (a', t') satisfies the incentive constraints, and gives all types $v \in V$ the same utility as in (a, t). Thus, (a', t') also satisfies the promise-keeping constraint. Finally, since a' has higher surplus than a, the mechanism (a, t) cannot be optimal. □

C.2 Proof of Lemma 4

Proof. First note for future reference that for each u,

$$\begin{aligned} u + RU(u) &= \sum_{v \geq p(u)} (v - p(u))f(v) + \sum_{v \geq p(u)} p(u)f(v) \\ &= \sum_{v \geq p(u)} vf(v). \end{aligned} \tag{20}$$

By Lemma 3, the transfer of a type v in an optimal mechanism is

$$-u(\underline{v}) + va(v) - \sum_{v' \leq v} (v - v')\Delta a(v') = -u(\underline{v}) + \sum_{v' \leq v} v' \Delta a(v').$$

Thus the expected revenue is

$$\begin{aligned} & -u(\underline{v}) + \sum_{v} f(v)\Big(\sum_{v' \leq v} v'\Delta a(v')\Big) \\ = & -u(\underline{v}) + \sum_{v} \Delta a(v)\Big(v \sum_{v' \geq v} f(v')\Big). \end{aligned} \tag{21}$$

And the expected utility is

$$u(\underline{v}) + \sum_{v} f(v) \sum_{v' \leq v} (v - v')\Delta a(v') = u(\underline{v}) + \sum_{v} \Delta a(v) \sum_{v' \geq v} (v' - v) f(v').$$

Thus the promise-keeping constraint is

$$\sum_{v} \Delta a(v) \sum_{v' \geq v} (v' - v) f(v') = \mathsf{PU} - u(\underline{v}). \tag{22}$$

As a result, the problem is to maximize (21), subject to (22), the individual rationality constraint $u(\underline{v}) \geq 0$, the monotonicity constraint $\Delta a(v) \geq 0$, and $\sum_v \Delta a(v) \leq 1$.

Now consider the following change of variables. Let $u = p^{-1}(v)$ and $\mu(u) = \Delta a(v)$. By definition we have $RU(u) = v \sum_{v' \geq v} f(v')$ and $u = \sum_{v' \geq v}(v' - v) f(v') = \sum_{v' > v}(v' - v) f(v')$. The problem becomes to maximize

$$-u(\underline{v}) + \sum_{u} \mu(u) RU(u), \tag{23}$$

subject to the promise-keeping constraint

$$\sum_{u} \mu(u) u = \mathsf{PU} - u(\underline{v}), \tag{24}$$

and the additional constraints that $u(\underline{v}) \geq 0$, $\mu(u) \geq 0$, and $\sum_u \mu(u) \leq 1$. We next show that the optimal solution must satisfy $\sum_u \mu(u) = 1$, and $u(\underline{v}) = 0$ unless $\mu(\bar{u}) = 1$.

We first argue that $\sum_u \mu(u) = 1$. Otherwise, it is possible to increase $\mu(0)$ without violating feasibility, since such a change does not affect the left hand side of the promise keeping constraint (24). Since $RU(0) > 0$ (posting the highest value in V as a take it or leave it price gives a strictly positive revenue), this change improves the objective and thus μ is not optimal.

We now argue that $u(\underline{v}) = 0$ unless $\mu(\bar{u}) = 1$. Assume for contradiction that $u(\underline{v}) > 0$ and $\mu(\bar{u}) < 1$. Since $\sum_u \mu(u) = 1$ as argued above, there must exists $u \neq \bar{u}$ such that $\mu(u) > 0$. We show that for δ small enough, it is feasible to decrease $\mu(u)$ by δ and increase $\mu(\bar{u})$ by δ. Notice that this change respects the two constraints $\mu(u) \geq 0$ and $\sum_u \mu(u) \leq 1$. The change in expected utility is $-\delta u + \delta \bar{u}$. Since $u(\underline{v}) > 0$, for small enough δ it is possible to add $\delta u - \delta \bar{u} < 0$ to $u(\underline{v})$ such that the promise-keeping constraint (24) stays satisfied as well. Now consider the change in objective. It is

$$\delta(\bar{u} - u) + \delta(RU(\bar{u}) - RU(u)) = \delta(\sum_{v \geq p(\bar{u})} vf(v) - \sum_{v \geq p(u)} vf(v)) > 0.$$

where the equality followed from (20), and the inequality followed since $p(\bar{u}) < p(u)$.

We now complete the proof. First, consider $\mathsf{PU} > \bar{u}$. For the promised utility constraint to be satisfied, we must have $u(\underline{v}) > 0$. Our discussion above shows that we must have $\mu(\bar{u}) = 1$.

Second, consider $\mathsf{PU} \leq \bar{u}$. For the promised utility constraint to be satisfied, we must have $\mu(\bar{u}) \leq 1$, and therefore $u(\underline{v}) = 0$ as argued above. Thus, the problem is to maximize

$$\sum_u \mu(u) RU(u), \tag{25}$$

subject to

$$\sum_u \mu(u) u = \mathsf{PU}, \tag{26}$$

$\mu(u) \geq 0$, and $\sum_u \mu(u) = 1$. In words, the objective is to maximize the expectation of RU over all distributions μ with expectation PU. The optimal value is equal to the concavification of RU at PU. $\square$

C.3 Proof of Corollary 1

Proof. Consider two promised utilities $\mathsf{PU}, \mathsf{PU}' \le \mathbf{E}[v_k]$. If the two promised utilities are in different concavified regions, that $p(\ell(\mathsf{PU})) \neq p(\ell(\mathsf{PU}))$ or $p(h(\mathsf{PU}')) \neq p(h(\mathsf{PU}'))$, then the allocation rules are different since they are the results of randomization over different prices. If $p(\ell(\mathsf{PU})) = p(\ell(\mathsf{PU}))$ and $p(h(\mathsf{PU}')) = p(h(\mathsf{PU}'))$, the the allocation rules are different since the probability of $p(\ell(\mathsf{PU}))$ is strictly decreasing in PU, and the probability of $p(\ell(\mathsf{PU}))$ is strictly increasing in PU'. $\square$

C.4 Proof of Lemma 5

We first establish a property of the continuation revenue problem. Namely, increasing the utility promise by δ may decrease the continuation revenue by at most δ. The is because the mechanism has the option to pay the extra promised utility back to the buyer with money.

Lemma 6. *For any i, and $\mathsf{PU} < \mathsf{PU}'$, the continuation revenue function satisfies $CR_i(\mathsf{PU}') - CR_i(\mathsf{PU}) \ge \mathsf{PU} - \mathsf{PU}'$.*

Proof. We prove the lemma inductively. Consider the last period and an optimal mechanism $(\mathcal{A}^{\mathsf{PU}}, \mathcal{U}^{\mathsf{PU}})$. Consider an alternative mechanism $(\mathcal{A}^{\mathsf{PU}}, \mathcal{U}^{\mathsf{PU}} + \mathsf{PU}' - \mathsf{PU})$. Notice that the alternative mechanism is feasible for the utility promise PU', and obtains a revenue that is equal to the revenue of mechanism $(\mathcal{A}^{\mathsf{PU}}, \mathcal{U}^{\mathsf{PU}})$ minus $\mathsf{PU}' - \mathsf{PU}$. Thus we must have $CR_k(\mathsf{PU}') - CR_k(\mathsf{PU}) \ge \mathsf{PU} - \mathsf{PU}'$.

Now consider a period i, and assume that $CR_i(\mathsf{PU}') - CR_i(\mathsf{PU}) \ge \mathsf{PU} - \mathsf{PU}'$. We show that the same holds for CR_{i-1}. The proof is similar to above. Giving all types an extra utility $\mathsf{PU}' - \mathsf{PU}$ results in a change in objective value that is equal to

$$\mathbf{E}\left[CR_i(\mathcal{U}^{\mathsf{PU}}(v) + \mathsf{PU} - \mathsf{PU}') - CR_i(\mathcal{U}^{\mathsf{PU}}(v))\right] \ge \mathsf{PU} - \mathsf{PU}',$$

by the induction hypothesis. Thus, $CR_{i-1}(\mathsf{PU}') - CR_{i-1}(\mathsf{PU}) \ge \mathsf{PU} - \mathsf{PU}'$. $\square$

We now prove Lemma 5.

Proof of Lemma 5. Consider any two mechanisms (a, t) and (a', t'), each satisfying the payment equation of Lemma 3 with the same utility for the lowest type $\underline{v}$. Assume further that $a'(v) - a(v) \ge 0$ for all v, which in turn implies that $u'(v) - u(v) \ge 0$. The difference between the objective values of the two mechanisms are

$$
\begin{aligned}
&\sum_v \Big(f(v)(va'(v) + CR(u'(v)))\Big) - \sum_v \Big(f(v)(va(v) + CR(u(v)))\Big) \\
&\geq \sum_v f(v)\Big(v(a'(v) - a(v)) - (u'(v) - u(v))\Big) \\
&= \sum_v f(v)(t'(v) - t(v)), \qquad (27)
\end{aligned}
$$

where the inequality followed from Lemma 6.

To prove the first statement of the lemma, consider an optimal mechanism (a, t). Consider an alternative mechanism (a', t') where a' is identical to a except that $a'(v) = 1$ for all $v \geq \underline{p}$. Notice that $a'(v) - a(v) \geq 0$ for all v. Thus, by (27), the objective value of mechanism (a', t') minus the objective value of the mechanism (a, t) is at least

$$\sum_v f(v)(t'(v) - t(v)) \geq 0.$$

Therefore, the mechanism (a', t') must also be optimal.

To prove the second statement, consider an optimal mechanism (a, t). Assume for contradiction that $a(v) < 1$ for some $v \geq \bar{p}$. Consider an alternative mechanism (a', t') where a' is identical to a except that $a'(v) = 1$ for all $v \geq \bar{p}$. Notice that $a'(v) - a(v) \geq 0$. Thus, by (27), the objective value of mechanism (a', t') minus the objective value of the mechanism (a, t) is at least

$$\sum_v f(v)(t'(v) - t(v)) > 0.$$

Therefore, the mechanism (a, t) cannot be optimal. □

C.5 Proof of Proposition 2

Proof. We prove the proposition by considering how the objective value changes in a and u. Thus let

$$R' := \frac{RU((\bar{v} - \underline{v})q_2) - RU(0)}{(\bar{v} - \underline{v})q_2}$$

be the slope of the first linear piece of the continuation revenue function (see Figure 5). Note that

$$R' = \frac{\underline{v} - \bar{v}q_2}{(\bar{v} - \underline{v})q_2} \geq \frac{\underline{v}q_2 - \bar{v}q_2}{(\bar{v} - \underline{v})q_2} = -1.$$

The subderivatives of CR_1 are either R' or -1. Let CR_1' be the derivative of CR_1 whenever the derivative exists, and the subderivatives of CR_1 otherwise.

For future reference, we calculate the derivative (or subderivatives) of the objective with respect to a and u. The subderivative with respect to a is

$$(1 - q_1)\underline{v} + q_1(\bar{v} - \underline{v})CR_1'(u + a(\bar{v}_1 - \underline{v}_1)). \tag{28}$$

and the subderivative with respect to u is

$$(1 - q_1)CR_1'(u) + q_1 CR_1'(u + a(\bar{v}_1 - \underline{v}_1)). \tag{29}$$

The right derivative of CR_1 at any value at or above $q_2(\bar{v} - \underline{v})$ is -1. Thus, if $u \geq q_2(\bar{v} - \underline{v})$, the right derivative of the objective with respect to u is negative. Therefore, any optimal solution must satisfy $u \leq q_2(\bar{v} - \underline{v})$.

Since $R' \geq -1$, the two possible subderivatives of the objective with respect to a satisfy $(1 - q_1)\underline{v} - q_1(\bar{v} - \underline{v}) \leq (1 - q_1)\underline{v} + q_1(\bar{v} - \underline{v})R'$. Consider the three possible cases for the signs of these two subderivatives.

1. $(1 - q_1)\underline{v} + q_1(\bar{v} - \underline{v})R' \leq 0$. Both subderivatives of the objective with respect to a are non-positive. As a result, it is optimal to set a as small as possible, $a = 0$. Notice also that in this case, $R' \leq 0$ and thus the subdrivatives of the objective with respect to u are also non-positive. Thus it is optimal to set $u = 0$. The optimal revenue is

$$(1 - q_1)(CR_1(0)) + q_1(\bar{v} + CR_1(0)) = q_1\bar{v} + q_2\bar{v}.$$

The optimal revenue is equal to the revenue of selling each product separately at price $\bar{v}$.

2. $(1 - q_1)\underline{v} - q_1(\bar{v} - \underline{v}) \leq 0$ and $(1 - q_1)\underline{v} + q_1(\bar{v} - \underline{v})R' \geq 0$. In this case, it is optimal to set

$u + a(\bar{v} - \underline{v}) = q_2(\bar{v} - \underline{v})$. Otherwise, if $u + a(\bar{v} - \underline{v}) > q_2(\bar{v} - \underline{v})$, a can be decreased without decreasing the objective. Similarly, if $u + a(\bar{v} - \underline{v}) < q_2(\bar{v} - \underline{v})$, a can be increased without decreasing the objective. Now consider increasing u by ϵ, and decreasing a by $\epsilon/(\bar{v} - \underline{v})$. The change in the objective value is

$$\epsilon(1 - q_1)(\frac{-\underline{v}}{\bar{v} - \underline{v}} + R').$$

If the expression above is positive, it is optimal to set $a = 0$ and $u = q_2(\bar{v} - \underline{v})$. Otherwise, it is optimal to set $a = q_2$ and $u = 0$. In the first case, the optimal revenue is

$$(1 - q_1)(0 + CR(q_2(\bar{v} - \underline{v}))) + q_1(\bar{v} + CR_1(q_2(\bar{v} - \underline{v}))) = q_1\bar{v} + \underline{v}.$$

Thus, the optimal revenue is equal to the revenue of selling the first product at price $\bar{v}$ and the second product at price $\underline{v}$. In the second case, the optimal revenue is

$$\begin{aligned}(1 - q_1)(q_2\underline{v} + CR_1(0)) + q_1(\bar{v} + CR_1(q_2(\bar{v} - \underline{v}))) &= (1 - q_1)(q_2\underline{v} + q_2\bar{v}) + q_1(\bar{v} + \underline{v}) \\ &= (\underline{v} + \bar{v})(q_1 + q_2 - q_1q_2).\end{aligned}$$

The optimal revenue is equal to the revenue of selling only the bundle of products at price $\underline{v} + \bar{v}$.

3. $(1 - q_1)\underline{v} - q_1(\bar{v} - \underline{v}) \geq 0$. Both subderivatives of the objective with respect to a are non-negative. Thus it is optimal to set a as large as possible, $a = 1$. To identify u, consider two cases. If $(1 - q_1)R' - q_1 \leq 0$, then it is optimal to set u as small as possible, $u = 0$. If $(1 - q_1)R' - q_1 \geq 0$, it is optimal to set u as large as possible, $u = q_2(\bar{v} - \underline{v})$. In the second case, the optimal revenue is

$$\begin{aligned}&(1 - q_1)(\underline{v} + CR(q_2(\bar{v} - \underline{v}))) + q_1(\bar{v} + CR_1((\bar{v} - \underline{v})(1 + q_2))) \\ &= (1 - q_1)(\underline{v} + \underline{v}) + q_1(\bar{v} + \underline{v} - (\bar{v} - \underline{v})) \\ &= 2\underline{v}.\end{aligned}$$

The optimal revenue is equal to the revenue of selling each product separately at price $\underline{v}$.

To prove the second statement, notice that $q_1 < \underline{v}/\bar{v}$ is equivalent to $(1-q_1)\underline{v} - q_1(\bar{v}-\underline{v}) > 0$ and $q_2 > \underline{v}(1-q_1)/(\bar{v}-q_1\underline{v})$ is equivalent to $(1-q_1)R' - q_1 < 0$. If $(1-q_1)\underline{v} - q_1(\bar{v}-\underline{v}) > 0$ and $(1-q_1)R' - q_1 < 0$, then following the analysis of the third case, the unique optimal mechanism satisfies $a = 1$ and $u = 0$. Conversely, if either $(1-q_1)\underline{v} - q_1(\bar{v}-\underline{v}) \leq 0$ or $(1-q_1)R' - q_1 \leq 0$, then the analysis above shows that a mechanism other than $a = 1$ and $u = 0$ is optimal. □

D Proofs from Section 5

D.1 Proof of Proposition 3

Proof. Assume first that the two conditions of the proposition hold. We show that any static mechanism is sub-optimal.

Assume for contradiction that a static mechanism is optimal. We first show that there must exist an optimal static mechanism (a^{ST}, t^{ST}) that further satisfies the following property: If changing v_1 does not affect the allocation of product 1, then it also does not affect the allocation of product 2. That is,

$$\text{if } a_1^{ST}(v_1, v_2) = a_1^{ST}(v_1', v_2) \text{ then } a_2^{ST}(v_1, v_2) = a_2^{ST}(v_1', v_2). \tag{30}$$

To see this, assume that $a_1(v_1, v_2) = a_1(v_1', v_2)$ and consider the incentive constraint of type $v = (v_1, v_2)$ for reporting (v_1', v_2),

$$v \cdot a(v) - t(v) \geq v \cdot a(v_1', v_2) - t(v_1', v_2).$$

Since $a_1(v_1, v_2) = a_1(v_1', v_2)$, the above inequality simplifies to

$$v_2 a_2(v) - t(v) \geq v_2 a_2(v_1', v_2) - t(v_1', v_2).$$

Similarly, the incentive constraint of type (v_1', v_2) for reporting v is

$$v_2 a_2(v_1', v_2) - t(v_1', v_2) \geq v_2 a_2(v) - t(v).$$

Thus, both incentive constraints must hold with equality. That is, each type is indifferent between her own allocation and the allocation of the other type. Therefore, we can assign both types the same allocation and transfer without changing revenue.

By Proposition 1, the induced separable mechanism of the static mechanism (a^{ST}, t^{ST}) is optimal. By Proposition 2, the induced separable mechanism (a^{ISP}, t^{ISP}) must be equal to the fifth mechanism identified in Proposition 2, that is, $a_1^{ISP}(v_1) = 1$ for all v_1, $a_2^{ISP}(v) = 1$ if $v \neq (\underline{v}, \underline{v})$, and $a_2^{ISP}(\underline{v}, \underline{v}) = 0$. Therefore, the allocation rule of the static mechanism must be as shown below.

v_1	v_2	a_1	a_2
$\underline{v}$	$\underline{v}$	1	0
$\underline{v}$	$\bar{v}$	1	1
$\bar{v}$	$\underline{v}$	1	1
$\bar{v}$	$\bar{v}$	1	1

Thus property in (30) is violated since $a_1^{ST}(\underline{v}, \underline{v}) = a_1^{ST}(\bar{v}, \underline{v})$ but $a_2^{ST}(\underline{v}, \underline{v}) \neq a_2^{ST}(\bar{v}, \underline{v})$.

Now assume that all static mechanisms are sub-optimal. Sub-optimality of all static mechanisms imply that in particular, the four static mechanisms of Proposition 2 must be sub-optimal. By Proposition 2, the two conditions of the proposition must hold. □

D.2 Proof of Proposition 4

Proof. Assume for contradiction that there exists a static IC mechanism (a, t) that is optimal.

We can assume without loss of generality that if changing v_1 does not affect the allocation of product 1, then it also does not affect the allocation of product 2. That is, if $a_1(v_1, v_2) = a_1(v_1', v_2)$ then $a_2(v_1, v_2) = a_2(v_1', v_2)$. To see this, assume that $a_1(v_1, v_2) = a_1(v_1', v_2)$ and consider the incentive constraint of type $v = (v_1, v_2)$ for reporting (v_1', v_2),

$$v \cdot a(v) - t(v) \geq v \cdot a(v_1', v_2) - t(v_1', v_2).$$

Since $a_1(v_1, v_2) = a_1(v_1', v_2)$, the above inequality simplifies to

$$v_2 a_2(v) - t(v) \geq v_2 a_2(v_1', v_2) - t(v_1', v_2).$$

Similarly, the incentive constraint of type (v_1', v_2) for reporting v is

$$v_2 a_2(v_1', v_2) - t(v_1', v_2) \geq v_2 a_2(v) - t(v).$$

Thus, both incentive constraints must hold with equality. That is, each type is indifferent between her own allocation and the allocation of the other type. Therefore, without loss of generality we can assume that both types have the same allocation and transfer.

We show in the following two paragraphs that optimality of (a, t) leads to two contradictory conclusions. The first paragraph shows that $u(\underline{v}, \underline{p}_2) > 0$, whereas the second paragraph shows that $u(\underline{v}, \underline{p}_2) \leq 0$.

First, it must be that $u(\underline{v}, \underline{p}_2) > 0$. By Proposition 1, the induced separable mechanism (a^{ISP}, t^{ISP}) of (a, p) must be optimal. By definition,

$$a_1^{ISP}(v_1) = \mathbf{E}\left[a_1(v_1, v_2)\right], \text{ and } \qquad a_2^{ISP}(v) = a_2(v).$$

By Lemma 5, we must have $a_1^{ISP}(v_1) = 1$ for all $v_1 \geq \bar{p}_1$. Since $a_1^{ISP}(v_1)$ is the expectation of $a_1(v_1, v_2)$ over v_2, we have

$$a_1(v_1, v_2) = 1, \forall v_1 \geq \bar{p}_1, v_2. \tag{31}$$

Now consider any pair of values $v_1 < v_1'$ that are at least as large as $\bar{p}_1$. Such a pair exists since $\bar{p}_1 < \underline{p}_2$ implies that $\bar{p}_1 < \bar{v}$ and therefore there exists two distinct values in V_1 that are at least as large as $\bar{p}_1$. The fact that $a_1(v_1, v_2) = a_1(v_1', v_2) = 1$ for all v_2 has two implications. First, by our discussion above, the allocations in the second period must be identical following v_1 and v_1', that is, $a_2(v_1, v_2) = a_2(v_1', v_2)$ for all v_2. Second, in the separable mechanism, the promised utility to interim type v_1 is strictly smaller than the promised utility to type v_1' since the promised utility is the area under the allocation rule a_1, which is positive at v_1. Corollary 1 implies that for the allocation in the second period to be equal following two different promised utilities, the allocation probability in the second period must be 1. That is,

$$a_2(v_1, v_2) = a_2^{ISP}(v_1, v_2) = 1, \forall v_1 \geq \bar{p}_1, v_2.$$

Together with (31), the above equality implies that in particular, $a(\bar{p}_1, \underline{v}) = (1, 1)$. By individual rationality, the payment $t(\bar{p}_1, \underline{v})$ of type $(\bar{p}_1, \underline{v})$ must be at most $\bar{p}_1 + \underline{v}$. Incentive compatibility of the static mechanism requires that the utility of type $(\underline{v}, \underline{p}_2)$ must be at least the utility it would get from the allocation and payment of type $(\bar{p}_1, \underline{v})$, that is, getting both products at price $t(\bar{p}_1, \underline{v})$. Therefore, we have

$$\begin{aligned} u(\underline{v}, \underline{p}_2) &\geq \underline{v} + \underline{p}_2 - t(\bar{p}_1, \underline{v}) \\ &\geq \underline{v} + \underline{p}_2 - (\bar{p}_1 + \underline{v}) \\ &= \underline{p}_2 - \bar{p}_1 \\ &> 0. \end{aligned}$$

We now show that $u(\underline{v}, \underline{p}_2) \leq 0$. The continuation revenue CR_2 is concave. Therefore, the promised utility to type v_1 in the induced separable mechanism cannot be larger than the level that maximizes CR_2, as otherwise we can decrease the utility of all types and increase the continuation revenue. As a result, by Lemma 4, following a history of report $v_1 = \underline{v}$, product 2 is sold by randomizing over at most two prices that are both at least as large as $\underline{p}_2$. Therefore, the ex post utility of type $(\underline{v}, \underline{p}_2)$ in the separable mechanism is zero. Since ex post utilities are equal in mechanisms (a, t) and its induced separable mechanism, we conclude that $u(\underline{v}, \underline{p}_2) \leq 0$. □