The Limits of Multi-product Price Discrimination*

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Abstract

We consider markets served by a multi-product seller who can engage in second and third degree price discrimination. We characterize markets for which the maximum consumer surplus across all possible segmentations equals the total surplus from the efficient allocation minus the profit for the seller in the unsegmented market. We show that this benchmark is achievable for all markets with a given set of consumer types if and only if the seller never finds it profitable to screen types by offering multiple products. The same condition also characterizes when the entire "surplus triangle" of Bergemann et al. (2015) is achievable.

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1 Introduction

Consider the set of consumer-producer surplus pairs that arise from all possible segmentations of a market served by a profit-maximizing monopolist. This surplus set is clearly a subset of the "surplus triangle" defined by the constraints that (i) consumer surplus is non-negative, (ii) producer surplus is no lower than in the unsegmented market, and (iii) total surplus is at most the surplus from the efficient allocation. Bergemann et al. (2015) showed that when there is a single product, the surplus set in fact coincides with the surplus triangle. In particular, "first-best consumer surplus" (FBCS), which corresponds to the efficient allocation and the monopolist obtaining the same surplus as in the unsegmented market, is achievable by some segmentation.

We ask whether and when FBCS and the surplus triangle are achievable when there are multiple products. To facilitate the comparison to the single product case we assume that there is a "best product," it is efficient to allocate this product to all consumers, and consumer types are ranked so that, for each product, the valuation of any consumer type is higher than the valuations of lower types. We also assume zero production costs. A leading example is digital goods, such as streaming services, where the best product corresponds to the "premium" or "full feature" version of the service.

A key feature of our setting is that the seller may screen consumers, that is, offer multiple products in a single market or segment. By combining market segmentation and screening within segments the seller engages in second and third degree price discrimination, and investigating this combination is part of the conceptual contribution of our analysis.[1] The main obstacle to determining whether the surplus triangle, and FBCS in particular, are achievable is that no general characterization of profit-maximizing mechanisms is known when there are multiple products. We develop an approach that sidesteps this difficulty.

Our main finding is that screening interferes with the achievability of FBCS and the surplus triangle. We first show that FBCS and the surplus triangle are not achievable for any non-trivial market for which screening is optimal. Of course, screening implies inefficiency because not all consumers obtain the best product; the result shows that it is impossible to achieve efficiency

[1]Ichihashi (2020) and Hidir and Vellodi (2020) consider maximum consumer surplus when a multi-product seller offers only one product in each market (but possibly different products in different markets). The only previous instance we are aware of in which a seller offers more than one product in a single market is Bergemann et al. (2015)'s parametric example with two types and non-linear valuations.

via segmentation without the seller appropriating some of the gains. This implies achievability for *all* markets with a given set of consumer types if and only if screening is not optimal for any market. In contrast, we show that achievability fails for *all* non-trivial markets if and only if for any market for which screening is not optimal there is a single optimal mechanism. These results imply that with two consumer types either achievability holds for all markets or for no non-trivial market. With three or more types achievability may hold for some markets and fail for others.

2 Model

There is a monopolistic seller, a mass 1 of consumers, and a set $T = 1, \ldots, n$ of consumer *types*. There is a set $A = 0, 1, \ldots, k$ of *products*, where $k \geq 1$ and product 0 is the outside option. A product can correspond to a particular quantity or quality of a good or a service or to a bundle of goods or services. For example, if a streaming service offers a movie subscription, a series subscription, and full-access subscription that combines both, then there are four products (including the outside option). The cost of production is 0. The valuation of type $i \in T$ for a product $a \in A$ is $v_a^i \geq 0$, with $v_0^i = 0$. We assume that some product $\bar{a} \in A$ is the "best product" that all consumers prefer, that is, $v_{\bar{a}}^i > v_a^i$ for all types i and products $a \neq \bar{a}$.[2] In the streaming setting the best product would be the full-access subscription. We place no restrictions on how products other than the best product are ranked by different types. We assume that types are ranked so that a higher type has a higher valuation for any product, that is, $v_a^1 < v_a^2 < \ldots < v_a^n$ for any product $a \neq 0$. In the streaming setting, the higher the consumer's type the more he likes to watch shows and movies, so the higher his valuation is for every kind of subscription. But some types of consumers may prefer a movie subscription to a series subscription while other types have the opposite preference.

An *allocation* $x \in X = \Delta(A)$ is a distribution over products, where x_a denotes the probability of product a. An allocation x is empty if $x_0 = 1$, and is non-empty otherwise. For each type the efficient allocation x satisfies $x_{\bar{a}} = 1$. The (expected) utility of a type i consumer from an

[2]The assumption implies that it efficient to allocate the best product to all types, that if the seller optimally offers a single product in a market (formally defined below), he offers the best product, and if the seller is restricted to offering a single product in a market, he offers the best product. This also means that if the seller offers more than one product and engages in second degree price discrimination, the outcome is necessarily inefficient.

allocation x and a payment p is $v^i \cdot x - p = (\sum_a v^i_a x_a) - p$.

A *mechanism* consists of an allocation function $x : T \to X$ and a payment function $p : T \to R$. Mechanism $M = (x, p)$ is incentive compatible (IC) if for all types i and j,

$$v^i \cdot x(i) - p(i) \geq v^i \cdot x(j) - p(j). \tag{1}$$

Mechanism M is individually rational (IR) if for all types i,

$$v^i \cdot x(i) - p(i) \geq 0. \tag{2}$$

Henceforth, by "mechanism" will refer to an IC and IR mechanism, unless otherwise stated.

A *market* $f \in \Delta(T)$ is a distribution over types, where f_i denotes the fraction of consumers with type i. The expected utility of consumers in market f with mechanism $M = (x, p)$ is $EU(f, M) = E_{i \sim f}[v^i \cdot x(i) - p(i)]$. A mechanism (x, p) is *optimal* for market f if it maximizes revenue

$$E_{i \sim f}[p(i)]$$

across all mechanisms. For a market f, let $ER(f)$ be the maximum expected revenue, $\mathcal{M}(f)$ be the set of optimal mechanisms, and $CS(f)$ be the highest consumer surplus (expected utility) across all optimal mechanisms,

$$CS\,(f) = \max_{M \in \mathcal{M}(f)} EU(f, M).$$

A *segmentation* $\mu \in \Delta(\Delta(T))$ of a market f is a distribution over markets that average to f, that is, $E_{f' \sim \mu}[f'] = f$. We refer to a market f' in the support of the segmentation μ as a market segment (or simply a segment). Let $SEG(f)$ denote the set of segmentations of f. Abusing notation, let $CS(\mu)$ be the consumer surplus in segmentation μ,

$$CS\,(\mu) = E_{f \sim \mu}[CS(f)].$$

When discussing segmentations of a given market f, we refer to f as the unsegmented market.

We often represent a mechanism indirectly by a *menu* of allocation-price pairs, where each type chooses a pair that maximizes its utility. If a type is indifferent between two allocation-price pairs, it chooses the one with a higher price. If, further, the prices are identical, then the tie breaking can be arbitrary.

A mechanism (x, p) is a *non-screening* mechanism if it can be represented by a menu with a single allocation-price pair, in addition to the outside option at price 0. Of particular interest is the set of non-screening mechanisms $\{N^i\}_{i\in T}$, where mechanism N^i offers the best product $\bar{a}$ at price $v_{\bar{a}}^i$. The allocation and payment functions (x,p) of mechanism N^i are as follows: $x_0(j) = 1$ and $p(j) = 0$ for all $j < i$, and $x_{\bar{a}}(j) = 1$ and $p(j) = v_{\bar{a}}^i$ for all $j \geq i$. Among all non-screening mechanisms, N^i is optimal for some i.[3] A mechanism is a *screening* mechanism if it is not a non-screening mechanism, that is, every menu that represents it includes at least two allocation-price pairs.

Finally, we say that market f is a *non-screening* market if for some i, mechanism N^i is optimal for f. Otherwise, that is, if N^i is not optimal for any i, we say f is a *screening* market.

2.1 Upper Bound on the Maximum Consumer Surplus

Given a market f, we denote by $MCS(f) = \max_{\mu\in SEG(f)} CS(\mu)$ the maximum consumer surplus across all segmentations of f, and refer to a segmentation that achieves the maximum as a *consumer-optimal segmentation.* By definition, $MCS(f) \geq CS(f)$. Also, $MCS(f)$ is at most the expected surplus of an efficient allocation, $E_{i\sim f}[v_{\bar{a}}^i]$, minus the maximum expected revenue, $ER(f)$. This is because for any segmentation, the sum of the expected revenue and the consumer surplus is at most $E_{i\sim f}[v_{\bar{a}}^i]$ and the expected revenue is at least $ER(f)$ (since the seller can offer a mechanism in $\mathcal{M}(f)$ for all market segments). We refer to this upper bound $E_{i\sim f}[v_{\bar{a}}^i] - ER(f)$ on consumer surplus as *first-best consumer surplus* (FBCS). The following lemma formalizes this discussion.

Lemma 1 *For any market f, $CS(f) \leq MCS(f) \leq E_{i\sim f}[v_{\bar{a}}^i] - ER(f)$.*

We study the conditions under which the upper bound is tight.

Definition 1 *1. FBCS is* achievable *for a market f if $MCS(f) = E_{i\sim f}[v_{\bar{a}}^i] - ER(f)$.*

2. A segmentation μ of market f achieves *FBCS if $CS(\mu) = E_{i\sim f}[v_{\bar{a}}^i] - ER(f)$.*

If a market f has an optimal mechanism with an efficient allocation, then the single-segment segmentation achieves FBCS: the surplus generated is $E_{i\sim f}[v_{\bar{a}}^i]$ and the seller's profit is $ER(f)$. Thus, $CS(f) = E_{i\sim f}[v_{\bar{a}}^i] - ER(f)$. We refer to such markets as efficient.

[3] Consider any non-screening mechanism M that offers a single allocation x at price p. The mechanism that offers $\bar{a}$ at price p obtains at least as much revenue. Further, a price p such that $v_{\bar{a}}^{i-1} < p < v_{\bar{a}}^i$ for some type i cannot be optimal, since offering $\bar{a}$ at price $v_{\bar{a}}^i$ generates more revenue. Thus, among all non-screening mechanisms it is optimal to offer $\bar{a}$ at price $v_{\bar{a}}^i$ for some i.

Definition 2 *A market f is* efficient *if $N^{\underline{i}(f)}$ is an optimal mechanism for the market, where $\underline{i}(f)$ is the lowest type in the support of f. Otherwise, the market is* inefficient.

2.2 The Surplus Triangle

Given a market f, denote by $\Gamma(f)$ the set of consumer-producer surplus pairs resulting from all possible segmentations of f. Abusing notation, let $ER(\mu) = E_{f\sim\mu}[ER(f)]$ be the producer surplus resulting from segmentation μ, and consider a consumer-producer surplus pair $(CS(\mu), ER(\mu))$. Since $ER(\mu) \geq ER(f)$, $CS(\mu) \geq 0$, and $CS(\mu) + ER(\mu) \leq E_{i\sim f}[v^i_{\bar{a}}]$, the set $\Gamma(f)$ is a subset of the "surplus triangle"

$$\Delta(f) = \{(a, b) : b \geq ER(f),\ a \geq 0,\ a + b \leq E_{i\sim f}[v^i_{\bar{a}}]\}.$$

Definition 3 *The surplus triangle is* achievable *for a market f if $\Gamma(f) = \Delta(f)$.*

Bergemann et al. (2015) showed that the surplus triangle is achievable for any market f with a single product. The surplus triangle is also obviously achievable for any "singleton market," which consists only of consumers of some single type i. In this case, the surplus triangle consists of the single pair $(0, v^i_{\bar{a}})$. For non-singleton markets, however, our results show that the surplus triangle is not always achievable when there are multiple products.

To proceed, observe that the surplus triangle is the convex hull of its vertices. Thus, to determine whether the surplus triangle is achievable it is enough to determine whether each of its vertices is generated by some segmentation. The vertex $(0, E_{i\sim f}[v^i_{\bar{a}}])$ is generated by first-degree price discrimination. The vertex $(ER(f), E_{i\sim f}[v^i_{\bar{a}}] - ER(f))$ is generated by segmentations that achieve FBCS. The vertex $(0, ER(f))$ generates the lowest possible total surplus of $ER(f)$.

Definition 4 *A segmentation μ of market f achieves the* lowest possible total surplus *(LPTS) if the resulting consumer-producer surplus pair is $(0, ER(f))$. If such a segmentation exists then LPTS is* achievable *for market f.*

The above discussion shows the following.

Lemma 2 *The surplus triangle is achievable for a market if and only if FBCS and LPTS are achievable for the market.*

2.3 Conditions for Achieving FBCS

We start by specifying two conditions, which are together necessary and sufficient for a segmentation to achieve FBCS. First, because the resulting allocation is efficient, every segment must be efficient (Definition 2). Second, the seller should not benefit from the segmentation, that is, every optimal mechanism for the unsegmented market must be optimal for every segment.[4]

Lemma 3 *For any segmentation μ of a market f, the following are equivalent:*

1. *μ achieves FBCS.*
2. *For some optimal mechanism M of f and every segment f' of μ, f' is efficient and has an optimal mechanism M.*
3. *For every optimal mechanism M of f and every segment f' of μ, f' is efficient and has an optimal mechanism M.*

3 Two Types

We first consider markets with only two types of consumers (but any number of products), and identify each market by its fraction $q \in [0, 1]$ of type 2 consumers. The following lemma shows that the set of markets $[0, 1]$ can be qualitatively divided into at most three regions. The first region consists of markets in which the fraction of type 1 consumers is high, so they are non-screening markets in which mechanism N^1 is optimal, and are efficient. The second region consists of markets in which the fraction of type 2 consumers is high, so they are non-screening markets in which mechanism N^2 is optimal, and are inefficient. The third region, which may be empty, consists of the remaining, intermediate markets. These markets are screening markets, that is, the allocations of the two types are different and non-empty. Moreover, the optimal mechanisms may vary across markets in this region. To formalize this, denote by $\mathcal{F}(M)$ the (possibly empty) set of markets for which a particular mechanism M is optimal.

Lemma 4 *There exist thresholds q_1 and q_2, $0 \leq q_1 \leq q_2 \leq 1$, such that $\mathcal{F}(N^1) = [0, q_1], \mathcal{F}(N^2) = [q_2, 1]$, and $\mathcal{F}(M) \subseteq [q_1, q_2]$ for any mechanism $M \neq N^1, N^2$.*

[4] Otherwise, there is a segment f' such that the seller can benefit by offering in f' an optimal mechanism for f' and offering in all other segments an optimal mechanism for the unsegmented market.

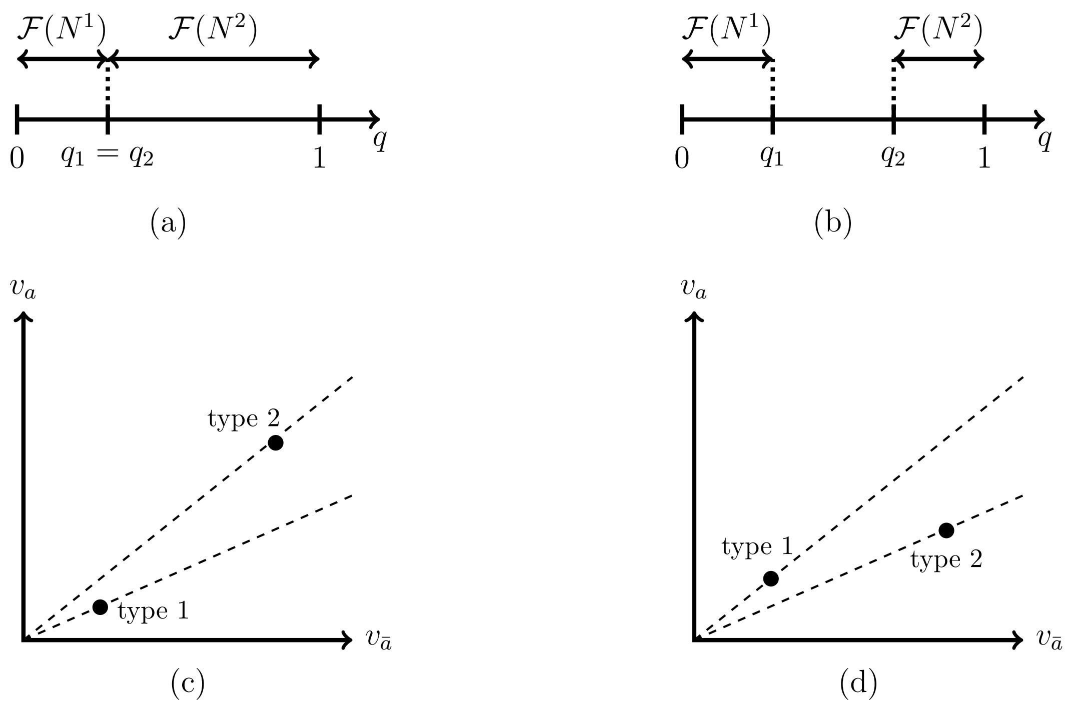

Figure 1: (a) $q_1 = q_2$. For any market, either N^1 or N^2 is optimal. (b) $q_1 < q_2$. Neither N^1 nor N^2 is optimal for markets in the interval (q_1, q_2). (c) $r_a^1 \le r_a^2$. (d) $r_a^1 > r_a^2$.

If $q_1 = q_2$, then all markets are non-screening markets, as shown in Figure 1 (a). Since the seller offers only the best product in each market, the setting is equivalent to one with a single product. Bergemann et al. (2015)'s result then shows that the surplus triangle, and FBCS in particular, are achievable for all markets. It is less clear what can be said about the achievability of FBCS if $q_1 < q_2$, which is shown in Figure 1 (b). The proposition below shows that in this case FBCS is unachievable for *any* inefficient market (the markets in $(q_1, 1)$).[5] As discussed in Section 2, a single-segment segmentation achieves FBCS for any efficient market. We thus obtain a characterization of the achievability of FBCS.

Proposition 1 *For any inefficient market q, FBCS is achievable if and only if $q_1 = q_2$.*

Proof. Suppose that $q_1 = q_2$. For completeness, we replicate Bergemann et al. (2015)'s result that FBCS is achievable for all markets. This is obviously true for the markets $[0, q_1] \cup \{1\}$ because they are efficient. Consider a market $q \in [q_2, 1]$, so mechanism N^2 is optimal for q, and a segmentation of q into $q' = 1$ and $q'' = q_1 = q_2$.[6] Both q' and q'' are efficient and have N^2 as

[5] A market $q < 1$ is efficient if and only if N^1 is optimal for the market, and these are the markets $[0, q_1]$. The singleton market 1 is clearly efficient.

[6] The segmentation assigns probability α to q', and probability $1 - \alpha$ to q'', where $\alpha = \frac{q - q_2}{1 - q_2}$.

an optimal mechanism, so the segmentation achieves FBCS by Lemma 3.

Now suppose that $q_1 < q_2$, and suppose that some segmentation μ of a market q achieves FBCS. We show that q is efficient, that is, q is in $[0, q_1] \cup \{1\}$. By Lemma 3, every segment in μ is efficient, and any optimal mechanism for q is optimal for every segment. The only optimal mechanism for market 1 is mechanism N^2. But since $q_1 < q_2$ and $\mathcal{F}(N^2) = [q_2, 1]$, N^2 is not optimal for any market in $[0, q_1]$. Therefore, either every segment of μ is equal to 1, in which case $q = 1$, or every segment is in $[0, q_1]$, in which case $q \in [0, q_1]$. Therefore, q is efficient. ■

We now turn to the achievability of the surplus triangle. Proposition 2 below shows that if $q_1 = q_2$ then the surplus triangle is achievable for all markets, and if $q_1 < q_2$ then the surplus triangle is not achievable for any non-singleton market, that is, for any market in $(0, 1)$. As discussed in Section 2, the surplus triangle is a singleton and is achievable for any singleton market. We thus obtain a characterization of the achievability of the surplus triangle.

Proposition 2 *For any non-singleton market q, the surplus triangle is achievable if and only if $q_1 = q_2$.*

Haghpanah and Hartline (2021) characterize the two cases, $q_1 = q_2$ or $q_1 < q_2$, in terms of the valuations of the two types, which are a primitive of the model. The characterization shows that $q_1 = q_2$ if and only if for any product a, type 2 has a higher ratio of valuations of product a to $\bar{a}$, that is, $r_a^1 \leq r_a^2$, where $r_a^i = v_a^i / v_{\bar{a}}^i$. Figure 1 (c) and (d) illustrate this inequality for the case of two products.

4 More than Two Types

We now consider markets with any number of types and any number of products. The logic of Bergemann et al. (2015) shows that if for a given set of types all markets are non-screening markets, then the surplus triangle (and thus FBCS) is achievable for every market with this set of types. We will show that this condition is in fact necessary by proving that FBCS, and thus the surplus triangle, are not achievable for any screening market. Of course, a screening mechanism is inefficient; What the result will show is that if a market is inefficient because it is a screening market, then it is impossible to achieve efficiency via segmentation without the seller appropriating some of the gains. This key result, Proposition 3, will also be useful in

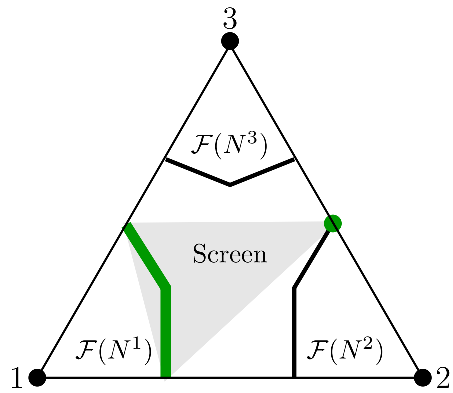

Figure 2: The set of markets with three types and the screening and non-screening regions. The convex hull (shaded gray) of the set of efficient markets for which a screening mechanism is also optimal (in green) includes some screening markets.

characterizing when FBCS and the surplus triangle are unachievable for *every* inefficient and non-singleton market, and not just screening markets, with a given set of types.

Proposition 3 *If FBCS is achievable for market f, then f is a non-screening market.*

The proof of Proposition 3 is not a simple generalization of parts of the proof of Proposition 1. That proof relies on the set of markets being an interval, which implies that all inefficient markets, and thus all screening markets, are higher than the highest efficient market q_1. Since the only efficient market higher than q_1 is the singleton market 1 that consists only of type 2 consumers, any segmentation of a screening market f into efficient markets must include market 1 as a segment. But no screening mechanism is optimal for market 1, so by Lemma 3 the segmentation does not achieve FBCS. With more than two types, the set of segmentations is a higher-dimensional simplex, so the convex hull of the set of efficient markets for which a screening mechanism is also optimal may include screening markets.[7] In Figure 2, this set is depicted in green and its convex hull is the shaded region, whose interior includes screening markets. Such screening markets could thus conceivably be segmented in a way that achieves FBCS. The proof of Proposition 3 shows this is not the case.

To prove Proposition 3, consider a market f with an optimal mechanism M, and suppose that FBCS is achievable for f. We will prove that f is a non-screening market by showing that mechanism N^j is also optimal for f, where j is the lowest type that is not excluded in M (a type

[7] With two types, this set is the singleton $\{q_1\}$.

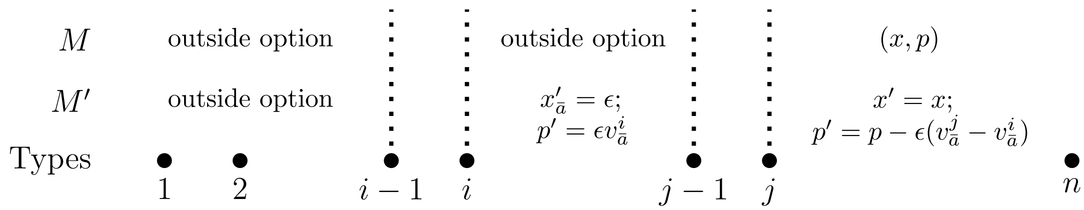

Figure 3: Construction of mechanism M' from mechanism M in the proof of Lemma 5.

is excluded if it gets an empty allocation). Consider a segmentation of market f that achieves FBCS, and take any segment f'. By Lemma 3, M is optimal for f'. Consider the lowest type $\underline{i}(f')$ in the support of f'. We must have $\underline{i}(f') \leq j$, otherwise every type in the support of f' gets strictly positive utility in M, since it can mimic type j and get strictly positive utility, so M is not optimal for market f'. By Lemma 3, f' is efficient so $N^{\underline{i}(f')}$ is also optimal for f'. The following lemma, which is the key to Proposition 3, shows that N^j is also optimal for f'.

Lemma 5 *Consider a market f' and an optimal mechanism $M = (x, p)$, and let j be the lowest type that gets a non-empty allocation in M, that is, $j = \min\{j' : x_0(j') < 1\}$. Suppose that for some $i \leq j$, N^i is also optimal for f'. Then, N^j is also optimal for f'.*

Proof. Assume without loss of generality that f' has full support on types 1 to n. Assume for contradiction that mechanisms M and N^i are optimal for f but mechanism N^j is not. We construct a mechanism M' that has a higher revenue than M. In M', types below i get an empty allocation (as they do in M). Types i to $j-1$ get product $\bar{a}$ with probability $\epsilon > 0$ and the outside option with probability $1-\epsilon$ and pay $\epsilon v^i_{\bar{a}}$. Types j to n have the same allocation as in M, but their payment is decreased by $\epsilon(v^j_{\bar{a}} - v^i_{\bar{a}})$ relative to their payment in M. See Figure 3.

Mechanism M' has a higher revenue than mechanism M. Compared to M, M' gains $\epsilon v^i_{\bar{a}}$ from every type $i' \geq i$ and loses $\epsilon v^j_{\bar{a}}$ from every type $i' \geq j$. The difference in revenue is $\epsilon v^i_{\bar{a}} \Pr[i' \geq i] - \epsilon v^j_{\bar{a}} \Pr[i' \geq j]$, which is ϵ times the difference between the revenue of mechanism N^i and the revenue of mechanism N^j. This difference is strictly positive by the assumption that N^i is optimal but N^j is not. It remains to show that M' is IR and IC for small enough $\epsilon > 0$.

IR holds for types $1, \ldots, i-1$ because they are excluded in M'. A type $i' = i, \ldots, j-1$ has utility $\epsilon v^{i'}_{\bar{a}} - \epsilon v^i_{\bar{a}} \geq 0$, and a type $i' \geq j$ has a higher utility in M' than in M. Thus, IR holds for any $\epsilon > 0$.

For IC, observe that M' coincides with M in the limit as ϵ goes to 0. Thus, if an IC constraint holds strictly in M, then it is satisfied in M' for small enough ϵ. In mechanism M a type i'

strictly prefers not to mimic another type i'' in two cases: (1) if $i' > j$ and $i'' < j$; (2) if $i' < j$ and $i'' \geq j$. In case (1), type i' has a strictly positive utility in M because it can mimic type j. Thus i' strictly prefers not mimic type i'' (and get utility 0) in M. In case (2), type i' gets a strictly negative utility from mimicking i'' because $v^{i'} \cdot x(i'') - p(i'') < v^j \cdot x(i'') - p(i'') \leq 0$, where the last inequality follows since the utility of type j is 0 and incentive compatibility of mechanism M implies that the utility of type j from mimicking type i'' cannot be positive.

We next verify the remaining IC constraints in mechanism M'. Consider a type $i' < j$. As discussed in case (2) above, such a type i' does not benefit from mimicking types $i'' \geq j$. Type i' prefers the allocation of types $1, \ldots, i-1$ (the outside option) to the allocation of types $i, \ldots, j-1$ if and only if $\epsilon(v^{i'}_{\bar{a}} - v^{i}_{\bar{a}}) \leq 0$, that is, $i' \leq i$. Thus truthtelling maximizes the utility of a type $i' < j$. For a type $i' \geq j$, note that mimicking a type $j, \ldots, n$ is not beneficial since M is IC and all such types get the same additional payment in M'. From case (1) above, a type $i' > j$ does not benefit from mimicking types $1, \ldots, j-1$. Finally, the utility of type j in M' is at least $\epsilon(v^{j}_{\bar{a}} - v^{i}_{\bar{a}}) > 0$, which is the utility it would get by mimicking types $i, \ldots, j-1$, and is no lower than the utility of 0 it would get by mimicking types $1, \ldots, i-1$. ■

Lemma 5 shows that mechanism N^j is optimal for every segment f'. The following result shows that N^j is also optimal for the original market f, which completes the proof of Proposition 3.

Lemma 6 *For any mechanism M, the set $\mathcal{F}(M)$ is convex.*

Lemma 6 follows from the observation that for any market f and any segmentation of f, the revenue from a mechanism M is the weighted average of the revenues in the segments. If M is optimal for the segments but not for f, some other mechanism would give a strictly higher revenue for f. The same must therefore be true for at least one of the segments, contradicting the optimality of M for the segments.

4.1 Achievability of FBCS and the Surplus Triangle

Proposition 3 and the logic of Bergemann et al. (2015) imply that FBCS and the surplus triangle are achievable for all markets with a given set of types T if and only if all markets with that set of types are non-screening markets, that is, $\cup_i \mathcal{F}(N^i) = \Delta(T)$. Whether non-screening is optimal for a *given* market is in general difficult to ascertain. But Haghpanah and Hartline (2021) provide

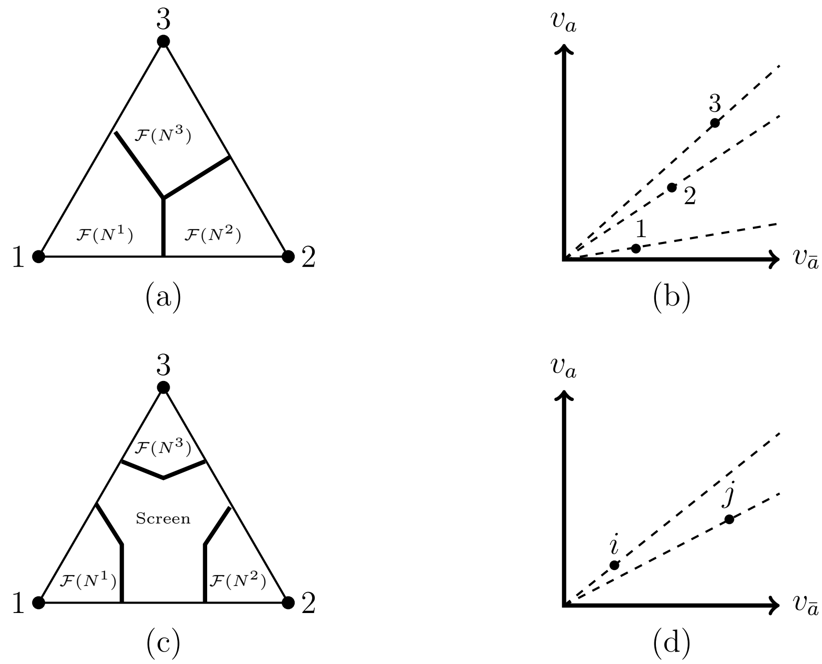

Figure 4: (a) Every market is a non-screening market (Statement (3) of Theorem 1). (b) The ratio of valuations increases in the valuation for the best product (Statement (4) of Theorem 1). (c) The boundaries of the non-screening regions $\mathcal{F}(N^1)$, $\mathcal{F}(N^2)$, and $\mathcal{F}(N^3)$ do not intersect (Statement (3) of Theorem 2). (d) The ratio of valuations decreases in the valuation for the best product (Statement (4) of Theorem 2).

a simple characterization of the sets of types for which *all* markets are non-screening markets, which is illustrated in Figure 4 (a) and (b). For the characterization, let $r_a^i = v_a^i / v_{\bar{a}}^i$ be the ratio between type i's valuations of products a and $\bar{a}$. Proposition 3 and this characterization lead to our first main result.

Theorem 1 *For any set of types T, the following are equivalent:*

1. *FBCS is achievable for every market.*
2. *The surplus triangle is achievable for every market.*
3. *Every market is a non-screening market.*
4. *The ratio r_a^i is non-decreasing in i for all a.*

Proof. (2) $\to$ (1): By definition.

(3) $\to$ (1) and (2): If offering only $\bar{a}$ is optimal for all markets, the setting is equivalent to one with a single product $\bar{a}$. The results of Bergemann et al. (2015) then imply (1) and (2).

(1) $\rightarrow$ (3): Immediate from Proposition 3.

(3) $\rightarrow$ (4) and (4) $\rightarrow$ (3): From Proposition 1 of Haghpanah and Hartline (2021). ■

Using the notation from Section 3, Theorem 1 states that with two types FBCS and the surplus triangle are achievable for every market if and only if $q_1 = q_2$. This generalizes parts of Proposition 1 and Proposition 2. However, Proposition 1 and Proposition 2 show that if $q_1 < q_2$, then FBCS is unachievable for *all* inefficient markets and the surplus triangle is unachievable for *all* non-singleton markets. Theorem 1 does not make such a statement, which is in fact not true with more than two types.

4.2 Unachievability of FBCS and the Surplus Triangle

In contrast to Proposition 1 and Proposition 2, with more than two types it may be that some markets are screening markets and yet FBCS and the surplus triangle are achievable for some inefficient and non-singleton non-screening markets with the same set of types. To identify the condition for unachievability for *all* inefficient and non-singleton markets, let us interpret the condition $q_1 < q_2$ in Proposition 1 and Proposition 2 as stating that the set of screening markets, (q_1, q_2), *separates* the sets $[0, q_1]$ and $[q_2, 1]$ of non-screening markets. Our second main result shows that this is the correct condition for any number of types. It is illustrated in Figure 4 (c) and (d).

Theorem 2 *For any set of types T, the following are equivalent:*

1. *FBCS is unachievable for every inefficient market.*

2. *The surplus triangle is unachievable for every non-singleton market.*

3. *For every market, N^i is optimal for at most one i.*

4. *For every pair of types $i < j$, there exists some product a such that $r_a^i > r_a^j$.*

To see why statement (3) implies statement (1) in Theorem 2, suppose that that FBCS is achievable for some inefficient market f. By Proposition 3, market f is a non-screening market, so for some $i > \underline{i}(f)$ mechanism N^i is optimal for f. At least one segment in any segmentation must include consumers of type $\underline{i}(f)$. By Lemma 3, both $N^{\underline{i}(f)}$ and N^i are optimal for that segment, so statement (3) does not hold.

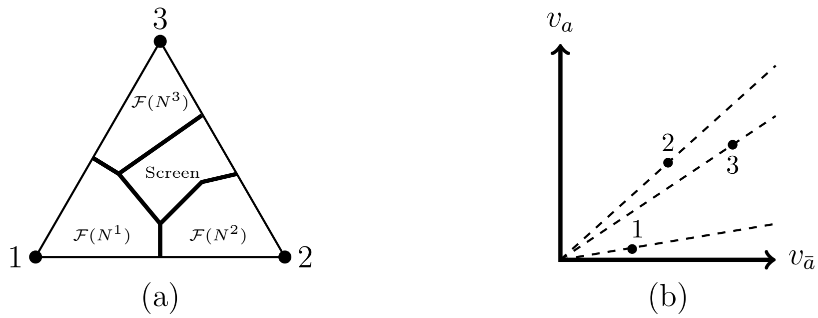

Figure 5: (a) Some markets are screening markets, but the screening region does not separate the non-screening regions. (b) The ratio of valuations decreases in the valuation for the best product for some but not all pairs of types.

To show that statement (4) implies statement (3) in Theorem 2, we cannot apply the results of Haghpanah and Hartline (2021) as we did in the proof of Theorem 1. Instead, we develop a new result that relates properties of type ratios to the set of non-screening mechanisms that may be optimal for any market. This is the content of the following lemma.

Lemma 7 *Consider a pair of types $i < j$ such that $r^i_a > r^j_a$ for some a. Then, for any market f, mechanisms N^i and N^j are not both optimal.*

The proof of Lemma 7 shows that, given a pair of types $i < j$ such that $r^i_a > r^j_a$ for some product a, if both N^i and N^j are assumed optimal, then there exists a mechanism that outperforms N^j.

5 Concluding remarks

We studied the achievability of FBCS and the surplus triangle in some multi-product environments. With more than two types, it is possible that some markets are non-screening markets but the set of non-screening markets does not separate the sets of screening markets. This is illustrated in Figure 5, which follows immediately from Theorem 1 and Theorem 2.

One direction for future research is to investigate the maximal consumer surplus when FBCS is not achievable. This will likely require a new approach; we consider a two-type, two-product example in the Online Appendix.

A Appendix

A.1 Proof of Lemma 4

We first show that for any mechanism M, $\mathcal{F}(M)$ is a closed interval. Indeed, if M is optimal for two markets q, q', then it is also optimal for any convex combination q'' of these markets, because for any mechanism, the revenue in q'' is the same convex combination of the revenues in q and in q'. And $\mathcal{F}(M)$ is closed because the revenue from any mechanism is continuous in the market q. We now argue that $q_1 \le q_2$ and for any $M \ne N^1, N^2$, we have $\mathcal{F}(M) \subseteq [q_1, q_2]$. To see this, consider any two mechanisms M, M' with payment rules $p \ne p'$. Then there is at most a single market q where the two mechanisms have the same revenue, $qp(1) + (1-q)p(2) = qp'(1) + (1-q)p'(2)$. Therefore, the intersection of $\mathcal{F}(M)$ and $\mathcal{F}(M')$ is at most a single market. The claim now follows from observing that for any mechanism $M \ne N^1, N^2$, the payment rules of M, N^1, and N^2 are all different.

A.2 Proof of Proposition 2

Suppose that $q_1 = q_2$. As noted by Bergemann et al. (2015), the same segmentation that achieves FBCS also achieves the surplus triangle.

Now suppose that $q_1 < q_2$. Consider any inefficient market $q > q_1$. By Proposition 1, FBCS, and therefore the surplus triangle, is unachievable for such a market. Now consider an efficient market $q \le q_1$ so mechanism N^1 is optimal for q. In this case LPTS is unachievable. This is because if the consumer surplus is 0 in some segment q', mechanism N^2 must be optimal for q'. Then mechanism N^1 is not optimal for q' and the segmentation increases producer surplus.

A.3 Proof of Lemma 7

Assume for contradiction that $r_a^i > r_a^j$ for some $i < j$ and a, and N^i and N^j are both optimal for a market f. Denote by q_i the fraction of types i and higher in market f, and by q_j the fraction of types j and higher in market f. For N^i and N^j to be both optimal, we must have $v_{\bar{a}}^i q_i = v_{\bar{a}}^j q_j$, that is $q_i = \frac{v_{\bar{a}}^j q_j}{v_{\bar{a}}^i}$. Thus we can write

$$v_a^i q_i = v_a^i \left(\frac{v_{\bar{a}}^j q_j}{v_{\bar{a}}^i}\right) = \left(\frac{v_{\bar{a}}^j v_a^i}{v_{\bar{a}}^i}\right) q_j > v_a^j q_j. \tag{3}$$

where the inequality followed from the assumption that $r_a^i > r_a^j$ (that is, $v_a^i / v_{\bar{a}}^i > v_a^j / v_{\bar{a}}^j$).

Construct a mechanism M that improves upon N^j as follows. Types $i, \ldots, j-1$ get product a with probability ϵ and pay ϵv_a^i. Types $j, \ldots, n$ get product $\bar{a}$ and pay $v_{\bar{a}}^j - \epsilon(v_a^j - v_a^i)$.

Let us compare the revenue of M with the revenue of N^j. Types $i, \ldots, j-1$ pay ϵv_a^i more in M than in N^j. Types j and higher pay $\epsilon(v_a^j - v_a^i)$ less in M than in N^j. The difference in expected revenue is

$$\epsilon v_a^i(q_i - q_j) - \epsilon(v_a^j - v_a^i)q_j = \epsilon(v_a^i q_i - v_a^j q_j) > 0,$$

where the inequality followed from inequality 3. So to complete the proof, we show that M is IC and IR, which contradicts the assumption that N^j is optimal.

Mechanism M is IR. Types lower than i are excluded. A type i' from i to $j-1$ has utility $\epsilon(v_a^{i'} - v_a^i) \geq 0$. Types j and higher have a higher utility in M than in N^j.

For IC, observe similarly to the proof of Lemma 5 that if an incentive constraint holds strictly in N^j, then it is satisfied in M for small enough $\epsilon > 0$. In particular, (1) a type $i' > j$ does not benefit from mimicking a type $i'' < j$, (2) a type $i' < j$ does not benefit from mimicking a type $i'' \geq j$.

We now verify the remaining incentive constraints. A type $i' < j$ prefers the allocation of types $i, \ldots, j-1$ to the outside option if and only if $\epsilon(v_a^{i'} - v_a^i) \geq 0$, that is, $i' \geq i$. Thus the incentive constraints are satisfied for types $i' < j$. For types $i' \geq j$, note that mimicking any type $j, \ldots, n$ is not beneficial since all such types have the same allocation and payment. Finally, the utility of type j in M is $\epsilon(v_a^j - v_a^i)$, which is the utility it would receive by mimicking types $i, \ldots, j-1$, and is strictly more than the utility it would receive by mimicking types $1, \ldots, i-1$.

A.4 Proof of Theorem 2

(3) $\rightarrow$ (2): That (3) implies (1) also shows that that if (3) holds, then the surplus triangle is not achievable for any inefficient market. So it remains to show that the surplus triangle is not achievable for any non-singleton efficient market. Consider a non-singleton efficient market f and suppose that a segmentation μ achieves LPTS (Definition 4). Consider a segment f' whose support includes type $\bar{i}(f)$, the highest type in the support of f. Because μ achieves LPTS, consumer surplus in f' is 0, so $N^{\bar{i}(f)}$ is optimal for f'. And since f is efficient, $N^{\underline{i}(f)}$ is optimal for f, where $\underline{i}(f)$ is the lowest type in the support of f. So by Lemma 3, $N^{\underline{i}(f)}$ is also optimal for f'.

(4) $\to$ (3): Directly from Lemma 7.

(1) $\to$ (4) and (2) $\to$ (4): Suppose for contradiction that for some $i < j$, $r_a^i \leq r_a^j$ for all a. Haghpanah and Hartline (2021) show that either N^i or N^j is optimal for any market with support in $\{i, j\}$. By Proposition 1 and Proposition 2, FBCS and the surplus triangle are achievable for every market with support in $\{i, j\}$.

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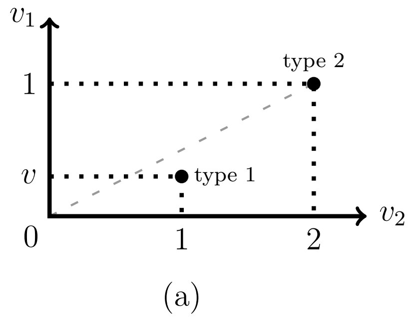

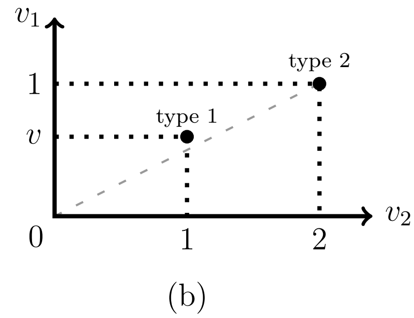

Figure 6: Type 1 has valuations $(v, 1)$ and type 2 has valuations $(1, 2)$ for product 1 and product 2. Case (a) corresponds to $v \leq 0.5$, and case (b) corresponds to $v \geq 0.5$.

B Online Appendix

B.1 A Two Type Example

In this section we discuss a parametric example to highlight our results. We directly calculate the closed form expression for the maximum consumer surplus and compare it to FBCS. Even though the calculations are straightforward, they are not easily extendable beyond this example.

Suppose that there are two products and two types. A type 1 consumer has valuation $v \in (0, 1)$ for one unit and valuation 1 for two units. A type 2 consumer has valuation 1 for one unit and valuation 2 for two units. The two types are illustrated in Figure 6, in which case (a) corresponds to $v \leq 0.5$ and case (b) corresponds to $v \geq 0.5$. A market q consists of a fraction $1 - q$ of type 1 consumers and a fraction q of type 2 consumers.

To identify maximum consumer surplus in different markets, it is useful to first identify the optimal mechanism in each market. Consider the following three mechanisms and their revenue in a market q. Mechanism N^1 offers product 2 at price 1. Mechanism N^2 offers product 2 at price 2. Mechanism S screens; it offers each consumer a choice between buying product 1 at price v or product 2 at price $v + 1$. It can be shown that for any market q, one of these three mechanisms is optimal, as illustrated in Figure 7. If $v \leq 0.5$, then mechanisms N^1 is optimal for markets in $[0, 0.5]$ and mechanism N^2 is optimal for markets in $[0.5, 1]$. If $v \geq 0.5$, then mechanism N^1 is optimal for markets in $[0, 1-v]$, mechanism S is optimal for markets in $[1-v, v]$, and mechanism N^2 is optimal for markets in $[v, 1]$.

Next, we compute the (average) consumer surplus in each market q generated by the optimal

Figure 7: Optimal mechanisms for the two cases (a) $v \leq 0.5$ and (b) $v \geq 0.5$.

Figure 8: (a) Consumer surplus $CS(q)$ for the two cases (a) $v \leq 0.5$ and (b) $v \geq 0.5$.

mechanism for that market.[8] Type 1 does not receive any information rents in any optimal mechanism. Thus, consumer surplus $CS(q)$ in market q is q times the utility of type 2 in the optimal mechanism for that market. Consumer surplus $CS(q)$ is illustrated in Figure 8.

A segmentation of market q is a distribution μ over markets $[0,1]$ such that $E_{q' \sim \mu}[q'] = q$. The maximum consumer surplus is $MCS(q) = \max_\mu E_{q' \sim \mu}[CS(q')]$, that is, the highest consumer surplus across all segmentations μ. The maximum consumer surplus is obtained by concavifying the function CS. That is, $MCS(q) = \overline{CS}(q)$, where $\overline{CS}$ is the lowest concave function that is point-wise at least as high as CS.

The maximum consumer surplus $MCS(q)$ is at least $CS(q)$ and at most first best consumer surplus $FBCS(q)$, which is the surplus from the efficient allocation (that is, product 2 for each type) minus the seller's revenue in market q. If the optimal mechanism for market q implements the efficient allocation, then the two bounds are equal, that is, $CS(q) = FBCS(q)$, so $CS(q) = MCS(q) = FBCS(q)$. This is the case for a market q for which mechanism N^1 is optimal and for market $q = 1$ which contains only type 2 consumers and for which mechanism N^2 is optimal. We refer to such markets as efficient, and otherwise as inefficient. If a market is efficient, then there is no scope for market segmentation to increase consumer surplus.

We can now address the possibility of achieving FBCS for all markets $q \in [0,1]$. The relationship between maximum consumer surplus, MCS, and first best consumer surplus, $FBCS$,

[8] If there is more than one optimal mechanism we choose the one with higher consumer surplus.

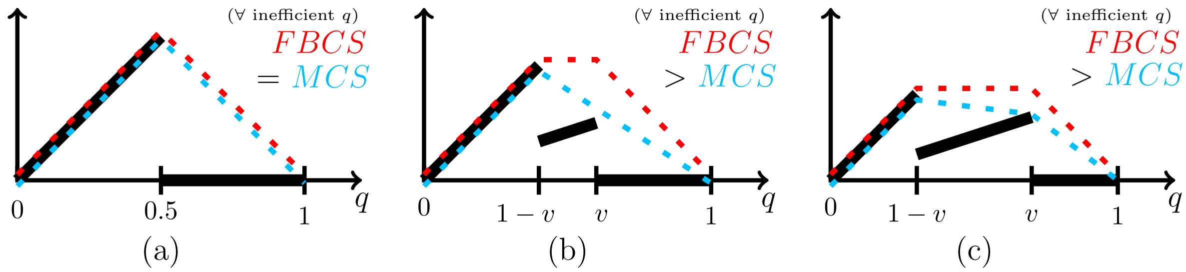

Figure 9: The relationship between CS, MCS, and $FBCS$: (a) $v \in (0, 0.5]$, (b) $v \in (0.5, \frac{\sqrt{5}-1}{2})$, and (c) $v \in [\frac{\sqrt{5}-1}{2}, 1)$.

is illustrated in Figure 9, and depends on the value of v. If v is in $(0.5, 1)$, as in Figure 9 (b) and (c), then FBCS is not achievable for any inefficient market. The only difference between cases (b) and (c) in Figure 9 is that in the former, $MCS(q)$ strictly exceeds $CS(q)$ for every inefficient market q whereas in the latter $MCS(q) = CS(q)$ for market $q = v$.[9] If $v \in (0, 0.5]$, as in Figure 9 (a), then FBCS is achievable for all markets. Equivalently, FBCS is achievable for all markets if and only if for every market either mechanism N^1 or N^2 is optimal, that is, the seller does not find it profitable to screen consumers.

This example can also be used to illustrate how close MCS is to $FBCS$ when $FBCS$ is not achievable. If v is in $(0.5, 1)$, as in Figure 9 (b) and (c), then the ratio $FBCS/MCS$ increases in q in the interval $(1-v, v)$ and decreases in the interval $(v, 1)$. Consider the maximal point $q = v$. At this point, we have $FBCS = 1 - v$; If $v \in (0.5, \frac{\sqrt{5}-1}{2})$, then $MCS = \frac{(1-v)^2}{v}$ so $FBCS/MCS = \frac{v}{1-v}$, which increases in v; If $v \in [\frac{\sqrt{5}-1}{2}, 1)$, then $MCS = (1-v)v$, so $FBCS/MCS = \frac{1}{v}$, which decreases in v.

What is the economic significance of v being greater than or smaller than 0.5? For type 2 consumers, product 2 is twice as valuable as product 1. For type 1 consumers, whether v is greater than or smaller than 0.5 determines whether product 2 is more than or less than twice as valuable as product 1. In other words, when $v \leq 0.5$ the second unit of the product is relatively more complementary to the first unit of the product for type 1 consumers than for type 2 consumers, and vice versa when $v > 0.5$.

Turning to the surplus triangle, it is trivially achievable for markets with a single type of consumer ($q = 0$ and $q = 1$). For all other markets, the same conditions that characterize achievability of FBCS also characterize when the surplus triangle is achievable for every market

[9]See Haghpanah and Siegel (2021) for a detailed investigation of when MCS strictly exceeds CS.

or no market (efficient or inefficient). Indeed, whenever FBCS is not achievable, the surplus triangle is clearly not achievable. And the results of Bergemann et al. (2015) show that when the seller does not find it optimal to screen, that is, in every market q only offers two units as a bundle, the entire surplus triangle is achievable.