543: Game Theory **Week 8**
Robust Contracts

Updated July 26, 2021.

Preliminary and incomplete.

Thanks to 534 Fall 2020 students for proofreading! (Remaining errors are mine.)

Source(s): [Carroll, 2015], [Kambhampati, 2020].

1 Robustness and Linear contracts

This is from [Carroll, 2015].

Setup:

- Principal, agent both risk neutral
- Output $y \in Y$ observable
- Action $(F, c) \in Y \times R^+$
- Agent has set of actions A, unknown to principal
- Principal knows A_0, and that $A_0 \subseteq A$, assume exists (F, c_0) s.t. $E_F[y] - c > 0$
- Contract $w : Y \to R^+$.
- Agent:

$$A^*(w|A) = \arg\max_{(F,c)\in A} (E_F[w(y)] - c)$$

$$V_A(w|A) = \max_{(F,c)\in A} (E_F[w(y)] - c).$$

- Principal

$$V_p(w|A) = \max_{(F,c)\in A^*(w|A)} E_F[y - w(y)]$$

$$V_P(w) = \inf_{A \supseteq A_0} V_P(w|A).$$

Can positive payoff be guaranteed?

- Yes. With a linear contract $w(y) = \alpha y, \alpha \in [0, 1]$.
- For any optimal (F, c)
$$E_f[w(y)] \geq E_f[w(y)] - c = V_A(w|A) \geq V_A(w|A_0)$$
With the linear contract we have
$$\alpha(y - w(y)) = (1 - \alpha)w(y).$$
So we can get a lower bound on principal's payoff
$$E_F[y - w(u)] \geq \frac{1-\alpha}{\alpha} E_F[w(y)] \geq \frac{1-\alpha}{\alpha} V_A(w|A_0)$$
$$V_P(w) \geq \frac{1-\alpha}{\alpha} V_A(w|A_0)$$
- If α close to 1, $V_A(w|A_0) > 0$, and thus principal guarantees positive payoff.

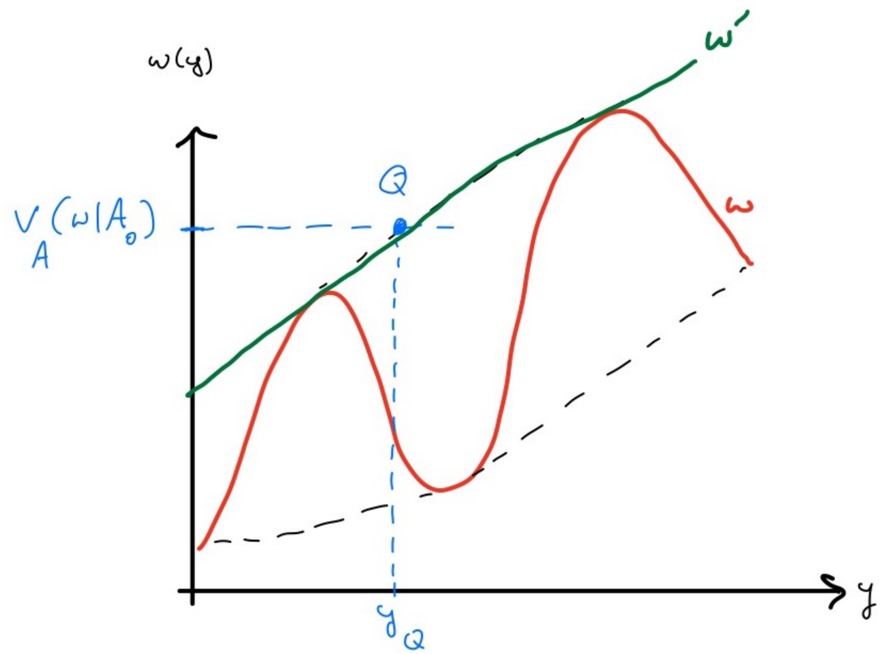
$$\begin{aligned}
V_P(w) &= \inf_{A \supseteq A_0} V_P(w|A) \\
&= \inf_{A \supseteq A_0} \max_{(F,c) \in A^*(w|A)} E_F[y - w(y)] \\
&= \inf_{A \supseteq A_0} \max_{(F,c) \in A^*(w|A)} \frac{1-\alpha}{\alpha} E_F[w(y)] \\
&\geq \inf_{A \supseteq A_0} \max_{(F,c) \in A^*(w|A)} \frac{1-\alpha}{\alpha} (E_F[w(y)] - c) \\
&= \inf_{A \supseteq A_0} \max_{(F,c) \in A} \frac{1-\alpha}{\alpha} (E_F[w(y)] - c) \\
&= \max_{(F,c) \in A_0} \frac{1-\alpha}{\alpha} (E_F[w(y)] - c) > 0
\end{aligned}$$

Theorem 1.1 *There exists a linear contract that maximizes V_P.*

Consider an arbitrary contract w. If the agent chooses F, then the point $(E_F[y], E_F[w(y)])$ is in the convex hull of w. For any A, the agent's payoff is at least $V_A(w|A_0)$. This defines the point $Q = (y_Q, V_A(w|A_0))$. The principal's worst payoff is $y_Q - V_A(w|A_0)$.

Now consider the afine contract w'. By the same argument, the principal's payoff is $y_Q - V_A(w|A_0)$.

So we have shown that there exists an optimal contract that is afine. Consider any afine contract $w(y) = \alpha y + \beta$. Since $w(0) \geq 0$, we have $\beta \geq 0$. Decreasing β doesn't change the agent's payoff, and only increases the principal's payoff. So the optimal contract is linear.



1.1 What is opt linear parameters?

For given α,

$$V_p(w) = \frac{1-\alpha}{\alpha} V_A(w|A_0)$$
$$= \max_{F,c \in A_0} ((1-\alpha)E_F[y] - \frac{1-\alpha}{\alpha} c)$$

So to maximize α

$$\max_{F,c,\alpha} ((1-\alpha)E_F[y] - \frac{1-\alpha}{\alpha} c)$$

For fixed F, c, opt over α is

$$\alpha = \sqrt{c/E_F[y]}.$$

So

$$E_F(y) + c - 2\sqrt{cE_F[y]} = (\sqrt{E_F[y]} - c)^2$$

1.2 Participation Constraints

Principal restricted to offering only w s.t. $E_F[w(y)] - c \geq \bar{U}_A$ for some $F, c \in A_0$

- The affine argument goes through
- But can't go from $\alpha y + \beta$ to αy.

Theorem 1.2 *There is an optimal contract that is linear.*

Proof:

- Affine is the same
- Opt β
$$\max_{F,c} \alpha E_F[y] - c + \beta = \bar{U}_A$$
$$\beta^* = \max(0, \bar{U}_A - \max_{F,c} \alpha E_F[y] - c).$$
- First case if $\alpha \geq \bar{\alpha}$, second case otherwise
- For any α, best affine contract guarantees
$$\max_{F,c \in A_0} ((1-\alpha)E_F[y] - \frac{1-\alpha}{\alpha} c) - \beta^*$$
- When $\alpha \geq \bar{\alpha}$, simplifies to
$$\max_{F,c} (E_F[y] - \frac{1}{\alpha} c) - \bar{U}_A,$$
which is increasing in α. So max attained at $\alpha \geq \bar{\alpha}$.

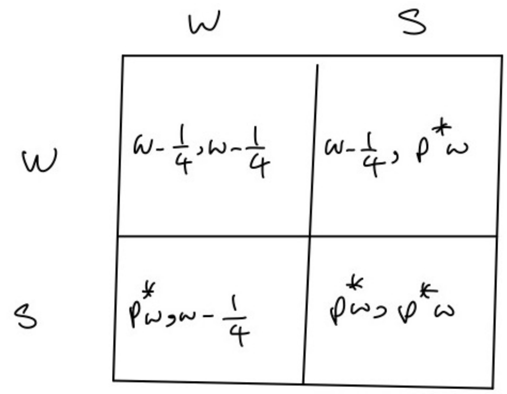
QED

2 Joint performance evaluation

This is from [Kambhampati, 2020].

- A principal
- Two agents
- Each agent chooses shirk or work
- Work costs 1/4, leads to success. Success have value 1 to principal
- Shirk costs 0, leads to success w.p. p^* (independent)

Principal offers a contract, which specifies non-negative wage for each profile of outcomes. For a given p^*, principal evaluates the contract at its worst NE in the induced game. The principal doesn't know p^*. So the payoff of a contract is evaluated at the worst p^*.



2.1 Independent performance evaluation

Wage = 0 if fail, $= w \in (1/4, 1)$ if succeed.

Game is as follows.

Agent has incentive to shirk if and only if

$$p^* w \geq w - 1/4 \Leftrightarrow p^* \geq 1 - 1/4w.$$

Worst-case is when $p^* = 1 - 1/4w$ and shirk,shirk is a NE, where principal's payoff is

$$(1 - 1/4w)(1 - w).$$

2.2 Joint performance evaluation

We now see that principal benefits from joint performance evaluation: wage = 0 if fail, $= w$ if succeed and the other agent also succeeds, $= w - \epsilon$ if succeed and the other agent fails.

Game is as follow.

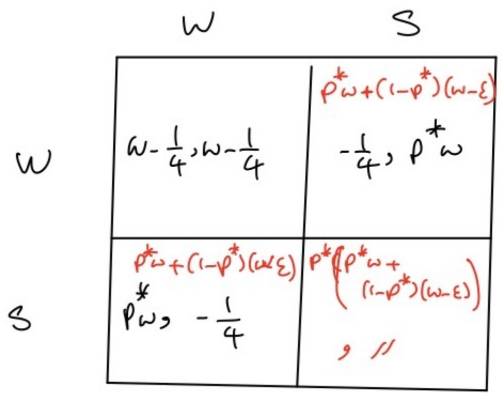
If the other agent works, the agent has incentive to shirk if, same as before,

$$p^* \geq 1 - 1/4w.$$

But now the advantage is that the principal pays strictly less when shirk,shirk.

Worst case is when p^* approaches $1 - 1/4w$ from above, in which case shirk,shirk is unique NE. Principals' payoff is

$$(1 - 1/4w) - (1 - 1/4w)(p^* w + (1 - p^*)(w - \epsilon)) > (1 - 1/4w)(1 - w).$$



References

[Carroll, 2015] Carroll, G. (2015). Robustness and linear contracts. American Economic Review, 105(2):536–63.

[Kambhampati, 2020] Kambhampati, A. (2020). Robust performance evaluation. working paper.