PROBLEMS FOR VIASM MINICOURSE: GEOMETRY OF MODULI SPACES LAST UPDATED: DEC 25, 2013

1. Problems on moduli spaces

The main text for this material is Harris & Morrison "Moduli of curves." (There are djvu files easily obtainable on the web.)

Problem 1.1. Let F be a moduli functor, and suppose $(M, \Psi : F \to h_M)$ is a fine moduli space. Show that (M, Ψ) is also a coarse moduli space.

Problem 1.2. Let F be a moduli functor, and suppose (M, Ψ) is a pair satisfying the universal property in the definition of a coarse moduli space: for any pair $(M', \Psi : F \to h_{M'})$, there is a morphism $M \to M'$ such that the transformation Ψ' factors uniquely as $F \to h_M \to h'_M$. Show that (M, Ψ) is unique up to unique isomorphism.

Problem 1.3. Let F be the functor of flat families of lines through the origin in $\mathbb{C}^2$. Let C be the projective curve with plane model $y^2 = x^3$. (That is, in projective space it is defined by $y^2z = x^3$). Show that there is no natural transformation $\Psi : F \to h_C$ making (C, Ψ) a coarse moduli space for F.

2. Problems on the geometry of a fixed curve

The classical reference for the study of a fixed curve is Arbarello, Cornalba, Griffiths & Harris "Geometry of Algebraic Curves," widely known as "ACGH." Chapter 1 alone has a great deal of relevant material and many good exercises. Again, a djvu file exists on the web.

Another much more elementary book is Rick Miranda's "Algebraic Curves and Riemann Surfaces." If you are unfamiliar with the correspondence between line bundles, divisors, sections of line bundles, and maps to projective space, this is an excellent first reference.

Chapter IV of Hartshorne can also be useful.

Problem 2.1. Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d. Use the adjunction formula to compute the genus of C.

Problem 2.2. Consider the plane curve C_d defined by $x^d + y^d + z^d = 0$ in $\mathbb{P}^2$.

(1) Show C_d is smooth.
(2) Use the Riemann-Hurwitz formula to compute the genus of C_d.
(3) Use part (2) to give another computation of the genus of an arbitrary smooth plane curve of degree d.

Problem 2.3. Let C be the cuspidal cubic curve in $\mathbb{P}^2$ with affine equation $y^2 = x^3$. Use the Riemann-Hurwitz formula to compute the geometric genus of C. (This curve is singular. The geometric genus of an irreducible singular curve is the geometric genus of its normalization. Here the normalization is $\mathbb{P}^1 \to C$, but this technique generalizes to cases where it is harder to identify the normalization.)

Problem 2.4. Work on exercise batch A at page 31 of ACGH to gain much more experience with applying Riemann-Hurwitz to compute the geometric genus.

Problem 2.5. Let $H_{d,N}$ be the Fermat hypersurface $x_0^d + \cdots + x_N^d = 0$ in $\mathbb{P}^N_{\mathbb{C}}$.

(1) Show $H_{d,N}$ is smooth.
(2) Compute the topological Euler characteristic $\chi(H_{d,N})$ by using induction on N.
(3) Compute the topological Euler characteristic of any smooth hypersurface of degree d in $\mathbb{P}^N$.

Problem 2.6. Show any smooth curve of genus 2 is hyperelliptic.

Problem 2.7. Show that there are smooth curves of genus 3 which are non-hyperelliptic. In particular, show that a smooth degree 4 plane curve is never hyperelliptic.

Problem 2.8. Show that if $g > 0$ then the canonical divisor K_C of a smooth curve C of genus g is basepoint-free. (In fact, if C is not hyperelliptic, then K_C is very ample.)

3. Problems on the geometry of surfaces

Standard references for surfaces include Beauville "Complex Algebraic Surfaces" and Chapter V of Hartshorne.

Problem 3.1. Show that any reduced, irreducible curve $C \subset \mathbb{P}^3$ of degree 2 is a plane conic curve.

Problem 3.2. Let $S \subset \mathbb{P}^3$ be a smooth cubic surface.

(1) Let $L \subset S$ be a line (in fact, S contains exactly 27 lines). Use the adjunction formula to compute $L \cdot L$.
(2) Let $C \subset S$ be a plane conic curve. Show that such curves exist. Furthermore, show that there exists a line $L \subset S$ such that $C + L \sim H$, where H is the hyperplane class. Compute $C \cdot C$.

Problem 3.3. Show that every elliptic curve C can be embedded in a quadric surface $Q \subset \mathbb{P}^3$ as a curve of class $2h_1 + 2h_2$, where the h_i are the classes of lines of the two rulings.

4. Problems on Hilbert schemes

References: Harris-Morrison, and Eisenbud-Harris "The Geometry of Schemes" for material on flat limits.

Problem 4.1. Let $X \subset \mathbb{P}^r$ be a hypersurface of degree d.

(1) Compute the Hilbert polynomial P_X.
(2) Show that if a closed subscheme $Y \subset \mathbb{P}^r$ has the same Hilbert polynomial as X, then Y is a hypersurface of degree d.

Problem 4.2. Let $X \subset \mathbb{P}^r$ be a linear subspace of dimension s.

(1) Compute the Hilbert polynomial P_X.
(2) Show that if a closed subscheme $Y \subset \mathbb{P}^r$ has the same Hilbert polynomial as X, then Y is a linear subspace of dimension s.

Problem 4.3. Let $M \subset \mathbb{A}^3$ be the union of the x-axis and the y-axis. Let L_t be the line given parametrically by the map

$$f_t : \mathbb{A}^1 \to \mathbb{A}^3$$
$$u \mapsto (u, u, tu),$$

and let X_t be the union of three lines $M \cup L_t$. Compute the flat limit $\lim_{t\to 0} X_t$.

Problem 4.4. Consider the zero-dimensional scheme $Z \subset \mathbb{P}^2$ given by $Z = V(x^2, y^2)$.

(1) Show that $P_Z(m) = 4$.

(2) Find a flat family $\mathscr{X}$ of subschemes of $\mathbb{P}^2$ parameterized by $\mathbb{A}^1$ such that $\mathscr{X}_0 = Z$ and $\mathscr{X}_1$ is a reduced collection of 4 points in the plane.

Problem 4.5. Let $C \subset \mathbb{P}^3$ be a twisted cubic curve. For each t, let π_t be the map given in affine coordinates by

$$\pi_t(x, y, z) = (x, y, tz).$$

Consider the family $X_t = \pi_t(C)$ parameterized by $\mathbb{A}^1 - \{0\}$.

(1) Describe the limit $\lim_{t\to 0} X_t$.
(2) Describe the limit $\lim_{t\to\infty} X_t$.

Problem 4.6. If $X \subset \mathbb{P}^r$ is a projective variety and P is a Hilbert polynomial, there is a closed subscheme $\mathrm{Hilb}_{P,X}$ of $\mathrm{Hilb}_{P,r}$ consisting of subschemes of X with Hilbert polynomial P. The tangent space to a point $[Z] \in \mathrm{Hilb}_{P,X}$ is naturally identified as $H^0(N_{Z/X})$.

Suppose $C \subset \mathbb{P}^r$ is a smooth curve. Let $P(m) = n$ be a constant polynomial, with $n > 0$. Show that the Hilbert scheme of n points in C, written as $C^{[n]} := \mathrm{Hilb}_{n,C}$, is a smooth, irreducible variety of dimension n.

(In fact, it can be shown that $C^{[n]} = \mathrm{Sym}^n C$ is just the symmetric product.)

Problem 4.7. Let $C \subset \mathbb{P}^r$ be a rational normal curve.

(1) Compute the Hilbert polynomial P of C.
(2) Compute the dimension of the component of $\mathrm{Hilb}_{P,r}$ containing $[C]$.
(3) Show $[C]$ is a smooth point of the Hilbert scheme.
(4) Does this Hilbert scheme have any other components?

Problem 4.8. Let $P(m) = 4m$, and consider the Hilbert scheme $\mathrm{Hilb}_{P,3}$. Show that this Hilbert scheme has a component whose general point is a smooth degree 4 elliptic curve which lies on a smooth quadric surface as a curve of class $2h_1 + 2h_2$. Furthermore, this component is smooth at any such point. (Hint: any such curve is a complete intersection of two quadric surfaces.)

5. Problems from 17/12

References: Harris-Morrison, ACGH, and Miranda.

Problem 5.1. Let L be a line bundle on a smooth curve C. Show that ϕ_L is an embedding iff

$$h^0(L(-p-q)) = h^0(L) - 2$$

for all points $p, q \in C$ (including the case $p = q$). In particular, given what was shown in class, show that ϕ_L is base-point free if the condition holds.

Problem 5.2. Determine necessary and sufficient conditions on a line bundle L of degree $2g$ on a smooth curve C of genus g such that ϕ_L is an embedding.

Problem 5.3. Let C be a smooth curve of genus 2. Show that C cannot be embedded in $\mathbb{P}^r$ as a curve of degree < 5. Show that if D is any divisor of degree 5, then ϕ_D gives an embedding. Describe what projective space the image lies in. Determine how many quadric and cubic surfaces contain the image. How many independent cubics are not in the ideal generated by the quadrics?

Problem 5.4. Compute the dimension of the locus H_3 of hyperelliptic curves in M_3, and show that it is irreducible.

Problem 5.5. Fix a genus g. Show that any automorphism of a curve C of genus g has at most $2g + 2$ fixed points. (See ACGH exercise batch F for hints.) Conclude that if $n \gg 0$ then no point in $M_{g,n}$ has any automorphisms. In fact, $M_{g,n}$ is a fine moduli space for sufficiently large n.

Problem 5.6. If it was new to you, review the construction of Riemann surfaces branched over a divisor $B \subset \mathbb{P}^1$ corresponding to given monodromy data in Miranda Chapter IV.

What is the degree of the branch divisor map $H_{d,g} \to (\mathrm{Sym}^b \mathbb{P}^1) - \Delta$ in case $d = 3$ and $g = 2$? (The combinatorics involved may or may not be messy; at least figure out how to set up the computation.)

6. Problems from 19/12

Problem 6.1. Show that a proper, connected, at-worst nodal curve of arithmetic genus $g \geq 2$ has finitely many automorphisms if and only if every smooth rational component intersects the rest of the curve in at least 3 points.

Problem 6.2. Generalize the previous exercise to marked curves in the appropriate way.

Problem 6.3. Combinatorially describe all stable marked curves in $\overline{M}_{0,5}$. For fixed combinatorial data (sometimes called the *topological type* of a curve) there is a stratum in $\overline{M}_{0,5}$ consisting of curves of that form. Determine the dimensions of the strata corresponding to each topological type, and determine which strata are contained in the closure of the others. (We will focus on more intricate questions of this type next lecture.)

Problem 6.4. Show that if C is a proper, connected, at worst nodal curve with components $C_1, \ldots, C_s$, then the arithmetic genus satisfies

$$p_a(C) = \sum_{i=1}^{s} p_g(C_i) + \delta - s + 1,$$

where δ is the number of nodes in C.

Problem 6.5. Let C be a smooth curve of genus $g \geq 1$. Suppose 2 moving marked points of C simultaneously collide with a third fixed point $p_0 \in C$. What are the possible stable limits of the corresponding family of marked curves in $\overline{M}_{g,3}$? (Hint: the answer will depend on the relative rates of approach of the marked points.)

Problem 6.6. Complete the argument started in class to show that a non-hyperelliptic smooth curve C of genus 4 cannot be embedded in projective space of a curve of degree ≤ 5. In particular, consider the case when the canonical curve lies on a singular quadric surface. What happens in the hyperelliptic case?

Problem 6.7. For each $g \geq 2$, what is the minimum value of n such that the map corresponding to nK_C is an embedding for all smooth curves C of genus g?

7. Problems from 24/12

Problem 7.1. Determine the possible topological types of a curve in $\overline{M}_2$. Which topological strata are contained in the closures of others?

The draft book "3264 and all that" by Harris and Eisenbud is a reference for the following material (although it goes much more in depth than the level of treatment we have given.) The lecture notes should be sufficient.

Problem 7.2. How many lines in $\mathbb{P}^3$ meet each of four general lines in $\mathbb{P}^3$?

Problem 7.3. Let $C \subset \mathbb{P}^3$ be a curve of degree d, and let $\Sigma_C \subset \mathbb{G}(1,3)$ be the locus of lines meeting C.

(1) What is the class in $H^*(\mathbb{G})$ corresponding to the fundamental class of Σ_C?

(2) Let $C_1, \ldots, C_4$ be sufficiently general curves of degrees $d_1, \ldots, d_4$. How many lines in $\mathbb{P}^3$ meet each of the C_i?

Problem 7.4. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface, and let $\Sigma \subset \mathbb{G}(1,3)$ be the locus of lines contained in Q. What is the class of Σ in $H^*(\mathbb{G})$?.

Problem 7.5. Let E be a rank 2 vector bundle on a smooth projective variety X. Use the splitting principle to compute the Chern classes of $\operatorname{Sym}^3 E$.

Problem 7.6. Let E be a rank r vector bundle on a smooth projective variety X, and let L be a line bundle on X. Use the splitting principle to determine a general formula for the Chern classes of $E \otimes L$.

Problem 7.7. Compute the cohomology ring of the Grassmannian $\mathbb{G}(1,4)$ of lines in $\mathbb{P}^4$. (This Grassmannian has a cell decomposition analogous to that of $\mathbb{G}(1,3)$, with cells $\sigma_{i,j}$ indexed by i, j with $0 \leq j \leq i \leq 3$.) How many lines do you expect lie on a smooth quintic threefold in $\mathbb{P}^4$?