Steiner bundles and divisors on the Hilbert scheme of points in $\mathbb{P}^2$

Jack Huizenga
Harvard University

Goals

- Determine the cone of effective divisors on the Hilbert scheme of n points in $\mathbb{P}^2$
- Find explicit constructions of boundary effective divisors and the moving curves they are dual to
- Better understand the structure of vector bundles on $\mathbb{P}^2$
- Learn about multiplication of polynomials: it's actually really hard!

Divisor classes on the Hilbert scheme

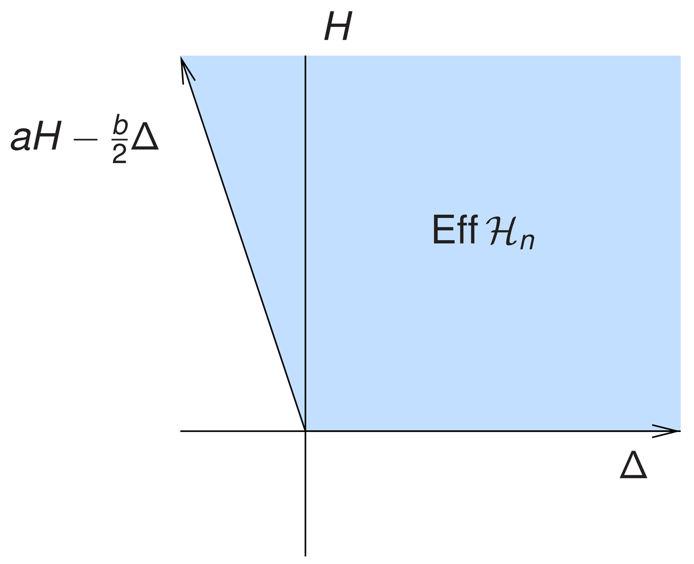
Denote by $\mathcal{H}_n$ the Hilbert scheme of n points in $\mathbb{P}^2$.

- H is the locus of subschemes $\Gamma \subset \mathbb{P}^2$ meeting a fixed line.
- Δ is the locus of nonreduced subschemes Γ.

We have

$$\operatorname{Pic} \mathcal{H}_n = \mathbb{Z}H \oplus \mathbb{Z}(\Delta/2)$$

and both H, Δ are effective. In fact Δ generates a boundary ray of the effective cone, but typically H is not extremal; the other boundary ray is spanned by a divisor of the form $aH - \frac{b}{2}\Delta$.



Toy examples of extremal divisors

- $n = 6$: do all six points lie on a conic?
- $n = 7$: do six of the seven points lie on a conic?
- $n = 8$: consider the pencil of cubics passing through 8 points. Does the 9th base point of this pencil lie on a fixed line?
- $n = 9$: given a fixed 10th point, do all 10 points lie on a cubic?
- $n = 10$: do all 10 points lie on a cubic?

In each case the dual moving curve is given by allowing the n points to move in a linear pencil on a smooth curve of some degree. In case $n = 6, 8$, or 9, it is given by allowing the points to move on a cubic. For $n = 7$, the points move on a quintic.

The Hilbert scheme of 12 points

A general collection $\Gamma \subset \mathbb{P}^2$ of 12 points lies on a 3-dimensional vector space of quartics and a 9-dimensional vector space of quintics. The multiplication map

$$H^0(\mathcal{I}_\Gamma(4)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{I}_\Gamma(5))$$

is an isomorphism for general Γ, and the locus D where it fails to be an isomorphism is an extremal effective divisor. It is dual to the moving curve given by letting 12 points move in a linear pencil on a smooth quartic.

Described differently, consider the vector bundle E given by a resolution

$$0 \to E \to \mathcal{O}_{\mathbb{P}^2}(4)^3 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(5) \to 0,$$

where M is a general matrix of linear forms. Then

$$D = \{\Gamma : H^0(E \otimes \mathcal{I}_\Gamma) \neq 0\}.$$

(Note that $E = T_{\mathbb{P}^2}(2)$)

Interpolation bundles

A vector bundle $E/\mathbb{P}^2$ of rank r has *interpolation for n points* if $h^0(E) = rn$ and $h^0(E \otimes \mathcal{I}_\Gamma) = 0$ for a general $\Gamma \in \mathcal{H}_n$.
For such a vector bundle, the locus

$$D_E = \{\Gamma \in \mathcal{H}_n : H^0(E \otimes \mathcal{I}_\Gamma) \neq 0\}$$

forms an effective divisor of class $c_1(E)H - \frac{r}{2}\Delta$.

Candidate interpolation bundles (works for 38% of all n)

Write

$$n = \frac{r(r+1)}{2} + s \qquad (0 \leq s \leq r).$$

Consider the general vector bundle E with resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(r-2)^s \to \mathcal{O}_{\mathbb{P}^2}(r-1)^{r+s} \to E \to 0.$$

If E has interpolation for n points, then D_E is an extremal effective divisor dual to the moving curve given by letting n points move in a linear pencil on a smooth r-ic.
We can show:

- If E has interpolation for n points, it is semistable.
- If for a general degree r map $f : \mathbb{P}^1 \to \mathbb{P}^2$ the bundle f^*E is balanced, then E has interpolation for n points.

Steiner bundles

A *general Steiner bundle* E on $\mathbb{P}^2$ is a bundle admitting a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}^s \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(1)^{r+s} \to E \to 0,$$

where M is a general matrix of linear forms. These bundles are natural generalizations of the tangent bundle.

Theorem (Brambilla). Such an E is semistable if and only if either $s/r \geq \varphi^{-1}$ or s/r is one of the convergents in the continued fraction expansion of φ^{-1}, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio.

Restrictions of Steiner bundles

Theorem (H.). Let E be a general Steiner bundle with resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}^{ks} \to \mathcal{O}_{\mathbb{P}^2}(1)^{k(r+s)} \to E \to 0.$$

If $f : \mathbb{P}^1 \to \mathbb{P}^2$ is a general degree r map and k is sufficiently large, then f^*E is balanced iff E is semistable.

This result determines the effective cone of $\mathcal{H}_n$ for roughly 38% of all values of n, when s is in the upper portion of $[0, r]$. Another 38% can be handled dually, when s is in the lower portion of this interval.

Multiplication of polynomials on $\mathbb{P}^1$

The previous restriction result is proved by answering a simple question about polynomials on $\mathbb{P}^1$. Let $V \subset H^0(\mathcal{O}_{\mathbb{P}^1}(r))$ be a fixed general 3-dimensional subspace.

Is it the case that for *every* $W \subset H^0(\mathcal{O}_{\mathbb{P}^1}(s-1))$ we have

$$\frac{\dim(V \cdot W)}{r+s} \geq \frac{\dim W}{s}?$$

In other words, does multiplication of W by V never decrease the *fraction* of the space of polynomials occupied by W? We show that the answer is yes if and only if $s/r \geq \varphi^{-1}$ or it is one of the convergents in the continued fraction expansion of φ^{-1}.

Questions for the remaining 24% of all n

When s lies near the middle of the interval $[0, r]$, things become much more complicated. For instance, when $n = 17$ (the first case not handled so far) the extremal divisor corresponds to the vector bundle with resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(2)^2 \to \mathcal{O}_{\mathbb{P}^2}(4)^{11} \to E \to 0,$$

where the map is a general matrix of quadrics. The moving curve dual to this is given by allowing 17 points to move in a linear pencil on an irreducible degree 9 curve having 4 nodes.

Question. If E is a stable vector bundle of rank r on $\mathbb{P}^2$ with $h^0(E) = rn$ and $h^1(E) = h^2(E) = 0$, does it (or a general deformation) have interpolation for n points? In the other direction, if E has minimal slope among bundles having interpolation for n points, must E be semistable?

An affirmative answer to this question would predict complicated behavior near where $s/r = 2/5$ or $s/r = 3/5$, as the classification of stable vector bundles becomes complicated in this region.