

RATIONALITY OF SESHADRI CONSTANTS ON GENERAL BLOW UPS OF \mathbb{P}^2

LUCJA FARNIK, KRISHNA HANUMANTHU, JACK HUIZENGA, DAVID SCHMITZ,
AND TOMASZ SZEMBERG

ABSTRACT. Let X be a projective surface and let L be an ample line bundle on X . The global Seshadri constant $\varepsilon(L)$ of L is defined as the infimum of Seshadri constants $\varepsilon(L, x)$ as $x \in X$ varies. It is an interesting question to ask if $\varepsilon(L)$ is a rational number for any pair (X, L) . We study this question when X is a blow up of \mathbb{P}^2 at $r \geq 0$ very general points and L is an ample line bundle on X . For each r we define a *submaximality threshold* which governs the rationality or irrationality of $\varepsilon(L)$. We state a conjecture which strengthens the SHGH Conjecture and assuming that this conjecture is true we determine the submaximality threshold.

1. INTRODUCTION

Let X be a smooth complex projective variety and let L be a nef line bundle on X . The *Seshadri constant* of L at $x \in X$ is defined as the real number

$$\varepsilon(X, L, x) := \inf_{x \in C} \frac{L \cdot C}{\text{mult}_x C},$$

where the infimum is taken over all irreducible and reduced curves passing through x . The Seshadri constants were defined by Demailly in [4], motivated by the Seshadri criterion for ampleness ([13, Theorem 7.1]) which says that L is ample if and only if $\varepsilon(X, L, x) > 0$ for all $x \in X$.

Seshadri constants have turned out to be fundamental to the study of positivity questions in algebraic geometry and a lot of research is currently focused on problems related to Seshadri constants. One such open problem is whether Seshadri constants can be irrational.

Assume that X is a surface. If L is an ample line bundle on X , then for any $x \in X$, we have $0 < \varepsilon(X, L, x) \leq \sqrt{L^2}$. The first inequality is the Seshadri criterion for ampleness and the second inequality is an easy observation. The largest and the smallest values of Seshadri constants as the point x varies are interesting and generally they behave very differently.

To be more precise, one has the following two definitions:

$$\begin{aligned} \varepsilon(X, L, 1) &:= \sup_{x \in X} \varepsilon(X, L, x), \\ \varepsilon(X, L) &:= \inf_{x \in X} \varepsilon(X, L, x). \end{aligned}$$

Date: January 4, 2019.

2010 Mathematics Subject Classification. Primary: 14C20. Secondary: 14H50, 14J26.

LF was partially supported by the National Science Centre, Poland, grant 2018/28/C/ST1/00339. KH was partially supported by a grant from Infosys Foundation and by DST SERB MATRICS grant MTR/2017/000243. JH was partially supported by the NSA Young Investigator Grant H98230-16-1-0306 and NSF FRG grant DMS 1664303. DS was partially supported by DFG grant PE 305/13-1.

It is known that $\varepsilon(X, L, 1) = \varepsilon(X, L, x)$ for very general points $x \in X$ (see [15]). It is also expected that $\varepsilon(X, L, 1) = \sqrt{L^2}$ in many situations. For example, let X be the blow up of \mathbb{P}^2 at at least 9 general points. If some well-known conjectures are true, then there exist ample line bundles on X such that $\varepsilon(X, L, 1) = \sqrt{L^2} \notin \mathbb{Q}$. See [5, 10] for more details.

On the other hand, $\varepsilon(X, L)$, called the global Seshadri constant, is usually attained at special points. In this context, [17, Question 1.6] asks whether $\varepsilon(X, L)$ is always rational for any pair (X, L) . In this paper we study this question in the case of blow ups of \mathbb{P}^2 at very general points. On the one hand, it is easy to exhibit ample line bundles L such that $\varepsilon(X, L)$ is rational. On the other hand, we state a strengthened version of the SHGH conjecture that implies that $\varepsilon(X, L)$ can be irrational for some line bundles L close to the boundary of the ample cone. See Example 4.7 for one such instance.

In fact, for $\mu \in \mathbb{Q}$ we study uniform line bundles $L = L(\mu) = \mu H - \sum_i E_i$ on blow ups of \mathbb{P}^2 at very general points and exhibit a threshold μ_0 such that $\varepsilon(X, L) \in \mathbb{Q}$ if $\mu \geq \mu_0$. This is proved in Theorem 2.5. We then state Conjecture 3.6 which strengthens the SHGH Conjecture. Assuming this conjecture is true, we show in Theorem 4.1 that if $\mu < \mu_0$ then $\varepsilon(X, L) \notin \mathbb{Q}$ unless $\sqrt{L^2} \in \mathbb{Q}$.

We will write $\varepsilon(L) = \varepsilon(X, L)$ when the variety X is clear.

Acknowledgements: We thank the Mathematisches Forschungsinstitut Oberwolfach for hosting the Mini-Workshop *Asymptotic Invariants of Homogeneous Ideals* during September 30 – October 6, 2018, where most of this work was done. The research stay of the second author was partially supported by the Simons Foundation and by the Mathematisches Forschungsinstitut Oberwolfach and he is grateful to them.

2. SUB-MAXIMALITY THRESHOLD

Let $p_1, \dots, p_r \in \mathbb{P}^2$ be very general points and let $X = \text{Bl}_{p_1, \dots, p_r} \mathbb{P}^2$ be the blowup of \mathbb{P}^2 at p_1, \dots, p_r . Let E_i be the exceptional divisor over p_i , and let $E = \sum_i E_i$. Let H denote the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$.

We will focus on *uniform* line bundles $L = dH - mE$ on X , i.e., such where all exceptional divisors appear with the same multiplicity m . We are interested in the rationality or irrationality of $\varepsilon(L)$. This only depends on the ratio $\mu = d/m$ and we work with the \mathbb{Q} -divisor $(d/m)H - E$. More generally, for $\mu \in \mathbb{R}$, let $L(\mu)$ be the \mathbb{R} -divisor $\mu H - E$. If $L(\mu)$ is ample then $\mu > \sqrt{r}$. If $r \geq 10$, then the converse is true if the Nagata conjecture holds.

In this paper, we discuss the following question.

Question 2.1. *Let $\mu \in \mathbb{Q}$ and suppose $L(\mu)$ is ample. Is $\varepsilon(L(\mu))$ rational?*

It is well-known that if L is an ample \mathbb{Q} -divisor and $\varepsilon(L, x) < \sqrt{L^2}$ then $\varepsilon(L, x)$ is achieved by a curve C containing x , and consequently, $\varepsilon(L, x) \in \mathbb{Q}$. So if $\varepsilon(L)$ is rational, then one of the following must be true:

- (1) $\varepsilon(L) = \sqrt{L^2} \in \mathbb{Q}$, or
- (2) $\varepsilon(L) < \sqrt{L^2}$ and there is a pair (C, x) where C is an irreducible and reduced curve containing a point x such that

$$\varepsilon(L) = \frac{L \cdot C}{\text{mult}_x C}.$$

A curve C satisfying

$$\frac{L \cdot C}{\text{mult}_x C} \stackrel{(-)}{<} \sqrt{L^2}$$

is called a (*weakly*) *submaximal curve* for L with respect to x (note that if equality holds, then $\sqrt{L^2}$ is rational). In light of this discussion, if L is ample then we have $\varepsilon(L) \in \mathbb{Q}$ if and only if either $\sqrt{L^2} \in \mathbb{Q}$ or there is a weakly submaximal curve.

When the number r of points is at most 9, a complete answer to Question 2.1 is given in the following theorem.

Theorem 2.2. *Let $r \leq 9$ and let $\mu \in \mathbb{Q}$ be such that $L(\mu)$ is ample. Then $\varepsilon(L(\mu)) \in \mathbb{Q}$.*

Proof. When $r \leq 8$, it is well-known that Seshadri constants of ample line bundles are rational.

For $r = 9$, the line bundle $L(\mu)$ is ample if and only if $\mu > 3$. We show that there is a weakly submaximal curve for $L(\mu)$. First, if $\mu \geq \sqrt{10}$ then an exceptional divisor E_i and a point $x \in E_i$ give a weakly submaximal curve for $L(\mu)$. Indeed, we have

$$1 = L(\mu) \cdot E_i \leq \sqrt{L(\mu)^2} = \sqrt{\mu^2 - 9}$$

whenever $\mu \geq \sqrt{10}$.

If instead $\mu \in (3, \sqrt{10})$, we need to give a different weakly submaximal curve. Consider the cubic $C = 3H - E$ through the 9 points, and let $x \in C$. Then C gives a weakly submaximal curve for $L(\mu)$ so long as

$$3\mu - 9 = L(\mu) \cdot C \leq \sqrt{L(\mu)^2} = \sqrt{\mu^2 - 9},$$

and this inequality holds for $\mu \in (3, 3.75]$. Therefore $\varepsilon(L(\mu)) \in \mathbb{Q}$. \square

Thus for the rest of the article we focus on the case $r \geq 10$. We can shift our focus to the existence of weakly submaximal curves.

Question 2.3. *For which real $\mu \geq \sqrt{r}$ does $L(\mu)$ admit a weakly submaximal curve?*

The answer to Question 2.3 is perhaps most interesting when μ is rational and $L(\mu)$ is ample, but there is no difficulty in stating or studying it more generally as we have done above. We first prove that there is a critical value $\mu_0 \geq \sqrt{r}$ such that L admits a weakly submaximal curve if $\mu \geq \mu_0$. It follows that if $L(\mu)$ is ample and $\mu \in \mathbb{Q}$ then $\varepsilon(L(\mu)) \in \mathbb{Q}$ for $\mu \geq \mu_0$.

Definition 2.4. Let $X = \text{Bl}_{p_1, \dots, p_r} \mathbb{P}^2$ be the blowup of \mathbb{P}^2 at r general points. A real number $\mu_0 \geq \sqrt{r}$ is called the *submaximality threshold* for r if

- (1) $L(\mu)$ does not admit a weakly submaximal curve for $\mu < \mu_0$, and
- (2) $L(\mu)$ does admit a weakly submaximal curve for $\mu \geq \mu_0$.

In Section 4, we prove that submaximality thresholds exist for $r \geq 10$, assuming a strengthening of the SHGH Conjecture. This in particular means that if $\sqrt{r} < \mu < \mu_0$ and $\sqrt{L(\mu)^2} \notin \mathbb{Q}$, then $\varepsilon(L(\mu)) \notin \mathbb{Q}$. See Conjecture 3.6 and Theorem 4.1.

Theorem 2.5. *Let $r \geq 1$ and let $\mu \in \mathbb{R}$. Then we have the following.*

- (1) *For any r , $L(\mu)$ admits a weakly submaximal curve for all $\mu \geq \sqrt{r+1}$.*
- (2) *If $r = 10$, then $L(\mu)$ admits a weakly submaximal curve for all $\mu \geq 77/24 \approx 3.208$.*

- (3) If $r = 11$, then $L(\mu)$ admits a weakly submaximal curve for all $\mu \geq 4 - \frac{\sqrt{3}}{3} \approx 3.422$.
(4) If $r = 13$, then $L(\mu)$ admits a weakly submaximal curve for all $\mu \geq \frac{1}{6}(26 - \sqrt{13}) \approx 3.732$.

Proof. (1) As in the proof of Theorem 2.2, an exceptional divisor E_i and a point $x \in E_i$ give the required weakly submaximal curve.

(2) For $r = 10$, consider the linear system of curves of class

$$10H - 4E_1 - 3 \sum_{i=2}^{10} E_i.$$

The expected dimension of this linear system is 1, so we can pick a pencil of curves of this class (in fact the system is nonspecial and there is precisely a pencil of curves of this class, but we will not need this). We claim the pencil has a singular member. Indeed, after blowing up the k base points of the pencil, we have a morphism $Y \rightarrow \mathbb{P}^1$. If all the fibers are smooth, they have arithmetic genus $3 = \binom{9}{2} - \binom{4}{2} - 9\binom{3}{2}$ and topological Euler characteristic -4 . Then $\chi_{\text{top}}(Y) = 2 \cdot (-4) = -8$ (see [7, Theorem 7.17]), but also

$$\chi_{\text{top}}(Y) = \chi_{\text{top}}(\mathbb{P}^2) + k = 3 + k.$$

This contradiction shows that there must be a singular member of the pencil.

Let C be a singular member of this pencil with singularity at $x \in C$. Then (C, x) gives a weakly submaximal curve for $L(\mu)$ if

$$\frac{10\mu - 31}{2} = \frac{L(\mu) \cdot C}{2} \leq \sqrt{L(\mu)^2} = \sqrt{\mu^2 - 10},$$

and this inequality holds for $\mu \in [\frac{77}{24}, \frac{13}{4}]$.

Since $\frac{13}{4} < \sqrt{11}$, we need to give a different weakly submaximal curve for $L(\mu)$ when $\mu \in (\frac{13}{4}, \sqrt{11})$. Consider a cubic through 9 of the 10 points, as in the proof of Theorem 2.2 in the $r = 9$ case. This gives a weakly submaximal curve for $L(\mu)$ if

$$3\mu - 9 = L(\mu) \cdot C \leq \sqrt{L(\mu)^2} = \sqrt{\mu^2 - 10},$$

and this inequality holds for $\mu \in [\frac{13}{4}, \frac{7}{2}]$. Thus, $L(\mu)$ admits a weakly submaximal curve for all $\mu \geq \frac{77}{24}$.

(3) For $r = 11$, there is a pencil of curves of class

$$4H - 2E_1 - \sum_{i=2}^{11} E_i.$$

By a similar computation as in the case $r = 10$, this pencil contains a singular curve C with a singular point $x \in C$. The pair (C, x) gives a weakly submaximal curve if

$$\frac{4\mu - 12}{2} = L(\mu) \cdot C \leq \sqrt{L(\mu)^2} = \sqrt{\mu^2 - 11},$$

and this inequality holds for $\mu \in [4 - \frac{\sqrt{3}}{3}, 4 + \frac{\sqrt{3}}{3}]$. Since $4 + \frac{\sqrt{3}}{3} > \sqrt{12}$, we are done.

(4) Finally, for $r = 13$, there is a pencil of curves of class

$$4H - \sum_{i=1}^{13} E_i.$$

Again as above, the pencil has a singular member C with singularity $x \in C$. It gives a weakly submaximal curve so long as $\mu \in [\frac{1}{6}(26 - \sqrt{13}), \frac{1}{6}(26 + \sqrt{13})]$, and since $\frac{1}{6}(26 + \sqrt{13}) > \sqrt{14}$ we are done. \square

3. A GENERALIZED SHGH CONJECTURE

In Theorem 2.5, we established upper bounds on the submaximality threshold. Conversely, to produce lower bounds on the submaximality threshold it is necessary to show that there are no weakly submaximal curves. We state a generalization of the SHGH conjecture which would guarantee that such curves cannot exist.

3.1. The SHGH conjecture. Suppose that we have integers $d \geq 0$ and $m_1, \dots, m_r \geq 0$. Consider the linear series

$$\mathcal{L} = |dH - m_1E_1 - \dots - m_rE_r|$$

on a general blowup $X = \text{Bl}_{p_1, \dots, p_r} \mathbb{P}^2$. The *expected dimension* of the series is defined to be

$$\text{edim } \mathcal{L} = \max \left\{ \binom{d+2}{2} - \sum_i \binom{m_i+1}{2} - 1, -1 \right\},$$

and the series is *nonspecial* if $\dim \mathcal{L} = \text{edim } \mathcal{L}$. There are many statements equivalent to the SHGH conjecture, but the following version is relevant for our purposes.

Conjecture 3.1 (SHGH). *If \mathcal{L} is special, then every divisor in \mathcal{L} is nonreduced.*

The contrapositive statement “if there is a reduced curve in \mathcal{L} then \mathcal{L} is nonspecial” is also often useful. Also note that if we add a very general simple point to the linear system \mathcal{L} , then the dimension and expected dimension drop exactly by 1. More precisely, we have $\dim \mathcal{L}' = (\dim \mathcal{L}) - 1$ and $\text{edim } \mathcal{L}' = (\text{edim } \mathcal{L}) - 1$, where \mathcal{L}' is the linear system $|dH - m_1E_1 - \dots - m_rE_r - E_{r+1}|$ on a general blow up $\text{Bl}_{p_1, \dots, p_r, p_{r+1}} \mathbb{P}^2$. Hence if Conjecture 3.1 is only stated for systems with $\text{edim } \mathcal{L} = -1$, then by imposing additional simple points the full conjecture follows.

More refined versions of Conjecture 3.1 discuss the structure of the base locus of \mathcal{L} more carefully and seek to completely classify the special systems. These various refinements have been stated and compared by various authors including Segre [16], Harbourne [11], Gimigliano [8] and Hirschowitz [14]. The various formulations are equivalent. See [3, 12] for more details.

The following stronger version of the SHGH conjecture easily follows from a conjecture attributed to Hirschowitz in [1, Conjecture 4.9]. It is also mentioned in [3, Conjecture 3.1 (iv)].

Conjecture 3.2. *If the general curve $C \in \mathcal{L}$ is reduced, then \mathcal{L} is nonspecial and C is smooth on X .*

More precisely, a slightly weaker version of the original conjecture from [1] reads as follows.

Conjecture 3.3 (Hirschowitz [1, Conjecture 4.9]). *Suppose \mathcal{L} is nonempty and nonspecial, and let $C \in \mathcal{L}$ be general. Suppose $p_a(C) \geq 0$ and C is reduced. Then C is smooth and irreducible on X .*

Remark 3.4. Let us show that Conjectures 3.1 and 3.3 imply Conjecture 3.2. By imposing additional simple points, it suffices to check Conjecture 3.2 in the case where $\text{edim } \mathcal{L} = 0$. Let $C \in \mathcal{L}$ be general and suppose it is reduced. By Conjecture 3.1, \mathcal{L} is nonspecial. If C is

irreducible, then $p_a(C) \geq 0$ and C is smooth by Conjecture 3.3. Suppose C is not irreducible. Then $C = C' + C''$ for some curves $C' \in \mathcal{L}'$ and $C'' \in \mathcal{L}''$. Since $\text{edim } \mathcal{L} = 0$ and C is reduced, we have $\mathcal{L} = \{C\}$ and therefore $\mathcal{L}' = \{C'\}$ and $\mathcal{L}'' = \{C''\}$. By Conjecture 3.1, we have $\text{edim } \mathcal{L}' = \text{edim } \mathcal{L}'' = 0$ and

$$\text{edim } \mathcal{L} = \text{edim } \mathcal{L}' + \text{edim } \mathcal{L}'' + C' \cdot C''.$$

Therefore $C' \cdot C'' = 0$, and if C' and C'' are smooth then so is C . By induction on the number of irreducible components, C is smooth.

3.2. A generalized SHGH conjecture. We now state a stronger SHGH conjecture by studying the loci in $\mathcal{L} = |dH - m_1E_1 - \cdots - m_rE_r|$ of curves with a singularity of some multiplicity $t \geq 2$. Fix a point $x \in X$. Then the expected codimension in \mathcal{L} of curves with a singularity of multiplicity t at x is $\binom{t+1}{2}$. As the point $x \in X$ varies, the expected codimension in \mathcal{L} of curves with a singularity of multiplicity t at some point is $\binom{t+1}{2} - 2$.

Various examples show that it is too much to hope for that the locus in \mathcal{L} of curves with a t -uple point always has the expected codimension. But, the source of these counterexamples seems to be nonreduced curves in the series.

Example 3.5. For example, let $r = 8$ and consider the series

$$\mathcal{L} = |6H - 2 \sum_{i=1}^8 E_i|.$$

The SHGH conjecture implies that $\dim \mathcal{L} = 28 - 24 - 1 = 3$. The expected codimension in \mathcal{L} of curves with a 4-uple point is $\binom{5}{2} - 2 = 8$, so we would expect that there are not any such curves. On the other hand, in the pencil of cubics through the 8 points there is a singular cubic, and its square is a member of \mathcal{L} with a 4-uple point.

In general, the locus in \mathcal{L} of nonreduced curves can be quite large and contain highly singular curves, but it seems possible that this is the only source of unexpectedly singular curves in linear series. We make the following conjecture.

Conjecture 3.6. *Let X be a blow up of \mathbb{P}^2 at $r \geq 0$ very general points. Suppose $d \geq 1$, $t \geq 1$, and $m_1, \dots, m_r \geq 0$ are integers such that*

$$\binom{d+2}{2} - \sum_{i=1}^r \binom{m_i+1}{2} \leq \max \left\{ \binom{t+1}{2} - 2, 0 \right\}.$$

Then any curve $C \in \mathcal{L}$ which has a point of multiplicity t is non-reduced.

Some initial cases of Conjecture 3.6 are well-known. In particular, the case $t = 1$ is equivalent to the $\text{edim } \mathcal{L} = -1$ case of Conjecture 3.1, so it is equivalent to Conjecture 3.1. When $t = 2$, the conjecture is the $\text{edim } \mathcal{L} = 0$ case of Conjecture 3.2, so it is equivalent to Conjecture 3.2.

Remark 3.7. We could weaken Conjecture 3.6 by changing the conclusion to “Then any curve $C \in \mathcal{L}$ which has a point of multiplicity t is non-reduced or non-irreducible.” This weakened version would still be strong enough to carry out the arguments in the next section. We highlight the stronger version instead since it is more analogous to the SHGH and Hirschowitz conjectures 3.1 and 3.2.

4. THE SUBMAXIMALITY THRESHOLD FOR 10 OR MORE POINTS

For the rest of the paper, we assume that Conjecture 3.6 is true. Under this assumption, we prove that Theorem 2.5 is sharp.

Theorem 4.1. *Suppose Conjecture 3.6 is true, and let $r \geq 10$. Then the submaximality threshold μ_0 for r exists, and*

$$\mu_0 = \begin{cases} \frac{77}{24} & \text{if } r = 10 \\ 4 - \frac{\sqrt{3}}{3} & \text{if } r = 11 \\ \frac{1}{6}(26 - \sqrt{13}) & \text{if } r = 13 \\ \sqrt{r+1} & \text{if } r = 12 \text{ or } r \geq 14. \end{cases}$$

If there is a weakly submaximal curve for $L(\mu)$ then there is an irreducible and reduced curve C and a point $x \in C$ such that

$$\frac{L(\mu) \cdot C}{\text{mult}_x C} \leq \sqrt{L(\mu)^2}.$$

Suppose C is not an exceptional divisor E_i . Then we can write $\mathcal{O}_X(dH - \sum m_i E_i) = \mathcal{O}_X(C)$ with $d > 0$ and $m_i \geq 0$, and we let $t = \text{mult}_x C$, so $1 \leq t \leq d$. Then by Conjecture 3.6 we have the simultaneous inequalities

$$(1) \quad \frac{\mu d - \sum_i m_i}{t} \leq \sqrt{\mu^2 - r}$$

$$(2) \quad \binom{d+2}{2} - \sum_{i=1}^r \binom{m_i+1}{2} > \max \left\{ \binom{t+1}{2} - 2, 0 \right\}.$$

In this section we prove the following result, which proves Theorem 4.1.

Proposition 4.2. *Let $r \geq 10$, and let μ_0 be the number in the statement of Theorem 4.1. If $\sqrt{r} \leq \mu < \mu_0$, then there is no pair (C, t) consisting of a curve class $C = dH - \sum_i m_i E_i$ and an integer $1 \leq t \leq d$ which satisfies (1) and (2).*

4.1. Bounding the multiplicities. Let $(dH - \sum_i m_i E_i, t)$ be a pair satisfying (1) and (2), and let $\bar{m} = \frac{1}{r} \sum_i m_i \in \mathbb{Q}$ be the average multiplicity. In this section we bound \bar{m} and t , in order to decrease the space in which we have to search for solutions to (1) and (2).

From (1) and (2) and Cauchy-Schwarz we conclude

$$(3) \quad \frac{\mu d - r\bar{m}}{t} \leq \sqrt{\mu^2 - r}$$

$$(4) \quad (d+2)(d+1) - r(\bar{m}+1)\bar{m} > (t+1)t - 4.$$

Rearrange (3) to get

$$d \leq \frac{r\bar{m} + t\sqrt{\mu^2 - r}}{\mu}.$$

Now we substitute this inequality into (4) and rearrange the terms to get a quadratic in \bar{m} and t :

$$Q(\bar{m}, t) := \left(\frac{r^2}{\mu^2} - r \right) \bar{m}^2 + \frac{2r\sqrt{\mu^2 - r}}{\mu^2} \bar{m}t - \frac{r}{\mu^2} t^2 + \left(\frac{3r}{\mu} - r \right) \bar{m} + \left(\frac{3\sqrt{\mu^2 - r}}{\mu} - 1 \right) t + 6 > 0.$$

The equation $Q(\bar{m}, t) = 0$ defines a parabola in the (\bar{m}, t) -plane, since the discriminant of the homogeneous degree 2 part is

$$\left(\frac{2r\sqrt{\mu^2 - r}}{\mu^2}\right)^2 + 4\left(\frac{r^2}{\mu^2} - r\right)\frac{r}{\mu^2} = 0.$$

Lemma 4.3. *Let $r \geq 10$. If $\sqrt{r} \leq \mu \leq \sqrt{r+1}$, then the inequalities $t \geq 1$, $\bar{m} \geq 0$, and $Q(\bar{m}, t) > 0$ describe a bounded region in the (\bar{m}, t) -plane. In particular, it is contained in the strip defined by the inequalities*

$$0 \leq \bar{m} \leq \frac{25}{4r - 12\sqrt{r}},$$

and if t is an integer then

$$\begin{aligned} t \in \{1, 2, 3, 4, 5\} & \quad \text{if } r = 10 \\ t \in \{1, 2, 3, 4\} & \quad \text{if } r = 11 \\ t \in \{1, 2, 3\} & \quad \text{if } r = 12 \\ t \in \{1, 2\} & \quad \text{if } r \geq 13. \end{aligned}$$

Proof. First notice that the point $(\bar{m}, t) = (0, 1)$ is in the region, since

$$Q(0, 1) = 5 - \frac{r}{\mu^2} + \frac{3\sqrt{\mu^2 - r}}{\mu} > 0$$

since $\mu > \sqrt{r}$.

Next we establish the bound on \bar{m} . View $\bar{m} > 0$ as fixed and consider the discriminant $\Delta_t(\bar{m})$ of the polynomial $Q(\bar{m}, t)$ of t :

$$\Delta_t(\bar{m}) = \frac{1}{\mu^2} \left(-(4r^2 - 12r\mu + 4r\sqrt{\mu^2 - r})\bar{m} + (15r + 10\mu^2 - 6\mu\sqrt{\mu^2 - r}) \right)$$

Then $\Delta_t(\bar{m})$ is decreasing in \bar{m} since $r \geq 10$ and $\mu^2 < r + 1$, and $\Delta_t(0) > 0$. For

$$\bar{m}_0(\mu) := \frac{15r + 10\mu^2 - 6\mu\sqrt{\mu^2 - r}}{4r^2 - 12r\mu + 4r\sqrt{\mu^2 - r}} > 0,$$

we have $\Delta_t(\bar{m}_0(\mu)) = 0$, so the parabola $Q(\bar{m}, t) = 0$ is tangent to and left of the vertical line $m = \bar{m}_0(\mu)$. The numerator in the quotient defining $\bar{m}_0(\mu)$ is decreasing in μ on $[\sqrt{r}, \sqrt{r+1}]$, and the denominator in the quotient is increasing in μ on $[\sqrt{r}, \sqrt{r+1}]$. This can be seen by differentiating the numerator and denominator with respect to μ and determining the signs of the derivatives on $[\sqrt{r}, \sqrt{r+1}]$. Thus $\bar{m}_0(\mu)$ is maximized on $[\sqrt{r}, \sqrt{r+1}]$ when $\mu = \sqrt{r}$, and for $\mu \in [\sqrt{r}, \sqrt{r+1}]$ we have

$$\bar{m}_0(\mu) \leq \frac{25}{4r - 12\sqrt{r}}.$$

Thus the region described by $Q(\bar{m}, t) > 0$ lies left of the line $\bar{m} = 25/(4r - 12\sqrt{r})$.

Suppose $t_0 > 1$ is a number such that $Q(\bar{m}, t_0) < 0$ for all $\bar{m} \geq 0$. Since $Q(0, 1) > 0$, the parabola $Q(\bar{m}, t) = 0$ crosses the t -axis at a point $(0, t_1)$ between $(0, 1)$ and $(0, t_0)$. Since the parabola is tangent to $\bar{m} = \bar{m}_0(\mu)$ at some point, the only possibility is that the point of tangency lies below the line $t = t_0$. Then it follows that the region bounded by $t \geq 1$, $\bar{m} \geq 0$, and $Q(\bar{m}, t) > 0$ is contained in the half-space $t \leq t_0$.

Thus to complete the proof, we must show that for all $\bar{m} \geq 0$ and $\sqrt{r} \leq \mu \leq \sqrt{r+1}$,

$$\begin{aligned} Q(\bar{m}, 6) &< 0 & \text{if } r = 10 \\ Q(\bar{m}, 5) &< 0 & \text{if } r = 11 \\ Q(\bar{m}, 4) &< 0 & \text{if } r = 12 \\ Q(\bar{m}, 3) &< 0 & \text{if } r \geq 13. \end{aligned}$$

Proving these inequalities is best left to the computer; for a given r and t_0 it is straightforward to maximize $Q(\bar{m}, t_0)$ on the region of (\bar{m}, μ) with $\bar{m} \geq 0$ and $\sqrt{r} \leq \mu \leq \sqrt{r+1}$. We carried this out to check the inequalities for $r \leq 19$.

Once $r \geq 20$, we can give a straightforward argument. For $\bar{m} \geq 0$ and $\sqrt{r} \leq \mu \leq \sqrt{r+1}$, we compute

$$\begin{aligned} -Q(\bar{m}, 3) &= \left(r - \frac{r^2}{\mu^2}\right) \bar{m}^2 + \left(r - \frac{3r}{\mu} - \frac{6r\sqrt{\mu^2 - r}}{\mu^2}\right) \bar{m} + \left(-3 - \frac{9\sqrt{\mu^2 - r}}{\mu} + \frac{9r}{\mu^2}\right) \\ &\geq \left(r - \frac{r^2}{r^2}\right) \bar{m}^2 + \left(r - \frac{3r}{\sqrt{r}} - \frac{6r}{r}\right) \bar{m} + \left(-3 - \frac{9}{\sqrt{r}} + \frac{9r}{r+1}\right) \\ &= (r - 3\sqrt{r} - 6)\bar{m} + \left(\frac{9r}{r+1} - \frac{9}{\sqrt{r}} - 3\right). \end{aligned}$$

Both coefficients of this linear polynomial are positive since $r \geq 20$, so $Q(\bar{m}, 3) < 0$ for all $\bar{m} \geq 0$. \square

4.2. Critical pairs. Suppose $(dH - \sum_i m_i E_i, t)$ satisfies (1) and (2). Write the multiplicities in decreasing order $m_1 \geq m_2 \geq \dots \geq m_r$. If $m_1 - m_r \geq 2$, we can replace m_1 by $m_1 - 1$ and m_r by $m_r + 1$. Then the resulting class still satisfies (1) and (2). Thus, if Proposition 4.2 is false, we can find a counterexample (C, t) where C is a *balanced curve class* of the form

$$(5) \quad dH - m(E_1 + \dots + E_s) - (m-1)(E_{s+1} + \dots + E_r)$$

We can compactly record a balanced class by the tuple $(d; m^s, (m-1)^{r-s})$, where $s > 0$ is as in (5).

Similarly, if we can increase either m_r or t by 1 without making (2) false, then inequality (1) only improves. Thus, to determine pairs (C, t) such that (1) and (2) are true, it is enough to study pairs where (2) is true and increasing either m_r or t makes (2) false. We call a pair

$$(C, t) = ((d; m^s, (m-1)^{r-s}), t)$$

consisting of a balanced curve class C and an integer $1 \leq t \leq d$ *critical* if it has this property.

Given integers $d \geq 1$ and $t \geq 1$, there is at most one critical pair $((d; m^s, (m-1)^{r-s}), t)$. Since t and the average multiplicity $\bar{m} = \frac{1}{r} \sum m_i$ have been bounded, there are not so many pairs to check for a given r . For each critical pair, we have to check that inequality (1) is false if $\mu < \mu_0$. Write $M = sm + (r-s)(m-1) = r\bar{m}$ for short. Then inequality (1) reads

$$d\mu - M \leq t\sqrt{\mu^2 - r}.$$

Both sides of the inequality are positive, so squaring both sides and rearranging shows this is equivalent to

$$(6) \quad R(\mu) := (d^2 - t^2)\mu^2 - 2dM\mu + (M^2 + t^2r) \leq 0.$$

Remark 4.4. Suppose (C, t) is a critical pair with $d = t$. Then if $t > 1$, the curve C is reducible, so one of its components computes a Seshadri quotient which is at least as small as that given by (C, t) . Thus the only pair we have to consider in this case is $((1; 1^2), 1)$, and in this case inequality (6) reads

$$-4\mu + 4 + r \leq 0,$$

which is only satisfied for

$$\mu \geq 1 + \frac{r}{4}.$$

For $r \geq 9$, it is easy to check that $1 + \frac{r}{4} > \sqrt{r+1}$. This gives $\mu > \sqrt{r+1}$. Since $\mu_0 \leq \sqrt{r+1}$, we conclude that inequality (6) (and hence inequality (1)) is false for any $\mu < \mu_0$. Therefore, we may assume $t < d$ in what follows.

For $t < d$, the graph of $R(\mu)$ is an upward parabola. Therefore inequality (6) is false for $\mu < \mu_0$ if either $R(\mu) = 0$ has no real roots, or if the smaller root is at least μ_0 . If $M^2 - r(d^2 - t^2) \geq 0$, let

$$\mu_- = \frac{dM - t\sqrt{M^2 - r(d^2 - t^2)}}{d^2 - t^2}$$

be the smaller of the two real roots. For each critical pair such that μ_- is defined, we check $\mu_- \geq \mu_0$.

Example 4.5. Let $r = 12$. In Table 1, we list all the critical pairs $((d; m^s, (m-1)^{12-s}), t)$ which are consistent with Lemma 4.3. According to the lemma, $t \in \{1, 2, 3\}$ and the total multiplicity M is bounded by 46. For each t , we increase d and list any corresponding critical class until M would exceed this bound. Whenever it is defined, we compute μ_- for each critical pair for $r = 12$. In fact, $R(\mu) = 0$ has no real roots for the vast majority of critical pairs (C, t) , and $\mu_- = 4 > \mu_0 = \sqrt{13}$ whenever it is defined. We also specify for which critical pairs μ_- does not exist (we write ‘‘DNE’’ for short). This proves Proposition 4.2 for $r = 12$.

For fixed r , it is easy to program a computer to generate the list of critical pairs (C, t) which satisfy the conclusion of Lemma 4.3. Then we can check that $\mu_- \geq \mu_0$ whenever μ_- is defined. We did this for $10 \leq r \leq 19$. On the other hand, once $r \geq 20$ we can give an argument that requires minimal computation.

Proposition 4.6. *Proposition 4.2 is true for $r \geq 20$.*

Proof. Suppose a critical pair $(C, t) = ((d; m^s, (m-1)^{r-s}), t)$ violates Proposition 4.2. Then Lemma 4.3 shows $t \in \{1, 2\}$ and $\bar{m} < 1$. For the last inequality, we use the hypothesis $r \geq 20$. Therefore $m = 1$ and $M = s < r$.

Note that the inequality (2) must be as sharp as possible for $((d; 1^s, 0^{r-s}), t)$; in other words, we have an equality $\binom{d+2}{2} - M = \max\{\binom{t+1}{2} - 2, 0\} + 1$. Indeed, if this fails then the inequality (2) is also satisfied by $((d; 1^{s+1}, 0^{r-s-1}), t)$, which contradicts the hypothesis that $((d; 1^s, 0^{r-s}), t)$ is critical.

Since $t \in \{1, 2\}$, it follows that

$$M = \frac{(d+2)(d+1)}{2} - t.$$

But then we claim that

$$M^2 - r(d^2 - t^2) < 0,$$

TABLE 1. Critical pairs for $r = 12$.

C	t	M	$M^2 - r(d^2 - t^2)$	μ_-
$(2; 1^5)$	1	5	-11	DNE
$(3; 1^9)$	1	9	-15	DNE
$(4; 2^1, 1^{11})$	1	13	-11	DNE
$(5; 2^4, 1^8)$	1	16	-32	DNE
$(9; 3^6, 2^6)$	1	30	-60	DNE
$(13; 4^8, 3^4)$	1	44	-80	DNE
$(3; 1^8)$	2	8	4	4
$(4; 1^{12})$	2	12	0	4
$(5; 2^3, 1^9)$	2	15	-27	DNE
$(6; 2^7, 1^5)$	2	19	-23	DNE
$(7; 2^{11}, 1^1)$	2	23	-11	DNE
$(8; 3^2, 2^{10})$	2	26	-44	DNE
$(9; 3^5, 2^7)$	2	29	-83	DNE
$(10; 3^9, 2^3)$	2	33	-63	DNE
$(11; 4^1, 3^{11})$	2	37	-35	DNE
$(12; 4^4, 3^8)$	2	40	-80	DNE
$(4; 1^{10})$	3	10	16	4
$(5; 2^2, 1^{10})$	3	14	4	4
$(6; 2^5, 1^7)$	3	17	-35	DNE
$(7; 2^9, 1^3)$	3	21	-39	DNE
$(8; 3^1, 2^{11})$	3	25	-35	DNE
$(9; 3^4, 2^8)$	3	28	-80	DNE
$(10; 3^8, 2^4)$	3	32	-68	DNE
$(11; 3^{12})$	3	36	-48	DNE
$(12; 4^3, 3^9)$	3	39	-99	DNE
$(13; 4^7, 3^5)$	3	43	-71	DNE
$(14; 4^{10}, 3^2)$	3	46	-128	DNE

so that $R(\mu) = 0$ has no real roots. If $d < 5$ then the only critical pairs are $((2; 1^5), 1)$, $((3; 1^9), 1)$, $((4; 1^{14}), 1)$, $((3; 1^8), 2)$, and $((4; 1^{13}), 2)$, and the inequality holds in these cases since $r \geq 20$. So, assume $d \geq 5$.

Now since $t \in \{1, 2\}$ and $d \geq 5$,

$$\frac{d^2 - t^2}{M} > \frac{2(d^2 - 4)}{(d+2)(d+1)} = \frac{2(d-2)}{(d+1)} \geq 1 > \frac{M}{r},$$

and therefore $M^2 - r(d^2 - t^2) < 0$. □

Example 4.7. Let $r = 10$ and let $L = 16H - 5E$. Then L is ample by [6], see also [9, Theorem 2.18]. After normalizing, we have $\mu = 3.2$. Suppose that Conjecture 3.6 is true. Since $\mu < 77/24 \approx 3.208$, by Theorem 4.1, there are no weakly submaximal curves for $L(\mu)$. Since $\sqrt{L(\mu)^2} = \sqrt{1.24} \notin \mathbb{Q}$, it follows that $\varepsilon(L(\mu)) \notin \mathbb{Q}$. Hence $\varepsilon(L) \notin \mathbb{Q}$.

REFERENCES

- [1] C. Ciliberto, Geometric aspects of polynomial interpolation in more variables and of Waring's problem, in *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, 289–316, Progr. Math., 201, Birkhäuser, Basel.
- [2] C. Ciliberto and R. Miranda, Degenerations of planar linear systems, *J. Reine Angew. Math.* **501** (1998), 191–220.
- [3] C. Ciliberto and R. Miranda, The Segre and Harbourne-Hirschowitz conjectures, in *Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001)*, 37–51, NATO Sci. Ser. II Math. Phys. Chem., 36, Kluwer Acad. Publ., Dordrecht.
- [4] J.-P. Demailly, Singular Hermitian metrics on positive line bundles, in *Complex algebraic varieties (Bayreuth, 1990)*, 87–104, Lecture Notes in Math., 1507, Springer, Berlin.
- [5] M. Dumnicki, A. Küronya, C. Maclean and T. Szemberg, Rationality of Seshadri constants and the Segre-Harbourne-Gimigliano-Hirschowitz conjecture, *Adv. Math.* **303** (2016), 1162–1170.
- [6] T. Eckl, Ciliberto-Miranda degenerations of \mathbb{CP}^2 blown up in 10 points, *J. Pure Appl. Alg.* **215** (2011), 672–696.
- [7] D. Eisenbud and J. Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016.
- [8] A. Gimigliano, *On linear systems of plane curves*, Ph.D. Thesis, Queen's University, Kingston, (1987).
- [9] K. Hanumanthu, Positivity of line bundles on general blow ups of \mathbb{P}^2 , *J. Algebra* **461** (2016), 65–86.
- [10] K. Hanumanthu and B. Harbourne, Single point Seshadri constants on rational surfaces, *J. Algebra* **499** (2018), 37–42.
- [11] B. Harbourne, The geometry of rational surfaces and Hilbert functions of points in the plane, in *Proceedings of the 1984 Vancouver conference in algebraic geometry*, 95–111, CMS Conf. Proc., 6, Amer. Math. Soc., Providence, RI.
- [12] B. Harbourne, Problems and progress: a survey on fat points in \mathbf{P}^2 , in *Zero-dimensional schemes and applications (Naples, 2000)*, 85–132, Queen's Papers in Pure and Appl. Math., 123, Queen's Univ., Kingston, ON.
- [13] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in Mathematics, Vol. 156, Springer-Verlag, Berlin, 1970.
- [14] A. Hirschowitz, Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles génériques, *J. Reine Angew. Math.* **397** (1989), 208–213.
- [15] K. Oguiso, Seshadri constants in a family of surfaces, *Math. Ann.* **323** (2002), no. 4, 625–631.
- [16] B. Segre, Alcune questioni su insiemi finiti di punti in geometria algebrica, in *Atti Convegno Internaz. Geometria Algebrica (Torino, 1961)*, 15–33, Rattero, Turin.
- [17] B. Strycharz-Szemberg and T. Szemberg, Remarks on the Nagata conjecture, *Serdica Math. J.* **30** (2004), no. 2-3, 405–430.

DEPARTMENT OF MATHEMATICS, PEDAGOGICAL UNIVERSITY OF CRACOW, PODCHORĄŻYCH 2, PL-30-084 KRAKÓW, POLAND

E-mail address: Lucja.Farnik@gmail.com

CHENNAI MATHEMATICAL INSTITUTE, H1 SIPCOT IT PARK, SIRUSERI, KELAMBAKKAM 603103, INDIA

E-mail address: krishna@cmi.ac.in

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802

E-mail address: huizenga@psu.edu

MATHEMATISCHES INSTITUT, UNIVERSITÄT BAYREUTH, D-95440 BAYREUTH

E-mail address: schmitzd@mathematik.uni-marburg.de

DEPARTMENT OF MATHEMATICS, PEDAGOGICAL UNIVERSITY OF CRACOW, PODCHORĄŻYCH 2, PL-30-084 KRAKÓW, POLAND

E-mail address: tomasz.szemberg@gmail.com