RATIONALLY CONNECTED VARIETIES

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Contents

1. Introduction

Recall that a variety X of dimension n is called *rational* if there is a birational map $\mathbb{P}^n \dashrightarrow X$, and *unirational* if there is a dominant map $\mathbb{P}^n \dashrightarrow X$. In the birational classification of curves and surfaces, these notions play a primary role; in each case

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the birational isomorphism classes of rational curves and surfaces are the simplest to describe. Rationality of a curve C is detected by the vanishing of the genus, while rationality of a surface S is detected by the vanishing of the second plurigenus $H^0(S, K_S^{\otimes 2})$ and the Hodge number $h^{1,0}(S)$. Moreover, in the case of curves and surfaces, the notions of rationality and unirationality coincide.

Unfortunately, when one moves into higher dimensions, these notions are not so well behaved. In dimensions 3 and greater, rationality and unirationality are no longer equivalent: the smooth cubic 3-fold in $\mathbb{P}^4$ is unirational, but not rational. It is also not known whether rationality is either a closed or an open property in families. To make matters worse, it is notoriously difficult to prove that a given variety is not rational or unirational. The proof that the smooth cubic 3-fold is not rational works by a miraculous argument involving its intermediate Jacobian; it does not seem to generalize to higher dimensions or degrees. We don't know if the general smooth cubic fourfold is rational or not, and we don't know if there are any smooth rational hypersurfaces in $\mathbb{P}^n$ of degree at least 4.

The goal of this work is to explore the concept of *rationally connected varieties*, as introduced independently by Kollár, Miyaoka, Mori [7] and Campana [2]. This notion can be seen as a generalization of unirationality which is more local in nature and better behaved in families.

Definition 1.1. A smooth projective variety X over $\mathbb{C}$ is *rationally connected* if any two general points of X can be joined by a rational curve in X.

Perhaps somewhat surprisingly, this condition is equivalent to the a priori weaker condition that a general pair of points can be joined by a connected curve, all of whose components are rational. It is also equivalent to the a priori stronger statement that *any* finite set of points can be joined by a single rational curve $f : \mathbb{P}^1 \to X$ such that f^*T_X is ample. For one more equivalence, X will be rationally connected if there is a *single* rational curve $f : \mathbb{P}^1 \to X$ such that f^*T_X is ample. We shall see that these various equivalences imply that rational connectivity is both an open and closed property in smooth, proper families.

A unirational variety is in fact rationally connected, so this notion in fact generalizes unirationality. Moreover, for curves and surfaces, rationality coincides with rational connectivity.

Whereas little is known about the rationality of smooth hypersurfaces in $\mathbb{P}^n$, we have a complete classification of the smooth rationally connected hypersurfaces. A smooth degree d hypersurface in $\mathbb{P}^n$ is rationally connected if and only if $d \leq n$. The "if" statement follows from the more general fact that if $-K_X$ is ample (i.e. if X is a *Fano variety*) then X is rationally connected.

There is also a variety $R(X)$, called the *maximal rationally connected (MRC) quotient* of X, which measures the failure of X to be rationally connected. Intuitively, points of $R(X)$ roughly correspond to rationally chain-connected components of X. The variety X is rationally connected if and only if its MRC quotient is a point.

One final nice result about rationally connected varieties is that they behave well with respect to fibrations: if $X \to Y$ is a morphism with Y rationally connected and the general fiber X_y rationally connected, then X is rationally connected.

The structure of the paper is as follows. In Section 2, we discuss various parameter spaces of rational curves on a variety, and some results from deformation theory that will be needed in the rest of the paper. We prove the classification theorem for vector bundles on $\mathbb{P}^1$ in Section 3, as it and its proof will be useful later. We introduce the notion of a *free rational curve* on a variety in Section 4; these are the rational curves such that f^*T_X is generated by global sections, and their importance is that they possess lots of deformations. In Section 5, we study the notion of *uniruledness*, which is a weaker condition than rational connectivity but has a very similar flavor. We finally prove the equivalence of the various conditions equivalent to rational connectivity introduced above in Section 6.

In Section 7, we outline a construction of the MRC quotient alluded to above. We use this to sketch a proof of the fact that Fano varieties are rationally connected in Section 8. We conclude the paper in Section 9 with a sketch of the proof that if $X \to Y$ is a morphism with Y rationally connected and rationally connected general fiber, then X is rationally connected.

We would have liked to be able to fix our ground field $k = \mathbb{C}$ throughout the paper, to avoid having to discuss the ground field at all. Unfortunately, this just isn't possible. The proof that Fano varieties are rationally connected demands that we use characteristic p techniques. It will therefore be necessary to allow k to be an arbitrary algebraically closed field in Sections 2 and 8; we can get away with assuming $k = \mathbb{C}$ everywhere else. We will remind the reader in Sections 2 and 8 that the field is not necessarily $\mathbb{C}$; otherwise $k = \mathbb{C}$ will be implicit.

Being a minor thesis and a survey paper, I of course make absolutely no claim of originality in this work. Proofs of results in Sections 2 and 4 through 8 mostly follow the proofs of corresponding results in Debarre [3], often times with little change; we do not cite Debarre every time we are referencing it since that work would probably be cited a hundred times in this paper. The interested reader should perhaps use this paper in consultation with Debarre in order to get a second perspective on the material. When possible, I have modified proofs that were intended to work over any field into simpler proofs that only work in characteristic zero. This is usually done by applying some principal like generic smoothness when it isn't strictly necessary, but gives many arguments a more geometric flavor. I have also attempted to add extra explanation to proofs where I thought it was warranted, and removed or simplified material in other spots. There is a very real possibility that I have introduced errors into arguments which were not present in the original sources; this is entirely my own fault.

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2. Parameter spaces of rational curves

The notions of uniruledness and rational connectivity are concerned with the configuration of rational curves on a given variety. As such, studying these notions demands we have some way of collecting the set of rational curves on a variety into some sort of parameter space. There are a few different ways of doing this, each of which has its own advantages and disadvantages.

For this section, k is an arbitrary algebraically closed field.

2.1. **Basic definitions.** The simplest parameter space to describe is the space $\operatorname{Rat} X$, which by definition is the union of the components of the Hilbert scheme of X whose general points correspond to reduced, connected curves with rational components. If we denote by $\operatorname{Rat}_d X$ the union of those components of $\operatorname{Rat} X$ corresponding to curves of degree at most d (with respect to some fixed ample divisor on X), then there are only finitely many components and so $\operatorname{Rat}_d X$ is a projective scheme. This parameter space will play an important role in the construction of the MRC quotient in Section 7.

Unfortunately, the parameter space $\operatorname{Rat} X$ usually does not meet our needs. For many arguments, it is beneficial to think of an irreducible rational curve in X as being the image of $\mathbb{P}^1$ in X under some morphism. In particular, even if two morphisms $f, g : \mathbb{P}^1 \to X$ have the same image, we would like the morphisms f and g to correspond to different points in the parameter space. Thus we would like a parameter space $\operatorname{Mor}(\mathbb{P}^1, X)$ for morphisms $\mathbb{P}^1 \to X$. This can be constructed as the open subset of the Hilbert scheme $\operatorname{Hilb}(\mathbb{P}^1 \times X)$ consisting of graphs of morphisms $\mathbb{P}^1 \to X$. Given a fixed ample divisor H on X, this parameter space decomposes as a disjoint union

$$\operatorname{Mor}(\mathbb{P}^1, X) = \coprod_{d \geq 0} \operatorname{Mor}_d(\mathbb{P}^1, X)$$

according to the degree of the pullback of H to $\mathbb{P}^1$. The components $\mathrm{Mor}_d(\mathbb{P}^1, X)$ are quasi-projective schemes.

In more generality, we will need parameter spaces for maps from a projective variety Y to a quasi-projective variety X. In this case, $\mathrm{Mor}(Y, X)$ is a locally Noetherian scheme with countably many components, corresponding to the Hilbert polynomials $P(m) = \chi(Y, mf^*H)$ of the morphisms $f : Y \to X$. The scheme $\mathrm{Mor}(Y, X)$ satisfies the obvious universal property, so that in particular there is a bijection between morphisms of schemes $T \to \mathrm{Mor}(Y, X)$ and T-morphisms $Y \times T \to X \times T$. Taking $T = \mathrm{Mor}(Y, X)$ and the map $T \to \mathrm{Mor}(Y, X)$ to be the identity, we get the *universal morphism*

$$Y \times \mathrm{Mor}(Y, X) \to X \times \mathrm{Mor}(Y, X).$$

Composing with the first projection gives the *evaluation map*

$$\mathrm{ev} : Y \times \mathrm{Mor}(Y, X) \to X.$$

A couple other versions of the space $\mathrm{Mor}(Y, X)$ will be needed as well. First, if B is a given subscheme of Y, and $g : B \to X$ is any morphism, we would like a space parameterizing morphisms $Y \to X$ which restrict to g on B. This parameter space can be described as the fiber of $[g]$ under the restriction map $\mathrm{Mor}(Y, X) \to \mathrm{Mor}(B, X)$. It will be denoted by $\mathrm{Mor}(Y, X; g)$.

We will also need a relative version of this construction. If Y, X, and B are all flat S-schemes and $g : B \to X$ is an S-morphism, we can form an S-scheme

$$\mathrm{Mor}_S(Y, X; g) \to S$$

with fibers

$$\mathrm{Mor}_S(Y, X; g)_s \cong \mathrm{Mor}(Y_s, X_s; g_s).$$

That is, flat families of schemes have their corresponding parameter spaces of morphisms fit together fiber by fiber.

There is one other parameter space parameterizing maps from curves to a variety X. Let C be a connected, complete, at-worst-nodal curve, and let $p_1, \ldots, p_n$ be an ordered set of n nonsingular *marked points* of C. A map $f : C \to X$ is called *stable* if C has only finitely many marking-preserving automorphisms preserving f. If $\beta \in N_1(X)$ is the numerical equivalence class of a 1-cycle on X, then the *Kontsevich moduli stack* $\overline{\mathscr{M}}_{g,n}(X, \beta)$ is the Deligne-Mumford stack parameterizing flat families of stable maps $f : C \to X$ such that $f_*[C] = \beta$, where C is any (connected, complete, at-worst-nodal) curve of arithmetic genus g with n marked points. We will have more occasion to work with the corresponding coarse moduli space $\overline{M}_{g,n}(X, \beta)$. One classical fact about this moduli space that we will need in Section 9 is that the coarse moduli space $\overline{M}_{g,0}(\mathbb{P}^1, d)$

parameterizing flat families of degree d stable branched covers of $\mathbb{P}^1$ by arithmetic genus g curves has a unique irreducible component whose general member consists of a stable map from a smooth genus g curve [4].

2.2. **Local structure.** The technical heart of the paper relies on an understanding of the local structure of the various parameter spaces of morphisms. Our first result describes the Zariski tangent space of $\mathrm{Mor}(Y, X)$.

Proposition 2.1. *Let $f : Y \to X$ be a map of varieties, with X quasi-projective and Y projective as before. There is an isomorphism*

$$T_{\mathrm{Mor}(Y,X),[f]} \cong H^0(Y, \mathscr{H}om(f^*\Omega^1_X, \mathcal{O}_Y)).$$

In particular, if X is smooth along the image of f, then

$$T_{\mathrm{Mor}(Y,X),[f]} \cong H^0(Y, f^*T_X).$$

Proof. The Zariski tangent space to $\mathrm{Mor}(Y, X)$ at $[f]$ is a vector space parameterizing morphisms $\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2) \to \mathrm{Mor}(Y, X)$ centered at $[f]$. By the universal property of $\mathrm{Mor}(Y, X)$, these are the same as $\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2)$-morphisms

$$f_\varepsilon : Y \times \mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2) \to X \times \mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2)$$

extending f, i.e. first-order deformations of f.

First assume that Y and X are affine, with $Y = \mathrm{Spec}\, B$ and $X = \mathrm{Spec}\, A$ and corresponding morphism $f^\# : A \to B$. The product $\mathrm{Spec}\, B \times \mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2)$ is just $\mathrm{Spec}\, B[\varepsilon]/(\varepsilon)^2$, so first-order deformations of f correspond to $k[\varepsilon]/(\varepsilon)^2$-algebra homomorphisms $f^\#_\varepsilon : A[\varepsilon] \to B[\varepsilon]$ which restrict to $f^\#$ on A after composing with $B[\varepsilon] \to B$ given by $\varepsilon \mapsto 0$. That is, the first-order deformations are the $k[\varepsilon]/(\varepsilon^2)$-algebra homomorphisms which satisfy

$$f^\#_\varepsilon(a) = f(a) + \varepsilon g(a)$$

for $a \in A$. Then clearly the requirement that $f^\#_\varepsilon(a)$ be a ring homomorphism is equivalent to the statement that $g : A \to B$ is a k-derivation of the A-module B. Hence $g : A \to B$ factors uniquely as $A \to \Omega_A \to B$, and $\mathrm{Hom}_A(\Omega_A, B) = \mathrm{Hom}_B(\Omega_A \otimes_A B, B)$ parameterizes first-order deformations of $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$.

This local result is easily seen to glue into the required global result. □

In the case when X is smooth along the image of f, this result gives us a tractable upper bound on the dimension of $\mathrm{Mor}(Y, X)$ at $[f]$. Similar results hold for the parameter spaces $\mathrm{Mor}(Y, X; g)$ and $\mathrm{Mor}_S(Y, X; g)$, where $g : B \to X$ and B is a subscheme of Y (with the necessary flatness conditions being implicit in the second case).

Proposition 2.2. *Let $f : Y \to X$ be a map of varieties (Y projective and X quasi-projective) extending a given map $g : B \to X$, where B is a subscheme of Y. If X is smooth along the image of f, then*

$$T_{\mathrm{Mor}(Y,X;g),[f]} \cong H^0(Y, f^*T_X \otimes \mathscr{I}_B),$$

where $\mathscr{I}_B$ is the ideal sheaf of B in Y.

In case Y, X, and B are flat over some scheme S and g is an S-morphism, $f_s : Y_s \to X_s$ extends g_s, and X_s is smooth along the images of f_s, the tangent space to $\mathrm{Mor}_S(Y, X)$ *at $[f_s]$ is given by*

$$T_{\mathrm{Mor}_S(Y,X),[f_s]} \cong T_{S,s} \oplus H^0(Y_s, f_s^*T_{X_s} \otimes \mathscr{I}_{B_s}).$$

Proof. The space $\mathrm{Mor}(Y, X; g)$ is the fiber of $[g]$ under the restriction $\mathrm{Mor}(Y, X) \to \mathrm{Mor}(B, X)$. If X is smooth along the image of f, the differential of this map at $[f]$ is the restriction

$$H^0(Y, f^*T_X) \to H^0(B, g^*T_X).$$

As $\mathrm{Mor}(Y, X; g)$ is the fiber of $[g]$, its tangent space is the kernel of this map, which is exactly $H^0(f^*T_X \otimes \mathscr{I}_B)$.

The relative statement is clear. □

Our next result gives us a lower bound for the dimension of $\mathrm{Mor}(Y, X)$ at a point $[f]$ such that X is smooth along the image of f, and also a criterion for $\mathrm{Mor}(Y, X)$ to be smooth at $[f]$. The proof is somewhat long, owing to the large amount of commutative algebra underpinning it.

Theorem 2.3. *Let $f : Y \to X$ be a morphism of projective varieties, such that X is smooth along the image of f. Put $h^i = h^i(Y, f^*T_X)$. In a neighborhood of $[f]$, the scheme* $\mathrm{Mor}(Y, X)$ *can be defined by at most h^1 equations in a nonsingular variety of dimension h^0. In particular,*

(1) any irreducible component of $\mathrm{Mor}(Y, X)$ *through $[f]$ has dimension at least $h^0 - h^1$, and*

(2) if $h^1 = 0$, then $\mathrm{Mor}(Y, X)$ *is smooth at $[f]$.*

Proof. Some affine neighborhood of $[f]$ can be embedded in an affine space $\mathbb{A}^n_k$, where it is defined by global polynomial equations $P_1, \ldots, P_m$. Without loss of generality, the equations $P_1, \ldots, P_r$ correspond to a submatrix of the Jacobian matrix $(\partial P_i/\partial x_j)([f])$ with the same rank r as the full Jacobian matrix at $[f]$. Then $P_1, \ldots, P_r$ define a subvariety V of $\mathbb{A}^n_k$ containing a neighborhood of $[f]$ in $\mathrm{Mor}(Y, X)$. The subvariety V is smooth at $[f]$, and its Zariski tangent space at $[f]$ equals the Zariski tangent space of $\mathrm{Mor}(Y, X)$ at $[f]$. It follows that the dimension of V equals h^0.

Let $I \subset \mathcal{O}_{V,[f]} = R$ be the ideal of the germ of $\mathrm{Mor}(Y,X)$ at $[f]$. We will prove that I can be generated by h^1 elements. Letting $\mathfrak{m}$ be the maximal ideal of R, we have $I \subset \mathfrak{m}^2$, for both $\operatorname{Spec} R$ and $\operatorname{Spec} R/I$ have the same Zariski tangent space. Then by Nakayama's lemma, it will be enough to prove $\dim I/\mathfrak{m}I \leq h^1$.

As in the first paragraph of the proof of Proposition 2.1, the morphism $\operatorname{Spec} R/I \to \mathrm{Mor}(Y,X)$ gives by the universal property of $\mathrm{Mor}(Y,X)$ an extension

$$f_{R/I} : Y \times \operatorname{Spec} R/I \to X \times \operatorname{Spec} R/I$$

of f. By Lemma 2.5, since $I^2 \subset \mathfrak{m}I$ we know that the obstruction to extending $f_{R/I}$ to a morphism

$$f_{R/\mathfrak{m}I} : Y \times \operatorname{Spec} R/\mathfrak{m}I \to X \times \operatorname{Spec} R/\mathfrak{m}I$$

lies in

$$H^1(Y, f^*T_X) \otimes_k I/\mathfrak{m}I.$$

We write this obstruction as a sum of simple tensors

$$\sum_{i=1}^{h^1} a_i \otimes \overline{b_i},$$

where the a_i form a basis for $H^1(Y, f^*T_X)$ and the b_i are elements of I. By the naturality of the obstruction, it vanishes modulo $(b_1, \ldots, b_{h^1})$, which means that we can lift $f_{R/I}$ to a morphism

$$f_{R/(\mathfrak{m}I+(b_1,\ldots,b_{h^1}))} : Y \times \operatorname{Spec} R/(\mathfrak{m}I + (b_1, \ldots, b_{h^1})) \to X \times \operatorname{Spec} R/(\mathfrak{m}I + (b_1, \ldots, b_{h^1})).$$

Thus the morphism $\operatorname{Spec}(R/I) \to \mathrm{Mor}(Y,X)$ lifts to a morphism

$$\operatorname{Spec}(R/(\mathfrak{m}I + (b_1, \ldots, b_{h^1}))) \to \mathrm{Mor}(Y,X),$$

which is to say that the identity $R/I \to R/I$ factors as

$$R/I \to R/(\mathfrak{m}I + (b_1, \ldots, b_{h^1})) \xrightarrow{\pi} R/I,$$

where π is the natural projection. This implies that $\mathfrak{m}I + (b_1, \ldots, b_{h^1}) = I$ by a simple application of Nakayama's lemma, so $I/\mathfrak{m}I$ is spanned by $\overline{b_1}, \ldots, \overline{b_{h^1}}$ and has dimension at most h^1. □

Before proceeding with the lemma used in the course of the proof, we need a result which allows us to deduce that certain local infinitesimal extensions of maps exist. The proof of this result follows an outline suggested in Exercise II.8.6 of [5].

Lemma 2.4. *Let A be a finitely generated k-algebra such that* $\operatorname{Spec} A$ *is a nonsingular variety over k. Let $0 \to I \to B' \xrightarrow{\pi} B \to 0$ be an exact sequence, where B' is a k-algebra and I is an ideal with $I^2 = 0$. If $f : A \to B$ is a k-algebra homomorphism, then there exists a k-algebra homomorphism $g : A \to B'$ such that $g = \pi \circ f$. Moreover, differences of liftings of f are parameterized by* $\operatorname{Hom}_A(\Omega_{A/k}, I)$.

Proof. Since $I^2 = 0$, we can regard I as a B-module, hence as an A-module. We prove the second statement first. Suppose that $g, g' : A \to B'$ are both liftings of f. Put $\theta = g - g'$. We claim that θ is a k-derivation of A into I. Since g and g' are k-algebra maps, it is clear that θ is k-linear and $\theta(k) = 0$. Also $\theta(A) \subset I$ since $\pi(\theta(A)) = 0$. And θ satisfies the derivation property, since

$$\begin{aligned} \theta(aa') &= g(aa') - g'(aa') = g(a)g(a') - g'(a)g'(a') \\ &= g(a')(g(a) - g'(a)) + g'(a)(g(a') - g'(a')) \\ &= g(a')\theta(a) + g'(a)\theta(a') = a'\theta(a) + a\theta(a') \end{aligned}$$

by the definition of the A-module structure on I.

Conversely, suppose θ is an arbitrary k-derivation of A into I. We claim that for any lifting g of f, $g' = g + \theta$ is another lifting of f. Clearly g' is k-linear, and still lifts f since $\pi \circ \theta = 0$. It is an algebra homomorphism, since

$$\begin{aligned} (g+\theta)(aa') &= g(a)g(a') + a\theta(a') + a'\theta(a) \\ &= g(a)g(a') + g(a)\theta(a') + g(a')\theta(a) + \theta(a)\theta(a') = (g+\theta)(a)(g+\theta)(a') \end{aligned}$$

in light of the fact that $I^2 = 0$. We conclude that the differences of liftings of f are exactly the k-derivations of A into I, and are thus parameterized by $\operatorname{Hom}_A(\Omega_{A/k}, I)$.

Let $P = k[x_1, \ldots, x_n]$ be a polynomial ring with quotient $A = P/J$. There exists a homomorphism $h : P \to B'$ making a commutative square

$$\begin{array}{ccc} P & \xrightarrow{h} & B' \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ A & \xrightarrow{f} & B. \end{array}$$

Indeed, we simply let $y_i \in B'$ be an arbitrary element mapping to $f(\pi'(x_i))$ for each i, and then define a k-algebra map $P \to B'$ by sending x_i to y_i. Since the diagram commutes on generators, it commutes.

We claim that h induces an A-linear map $\overline{h} : J/J^2 \to I$. Note that J/J^2 has a natural structure of $P/J = A$-module. Since $\pi'(J) = 0$, we have $h(J) \subset I$. This implies

$h(J^2) \subset I^2 = 0$, so the map $h|_J : J \to I$ factors through J/J^2 to give $\overline{h} : J/J^2 \to I$. The map is A-linear, more or less by the definition of the A-module structure on I.

Now, because $\operatorname{Spec} A$ is nonsingular, we have an exact sequence

$$0 \to \mathscr{J}/\mathscr{J}^2 \to \Omega_{\operatorname{Spec} P/k} \otimes \mathcal{O}_{\operatorname{Spec} A} \to \Omega_{\operatorname{Spec} A/k} \to 0$$

of sheaves on A, where $\Omega_{\operatorname{Spec} A/k}$ is locally free. Since $\operatorname{Spec} A$ is affine and these sheaves are coherent, this yields an exact sequence

$$0 \to J/J^2 \xrightarrow{\delta} \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0$$

of A-modules. The module $\Omega_{A/k}$ is locally free, hence flat. Now $\Omega_{P/k}$ is a free P-module of rank n, so $\Omega_{P/k} \otimes_P A$ is a free A-module of rank n. The submodule J/J^2 is finitely generated since A is Noetherian, so $\Omega_{A/k}$ is finitely presented by Exercise 2.5b in [8]. Then by Corollary 7.12 in [8], $\Omega_{A/k}$ is a projective A-module. Therefore $\operatorname{Ext}^1_A(\Omega_{A/k}, I) = 0$, and we obtain an exact sequence

$$0 \to \operatorname{Hom}_A(\Omega_{A/k}, I) \to \operatorname{Hom}_P(\Omega_{P/k}, I) \xrightarrow{\delta^*} \operatorname{Hom}_A(J/J^2, I) \to 0$$

by applying $\operatorname{Hom}_A(\cdot, I)$. Thus some $\alpha \in \operatorname{Hom}_P(\Omega_{P/k}, I)$ maps to $\overline{h} \in \operatorname{Hom}_A(J/J^2, I)$.

Now consider the derivation $\theta = \alpha \circ d$, where $d : P \to \Omega_{P/k}$ is the natural map. Since α maps to $\overline{h}$, we see that $\theta = h$ on J. Therefore $h' = h - \theta$ is a map of algebras $P \to B'$ vanishing on J, and such that the above diagram still commutes. Passing to the quotient A gives the required lift of f. □

We now prove the lemma that was used in the proof of Theorem 2.3.

Lemma 2.5. *Let R be a finitely generated local k-algebra with maximal ideal $\mathfrak{m}$ and residue field k, and let I be an ideal contained in $\mathfrak{m}$ such that $\mathfrak{m}I = 0$. Let $f : Y \to X$ be a morphism and let*

$$f_{R/I} : Y \times \operatorname{Spec}(R/I) \to X \times \operatorname{Spec}(R/I)$$

be an extension of f. Assume X is smooth along the image of f. The obstruction to extending $f_{R/I}$ to a morphism

$$f_R : Y \times \operatorname{Spec} R \to X \times \operatorname{Spec} R$$

lies in

$$H^1(Y, f^*T_X) \otimes_k I.$$

Proof. First assume $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. A lifting of $f_{R/I}$ to a morphism f_R corresponds to an R-algebra map $f_R^\#$ such that the diagram

$$\begin{array}{ccc} A \otimes_k R & \overset{f_R^\#}{\dashrightarrow} & B \otimes_k R \\ \downarrow & & \downarrow \\ A \otimes_k R/I & \xrightarrow{f_{R/I}^\#} & B \otimes_k R/I \end{array}$$

commutes. If we first consider the diagram

$$\begin{array}{ccc} A & \dashrightarrow & B \otimes_k R \\ \downarrow & & \downarrow \\ A \otimes_k R/I & \xrightarrow{f_{R/I}^\#} & B \otimes_k R/I \end{array}$$

then since $I^2 = 0$ Lemma 2.4 assures us that there is always such a dotted map, although it is only a map of k-algebras. The various possible dotted maps all differ by k-derivations of A into $B \otimes_k I \subset B \otimes_k R$, which is to say elements of $\operatorname{Hom}_A(\Omega_{A/k}, B \otimes_k I)$. But then by applying $\cdot \otimes_k R$ to the upper left corner we get the required map $f_R^\#$, and the various possible maps differ by R-derivations of $A \otimes_k R$ into $B \otimes_k I$, which are parameterized by $\operatorname{Hom}_{A \otimes_k R}(\Omega_{A \otimes_k R/R}, B \otimes_k I)$. Now $\Omega_{A \otimes_k R/R} \cong \Omega_{A/k} \otimes_k R$, so

$$\operatorname{Hom}_{A \otimes_k R}(\Omega_{A \otimes_k R/R}, B \otimes_k I) \cong \operatorname{Hom}_{A \otimes_k R}(\Omega_{A/k} \otimes_k R, B \otimes_k I),$$

and since $\mathfrak{m} I = 0$ and $R/\mathfrak{m} = k$ we have

$$\begin{aligned} \operatorname{Hom}_{A \otimes_k R}(\Omega_{A/k} \otimes_k R, B \otimes_k I) &\cong \operatorname{Hom}_{A \otimes_k R/\mathfrak{m}}(\Omega_{A/k} \otimes_k R/\mathfrak{m}, B \otimes_k I) \\ &\cong \operatorname{Hom}_A(\Omega_{A/k}, B \otimes_k I). \end{aligned}$$

We can write this further as

$$\begin{aligned} \operatorname{Hom}_A(\Omega_{A/k}, B \otimes_k I) &\cong \operatorname{Hom}_B(B \otimes \Omega_{A/k}, B \otimes_k I) \\ &\cong H^0(Y, \mathscr{H}om(f^*\Omega^1_X, \mathscr{O}_Y)) \otimes_k I \\ &\cong H^0(Y, f^*T_X) \otimes_k I \end{aligned}$$

With the local lifting problem dealt with, we consider the global one. Cover Y by open affines $\mathscr{V} = \{V_i\}$. Choose arbitrary local lifts f_R^i of $f_{R/I}$ over each open set V_i. The differences over pairwise intersections define an element α of the Cech group $C^1(\mathscr{V}, f^*T_X \otimes_{\mathscr{O}_Y} I)$ satisfying $d\alpha = 0$, hence an element of $H^1(Y, f^*T_X) \otimes_k I$. This element vanishes if and only if there are sections of $f^*T_X \otimes I$ over each V_i defining an element β of $C^0(\mathscr{V}, f^*T_X \otimes I)$ such that $d\beta = \alpha$. Denote by θ^i the section of $f^*T_X \otimes I$ over V_i corresponding to β. Thinking of θ^i as R-derivations of $A \otimes_k R$ into $B \otimes_k I$, we put $g_R^i = f_R^i - \theta^i$. Then for $i < j$,

$$(g_R^i - g_R^j)|_{V_i \cap V_j} = (f_R^i - f_R^j + \theta^j - \theta^i)|_{V_i \cap V_j} = \alpha^{ij} - (d\beta)^{ij} = 0.$$

Hence the g_R^i's glue into a lift of $f_{R/I}$. Conversely, if there are compatible choices of local lifts the corresponding class α obviously vanishes. Thus there is an element of $H^1(Y, f^*T_X) \otimes_k I$ whose vanishing is equivalent to the existence of an extension f_R of $f_{R/I}$. Notice that if we choose our local lifts f_R^i differently, the differences form an element of $C^0(\mathscr{V}, f^*T_X \otimes I)$ whose differential is the difference of the elements α corresponding to the choices of local liftings. Thus the constructed element of $H^1(Y, f^*T_X) \otimes_k I$ is actually independent of the initial choice of local liftings. □

As with Proposition 2.1, there are also versions of Theorem 2.3 for the parameter spaces $\mathrm{Mor}(Y, X; g)$ and $\mathrm{Mor}_S(Y, X; g)$, when those spaces are defined.

Theorem 2.6. *The dimension of* $\mathrm{Mor}(Y, X; g)$ *at a point* $[f]$ *such that* X *is smooth along the image of* f *is bounded below by* $\chi(Y, f^*T_X \otimes \mathscr{I}_B)$*. If* $H^1(Y, f^*T_X \otimes \mathscr{I}_B) = 0$*, then* $\mathrm{Mor}(Y, X; g)$ *is smooth at* $[f]$*.*

Similarly, the dimension of $\mathrm{Mor}_S(Y, X; g)$ *at a point* $[f_s]$ *such that* X_s *is smooth along the image of* f *is bounded below by* $\chi(Y_s, f_s^*T_{X_s} \otimes \mathscr{I}_{B_s}) + \dim_s S$*. If* $H^1(Y_s, f^*T_{X_s} \otimes \mathscr{I}_{B_s})$ *vanishes, then* $\mathrm{Mor}_S(Y, X; g)$ *is smooth over* S *at* $[f_s]$*.* □

The case we shall be most interested is when Y (or Y_s) is a curve C (resp. C_s) and B (resp. B_s) is a finite collection of points. In this case, our lower bounds for the dimension of $\mathrm{Mor}(Y, X; g)$ and $\mathrm{Mor}_S(Y, X; g)$ take on a particularly nice form, via the Riemann-Roch theorem.

Corollary 2.7. *Keeping all our previous hypotheses,*

$$\dim_{[f]} \mathrm{Mor}(C, X; g) \geq -K_X \cdot f_*C + (1 - g(C) - \#B) \dim X$$

and

$$\dim_{[f_s]} \mathrm{Mor}_S(C, X; g) \geq -K_{X_s} \cdot f_{s*}C_s + (1 - g(C_s) - \#B_s) \dim X_s + \dim S.$$

Proof. In the first case, Riemann-Roch for vector bundles on a curve gives

$$\begin{aligned} \chi(C, f^*T_X \otimes \mathscr{I}_B) &= c_1(f^*T_X \otimes \mathcal{O}_C(-B)) + (1-g) \dim X \\ &= c_1(f^*T_X) + (1 - g + c_1(\mathcal{O}_C(-B))) \dim X \\ &= -K_X \cdot f_*C + (1 - g - \#B) \dim X. \end{aligned}$$

The relative case is identical. □

3. Vector bundles on $\mathbb{P}^1$

The following result completely classifies vector bundles on $\mathbb{P}^1$, and will prove useful in the next section. We follow the proof given in [9].

Proposition 3.1. *Any vector bundle $\mathscr{E}$ on $\mathbb{P}^1$ splits as a direct sum of line bundles. Thus there is a unique nonincreasing sequence of integers $a_1 \geq a_2 \geq \cdots \geq a_s$ such that*

$$\mathscr{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_s).$$

Proof. We prove the result by induction on the rank s of $\mathscr{E}$. Assume that $s \geq 2$. A theorem of Serre implies there is a smallest integer a such that $H^0(\mathbb{P}^1, \mathscr{E}(-a)) \neq 0$. Without loss of generality, $a = 0$, and thus there is a nonzero section σ of $\mathscr{E}$.

A priori, the section σ might have zeroes. It then defines a rational section $\mathbb{P}^1 \dashrightarrow \mathbb{P}\mathscr{E}$, which extends to a global section $\mathbb{P}^1 \to \mathbb{P}\mathscr{E}$ since any rational map from a smooth projective curve to a projective variety is regular. This defines a line subbundle $\mathscr{L}$ of $\mathscr{E}$. This line bundle has a global section (namely σ), so is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a)$ for some $a \geq 0$. But then the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(a-1) \to \mathscr{E}(-1)$$

shows that $H^0(\mathcal{O}_{\mathbb{P}^1}(a-1)) = 0$, which implies $a \leq 0$. Hence $a = 0$, $\mathscr{L}$ is trivial, and σ doesn't actually have any zeroes.

Now consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathscr{E} \to \mathscr{F} \to 0,$$

with $\mathscr{F}$ defined by the sequence. The sheaf $\mathscr{F}$ is locally free. By the induction hypothesis, $\mathscr{F} \cong \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_s)$ for some nonincreasing integers a_i. The exact sequence

$$0 = H^0(\mathscr{E}(-1)) \to H^0(\mathscr{F}(-1)) \to H^1(\mathcal{O}(-1)) = 0$$

shows $H^0(\mathscr{F}(-1)) = 0$, so $a_i \leq 0$ for all i.

By Lemma 3.2, the obstruction to splitting

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathscr{E} \to \mathscr{F} \to 0$$

lies in $H^1(\mathbb{P}^1, \mathscr{H}om(\mathscr{F}, \mathcal{O}_{\mathbb{P}^1}))$. In our present case, this obstruction group vanishes. For

$$H^1(\mathbb{P}^1, \mathscr{H}om(\mathscr{F}, \mathcal{O}_{\mathbb{P}^1})) = H^1(\mathbb{P}^1, \mathcal{O}(-a_2) \oplus \cdots \oplus \mathcal{O}(-a_s)) = 0$$

since $a_i \leq 0$ for all i. Therefore the exact sequence splits, and

$$\mathscr{E} \cong \mathcal{O} \oplus \mathcal{O}(-a_2) \oplus \cdots \oplus \mathcal{O}(-a_s).$$

Uniqueness of the sequence is straightforward. □

We now prove the lemma used in the proof; we will find this lemma useful later, as well.

Lemma 3.2. *Let X be a locally Noetherian scheme, and*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

a short exact sequence of coherent sheaves on X, with $\mathscr{F}''$ locally free. The obstruction to splitting this sequence lies in $H^1(X, \mathscr{H}om(\mathscr{F}'', \mathscr{F}'))$.

Proof. Since $\mathscr{F}''$ is locally free, $\mathscr{H}om(\mathscr{F}'', \cdot)$ is an exact covariant functor. Applying this functor to the short exact sequence and taking the long exact sequence in cohomology, we get a short exact sequence

$$\mathrm{Hom}(\mathscr{F}'', \mathscr{F}) \to \mathrm{Hom}(\mathscr{F}'', \mathscr{F}'') \xrightarrow{\delta} H^1(X, \mathscr{H}om(\mathscr{F}'', \mathscr{F}')).$$

Now there is a splitting $\mathscr{F}'' \to \mathscr{F}$ if and only if the identity map $\mathscr{F}'' \to \mathscr{F}''$ is in the image of $\mathrm{Hom}(\mathscr{F}'', \mathscr{F}) \to \mathrm{Hom}(\mathscr{F}'', \mathscr{F}'')$ if and only if $\delta(\mathrm{id}) = 0$. Hence the obstruction to splitting the exact sequence is $\delta(\mathrm{id}) \in H^1(X, \mathscr{H}om(\mathscr{F}'', \mathscr{F}'))$. □

4. Free rational curves

One technique for showing that a variety has lots of rational curves is to show that it has some rational curve with lots of deformations. Let $f : \mathbb{P}^1 \to X$ be a rational curve on a smooth quasi-projective variety X. By Proposition 3.1, we can write

$$f^*T_X \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_s)$$

for a nonincreasing sequence of integers a_i. If $a_s \geq -1$, then $H^1(\mathbb{P}^1, f^*T_X) = 0$, and $\mathrm{Mor}(\mathbb{P}^1, X)$ is smooth at $[f]$: no infinitesimal deformations of f are obstructed. Moreover, if a_s is large then f^*T_X has lots of global sections. Thus $\mathrm{Mor}(\mathbb{P}^1, X)$ has large dimension at $[f]$ when a_s is large, and f has lots of deformations.

We can actually compute the number a_s in a way that does not rely on decomposing f^*T_X as a direct sum of line bundles. The number a_s is the largest number such that $f^*T_X(-a_s)$ is generated by global sections.

Definition 4.1. A rational curve $f : \mathbb{P}^1 \to X$ is *r-free* if $f^*T_X(-r)$ is generated by global sections. We call f *free* if it is 0-free, and *very free* if it is 1-free.

The next result justifies our statement that freeness measures the extent to which a rational curve can be deformed. It says that a free curve can be deformed in such a way that any one of its points moves in a specified direction, while a very free curve can be deformed in such a way that any one of its points remains fixed and any other point moves in a specified direction.

Proposition 4.2. *Let X be a smooth quasi-projective variety, let r be a nonnegative integer, let $f : \mathbb{P}^1 \to X$ be an r-free rational curve and let B be a finite subscheme of $\mathbb{P}^1$ of length $b \leq r$. If s is a positive integer such that $b + s - 1 \leq r$, then the evaluation map*

$$\mathrm{ev} : (\mathbb{P}^1)^s \times \mathrm{Mor}(\mathbb{P}^1, X; f|_B) \to X^s$$

is smooth at $(t_1, \ldots, t_s, [f])$ when $t_1, \ldots, t_s$ are not in the support of B.

Proof. Since $a_n \geq b$, we see that $H^1(\mathbb{P}^1, f^*T_X(-B)) = 0$. Therefore $\mathrm{Mor}(\mathbb{P}^1, X; f|_B)$ is smooth at $[f]$ by Theorem 2.6, and the statement that ev is smooth is the same as the statement that its differential is surjective.

The differential of ev at $(t_1, \ldots, t_s, [f])$ is the map

$$\bigoplus_{i=1}^{s} T_{\mathbb{P}^1, t_i} \oplus H^0(\mathbb{P}^1, f^*T_X(-B)) \to \bigoplus_{i=1}^{s} (f^*T_X)_{t_i}$$

given by

$$(u_1, \ldots, u_s, \sigma) \mapsto (df_{t_1}(u) + \sigma(t_1), \ldots, df_{t_s}(u) + \sigma(t_s)).$$

This will certainly be surjective if the map $H^0(\mathbb{P}^1, f^*T_X(-B)) \to \bigoplus (f^*T_X)_{t_i}$ given by $\sigma \mapsto (\sigma(t_1), \ldots, \sigma(t_s))$ is surjective. To see this, we show that each factor $(f^*T_X)_{t_i}$ is contained in the image. Since $b + s - 1 \leq r$, we see that $f^*T_X(-B - t_1 - \cdots - t_s)$ is generated by global sections. Hence there is a global section of $f^*T_X(-B)$ vanishing at each point of $t_1, \ldots, t_{i-1}, t_{i+1}, t_s$, and taking on any value we please at t_i. □

The next result gives a partial converse to the previous one.

Proposition 4.3. *Let X be a smooth quasi-projective variety, let $f : \mathbb{P}^1 \to X$ be a rational curve, and let B be a finite subscheme of $\mathbb{P}^1$ of length b. If the differential of*

$$\mathrm{ev} : (\mathbb{P}^1)^s \times \mathrm{Mor}(\mathbb{P}^1, X; f|_B) \to X^s$$

is surjective at some point $(t_1, \ldots, t_s, [f])$, then f is $\min\{2, b + s - 1\}$-free.

Proof. Without loss of generality, $b + s - 1 \leq 2$. We may assume $\dim X \geq 2$, which implies $t_1, \ldots, t_s \notin B$. We are given that

$$\bigoplus_{i=1}^{s} T_{\mathbb{P}^1, t_i} \oplus H^0(\mathbb{P}^1, f^*T_X(-B)) \to \bigoplus_{i=1}^{s} (f^*T_X)_{t_i}$$

is surjective. This implies the map

$$H^0(\mathbb{P}^1, f^*T_X(-B)) \to \bigoplus_{i=1}^{s} (f^*T_X)_{t_i}/(\operatorname{im} df_{t_i})$$

is surjective. In the commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^1, f^*T_X(-B)) & \xrightarrow{e} & \bigoplus_{i=1}^{s} (f^*T_X)_{t_i} \\ \uparrow & & \uparrow {\scriptstyle \bigoplus_{i=1}^{s} df_{t_i}} \\ H^0(\mathbb{P}^1, T_{\mathbb{P}^1}(-B)) & \xrightarrow{e'} & \bigoplus_{i=1}^{s} T_{\mathbb{P}^1, t_i} \end{array}$$

the map e' is surjective since $T_{\mathbb{P}^1}(-B - t_1 - \cdots - \hat{t}_i - \cdots - t_s)$ is generated by global sections for each i and $t_i \notin B$. Thus e is actually surjective. This implies in particular that $H^0(\mathbb{P}^1, f^*T_X(-B-t_1-\cdots-t_{s-1})) \to (f^*T_X)_{t_s}$ is surjective, so by Nakayama's lemma $f^*T_X(-B-t_1-\cdots-t_{s-1})$ is generated by global sections at t_s. A line bundle on $\mathbb{P}^1$ that is generated by global sections at a point is generated by global sections everywhere. By the classification of vector bundles on $\mathbb{P}^1$, the same result holds for arbitrary vector bundles. Thus $f^*T_X(-B - t_1 - \cdots - t_{s-1})$ is generated by global sections, and f is $(b + s - 1)$-free. □

5. Uniruledness

We have now developed the necessary results from deformation theory to be able to study the notions we set out to study. In this section, we study varieties which are *uniruled*. In nice circumstances, these are the varieties which are covered by rational curves. In the first subsection, we discuss some equivalent conditions to uniruledness. We will explore some consequences of uniruledness in the second section.

5.1. Definitions equivalent to uniruledness.

To have a starting point, we adopt the following definition.

Definition 5.1. A variety X of dimension n is *uniruled* if there is a variety Y of dimension $n-1$ together with a dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow X$.

Notice that a point is not uniruled, according to this definition. We will more or less adopt the convention that our varieties are not points, to eliminate pathologies.

Under reasonable circumstances, this definition is equivalent to the existence of a rational curve through any point of X. This is one of the statements of the next theorem.

Theorem 5.2. *Let X be a smooth projective variety over an uncountable algebraically closed field of characteristic zero. The following conditions are equivalent.*

(1) *X is uniruled.*
(2) *There exists a variety Y and a dominant rational map $e : \mathbb{P}^1 \times Y \dashrightarrow X$ such that $e|_{\mathbb{P}^1 \times \{y\}}$ is a nonconstant rational map for some $y \in Y$.*
(3) *There exists a variety Y of dimension $n-1$ and a dominant morphism $e : \mathbb{P}^1 \times Y \to X$.*
(4) *A general point of X is contained in a rational curve.*
(5) *Every point of X is contained in a rational curve.*
(6) *There is some integer d such that every point of X is contained in a rational curve of degree at most d.*
(7) *There is a free rational curve on X.*
(8) *A general point of X is contained in a free rational curve.*

Proof. The implications (1)⇒(2) and (3)⇒(1) are obvious. We start by proving their converses.

(1)⇒(3). First compactify and normalize Y. The exceptional locus of $e : \mathbb{P}^1 \times Y \dashrightarrow X$ has codimension at least 2 in $\mathbb{P}^1 \times Y$ since $\mathbb{P}^1 \times Y$ is normal and X is projective, so by shrinking Y we may assume that $e : \mathbb{P}^1 \times Y \to X$ is a morphism.

(2)⇒(1). Without loss of generality, Y is affine of dimension $m \geq n$. Since (a dense subset of) some curve $\mathbb{P}^1 \times \{y\}$ is not contracted by e, the general curve $\mathbb{P}^1 \times \{y'\}$ is not contracted by e. By generic smoothness, we can find a point $(t, y) \in \mathbb{P}^1 \times Y$ such that $de_{(t,y)} : T_{\mathbb{P}^1,t} \oplus T_{Y,y} \to T_{X,e(t,y)}$ is surjective and $de_{(t,y)}|_{T_{\mathbb{P}^1,t}}$ is nonzero. Suppose we know there is an $(n-1)$-dimensional subspace W of $T_{Y,y}$ such that $de_{(t,y)}|_{T_{\mathbb{P}^1,t} \oplus W}$ is still surjective. We can then choose a plane of some dimension in affine space which meets $T_{Y,y}$ transversely in W. The corresponding plane section Y' of Y will have dimension $n-1$, and the restriction of e to $\mathbb{P}^1 \times Y'$ will be smooth at (t, y). Hence $\mathbb{P}^1 \times Y' \to X$ will be dominant.

All that remains is to produce W; this is entirely linear algebra. Consider the map $T_{Y,y} \to T_{X,e(t,y)}/de_{(t,y)}(T_{\mathbb{P}^1,t})$. It is necessarily surjective. Choosing arbitrary preimages of a basis for $T_{X,e(t,y)}/de_{(t,y)}(T_{\mathbb{P}^1,t})$ gives a basis of an $(n-1)$-dimensional subspace W of $T_{Y,y}$ such that $T_{\mathbb{P}^1,t} \oplus W \to T_{X,e(t,y)}$ is surjective.

Having established the equivalence of (1), (2), and (3), we next show that these conditions are equivalent to (4), (5), and (6). We will do this by showing (3)⇒(6) and (4)⇒(2).

(3)⇒(6). Without loss of generality, the variety Y given to us by (3) is irreducible. The universal property of $\mathrm{Mor}(\mathbb{P}^1, X)$ gives us a factorization

$$\mathbb{P}^1 \times Y \to \mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X) \to X.$$

The image of $\mathbb{P}^1 \times Y \to \mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X)$ lies in some component $\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)$. This says exactly that a general point of X is contained in a rational curve of degree d.

For the more precise statement that every point is contained in a rational curve of degree d, we need to see that if $M_d = \coprod_{i=1}^d \mathrm{Mor}_i(\mathbb{P}^1, X)$ then the natural evaluation map $\mathbb{P}^1 \times M_d \to X$ has closed image. But this is true more or less because a rational curve of degree d can only degenerate to a reducible rational curve, all of whose components have degree less than d.

(4)⇒(2). By the last paragraph, the images of the evaluation maps $\mathbb{P}^1 \times M_d \to X$ form an increasing chain of closed subvarieties of X. Since k is uncountable, the only way their union can contain a dense open set is if some $\mathbb{P}^1 \times M_d \to X$ is surjective. Then some evaluation map $\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \to X$ must itself be dominant, giving (2).

We now conclude the proof by showing (7)⇒(2) and (6)⇒(8).

(7)⇒(2). Let $f : \mathbb{P}^1 \to X$ be free. By Proposition 4.2, the differential of the evaluation map $\mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X) \to X$ is smooth at $[f]$. Since X is irreducible, the restriction of this map to the component containing $[f]$ is dominant.

(6)⇒(8). Choose d such that the evaluation map $\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \to X$ is dominant. By generic smoothness, there is a dense open subset U of $\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)$ where this evaluation map is smooth. The projection $U \to \mathrm{Mor}_d(\mathbb{P}^1, X)$ is dominant, say with image V, and Proposition 4.3 implies that every $[f] \in V$ is free. But $\mathbb{P}^1 \times V \to X$ is still dominant, so a general point of X is contained in a free rational curve. □

5.2. **Examples and consequences of uniruledness.** Now that we have several criteria for uniruledness, we will explore some of the basic examples.

Example 5.3. Any variety dominated by a uniruled variety is uniruled. In particular, unirational varieties are uniruled, since $\mathbb{P}^n$ is uniruled. Uniruled varieties are not necessarily unirational: the ruled surface $\mathbb{P}^1 \times C$, with C an irrational curve, is uniruled but not unirational.

Example 5.4. A curve is uniruled if and only if it is rational.

Example 5.5. A finite étale cover of a smooth projective uniruled variety is uniruled. This follows directly from the fact that $\mathbb{P}^1$ is simply connected.

Example 5.6. *Fano varieties* are smooth projective varieties such that $-K_X$ is ample. We shall see in Section 8 that they are uniruled.

Example 5.7. Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of hypersurfaces of degrees $d_1, \ldots, d_s$. Put $d = d_1 + \cdots + d_s$. Then X is uniruled if and only if $d \leq n$.

Indeed, it follows from the adjunction formula that $K_X = \mathcal{O}_X(d - n - 1)$, so $-K_X$ is ample if and only if $d \leq n$. Hence if $d \leq n$, X is a Fano variety, and is uniruled. Now suppose that $d > n$, and that $f : \mathbb{P}^1 \to X$ is a free rational curve. Writing $f^*T_X = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{n-s})$, we know that $a_1 \geq 2$ since f^*T_X contains $T_{\mathbb{P}^1}$ as a subbundle. Then we have

$$K_X \cdot f_*\mathbb{P}^1 = c_1(f^*K_X) = -c_1(f^*T_X) = -\sum a_i \leq -2.$$

But K_X is either trivial or ample, so this is absurd. Hence X contains no free rational curve if $d > n$, and X is not uniruled.

Another consequence of the observation that $K_X \cdot f_*\mathbb{P}^1 \leq -2$ for a free curve f on X is the following result.

Proposition 5.8. *For a smooth projective uniruled variety over a field of characteristic zero, the plurigenera $h^0(X, K_X^{\otimes m})$ all vanish for $m \geq 1$.*

Proof. Suppose σ is a section of $K_X^{\otimes m}$ for some m. If $f : \mathbb{P}^1 \to X$ is a free rational curve, then $mK_X \cdot f(\mathbb{P}^1) < 0$, which is to say that $K_X^{\otimes m}|_{f(\mathbb{P}^1)}$ has no nonzero global sections. Thus $\sigma|_{f(\mathbb{P}^1)}$, being a global section of $K_X^{\otimes m}|_{f(\mathbb{P}^1)}$ must vanish along $f(\mathbb{P}^1)$. But free rational curves cover a dense open subset of X, so σ vanishes on a dense subset of X, hence vanishes everywhere. $\square$

The converse of this result has been conjectured to hold by Mori; it is one of the hardest open problems in higher dimensional algebraic geometry. It is known to hold up to dimension 3, where the Mori program has been successful.

Example 5.9. A surface is uniruled if and only if it is ruled. A ruled surface is obviously uniruled, while the converse is provided by Enriques' theorem [1, Theorem VI.17] stating that a surface with vanishing plurigenera is ruled.

Uniruledness behaves exceptionally well in families: it is both a closed and an open condition. We elaborate on the proof given in [6].

Proposition 5.10. *If $\mathscr{X} \to B$ is a smooth, proper morphism, then the locus $\{b \in B : X_b$ is uniruled$\}$ is both open and closed in B.*

Proof. To see the locus is closed, it is enough to consider the case where B is an analytic disc. Let $x_0 \in X_0$ be a point of the special fiber. Perhaps after performing a base change, we can find a section $\sigma : B \to \mathscr{X}$ passing through x_0. We can look at the family $\mathrm{Hom}_B(\mathbb{P}^1 \times B, \mathscr{X}; (0,b) \mapsto \sigma(b))$ of rational curves contained in fibers and passing through the section. Passing to a 1-dimensional subvariety of this family that is transverse to the fibers, we can specialize to a curve on X_0 passing through x_0, the components of which are all rational.

On the other hand, if X_0 is uniruled, fix a free rational curve $f : \mathbb{P}^1 \to X_0$. Consider the exact sequence

$$0 \to f^*T_{X_0} \to f^*T_X \to \bigoplus_{j=1}^{r} \mathcal{O}_{\mathbb{P}^1} \to 0,$$

where r is the dimension of B at 0. If $f^*T_{X_0} \cong \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^1}(a_i)$, We calculate

$$\mathscr{H}om(\bigoplus_{j=1}^{r} \mathcal{O}_{\mathbb{P}^1}, f^*T_X) \cong \bigoplus_{i,j} \mathscr{H}om(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(a_i)) \cong \bigoplus_{i,j} \mathcal{O}_{\mathbb{P}^1}(a_i).$$

Since f is free, the a_i are all nonnegative, and therefore H^1 of this sheaf vanishes. By Lemma 3.2, the exact sequence splits. This implies that $H^1(f^*T_X) = 0$, so there are no obstructions to extending first order deformations of f, and $H^0(f^*T_X) = H^0(f^*T_{X_0}) \oplus H^0(\bigoplus \mathcal{O}_{\mathbb{P}^1})$, so first order deformations of f decompose into first order deformations within the fibers and first order deformations transverse to the fibers. Thus we can move f into nearby fibers. Whether or not the deformed curves are free is an open condition, since it can be stated in terms of the vanishing of a cohomology group. Thus the general deformed curve is free, and uniruledness is an open condition in smooth families. $\square$

We conclude this section with a somewhat technical result on uniruled varieties which will be important in our study of rationally connected varieties. It essentially states that on a uniruled variety, the vast majority of rational curves are free.

Lemma 5.11. *Let X be a smooth projective uniruled variety over a field of characteristic zero. There is a countable collection $\{U_i\}$ of dense open subsets of X such that every rational curve in X meeting $X_{\mathrm{free}} = \bigcap U_i$ is free.*

Proof. Let $\mathrm{ev}_d : \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \to X$ be the evaluation map. By generic smoothness, there is a dense open subset $U_i \subset X$ such that the restriction of ev_d to $\mathrm{ev}_d^{-1}(U_i)$ is smooth. It follows from Proposition 4.3 that any rational curve in X meeting X_{free} is free. $\square$

6. Rational connectivity

In this section, we finally begin our study of rationally connected varieties. The initial formalisms are very similar to uniruledness, with identical proofs. We will therefore skip the proofs, except where things are genuinely different. As with uniruledness, we will start with a number of equivalent definitions for rational connectivity. We will then investigate some examples and basic consequences. The heart of this section lies in the second subsection, where we will sketch a proof of the fact that rationally chain-connected varieties are rationally connected.

6.1. **Definitions equivalent to rational connectivity.**

We can view rational connectivity as a generalization of uniruledness. While single points of a uniruled variety are covered by rational curves, pairs of points of a rationally connected variety are covered by rational curves.

Definition 6.1. A variety X is *rationally connected* if there is a variety Y and a rational map $e : \mathbb{P}^1 \times Y \dashrightarrow X$ such that

$$(\mathbb{P}^1 \times Y) \times_Y (\mathbb{P}^1 \times Y) \dashrightarrow X \times X$$

is dominant.

Note that, by way of analogy, we could have defined a uniruled variety to be one such that there is a dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow X$ such that the image of the corresponding map $(\mathbb{P}^1 \times Y) \times_Y (\mathbb{P}^1 \times Y) \dashrightarrow X \times X$ is not contained in the diagonal. This is easily seen to be equivalent to criterion (2) of Theorem 5.2.

Most of the equivalences of the next theorem follow as in Theorem 5.2. We shall prove some highly nontrivial equivalences later, in Subsection 6.2.

Theorem 6.2. *For a smooth projective variety X (over $\mathbb{C}$), the following are equivalent.*

1. *X is rationally connected.*
2. *There exists a variety Y and a morphism $\mathbb{P}^1 \times Y \to X$ such that $(\mathbb{P}^1 \times Y) \times_Y (\mathbb{P}^1 \times Y) \to X \times X$ is dominant.*
3. *A general pair of points of X is contained in a rational curve.*
4. *A general r-tuple of points of X is contained in a rational curve.*
5. *There is some integer d such that a general r-tuple of points of X is contained in a rational curve of degree at most d.*
6. *X contains a very free rational curve.*
7. *A general r-tuple of points of X is contained in a very free rational curve.*

Proof. There is one "trick" involved in the proofs. If $f : \mathbb{P}^1 \to X$ is a very free rational curve, then by precomposing with an r-sheeted branched cover $\mathbb{P}^1 \to \mathbb{P}^1$ we get an r-free rational curve on X. This trick, together with Proposition 4.2, allows us to show that rational curves pass through arbitrarily large general finite subsets. □

Unlike with uniruledness, it is not a priori clear that arbitrary pairs of points can be joined by rational curves, since the limit of a family of irreducible rational curves need not be irreducible. This will turn out to be true, as we will see in Corollary 6.8.

6.2. **Rational chain-connectivity implies rational connectivity.** In the previous section, we studied formal properties of rational connectivity that were essentially analogous to properties of uniruledness. In this section we shall show that rational connectivity is actually equivalent to an a priori much weaker condition. The notion of rational chain-connectivity is only well-behaved for smooth varieties over an uncountable field of characteristic zero, so we will only discuss that case.

Definition 6.3. A *chain* of rational curves on X is a map $f : C \to X$ from a connected projective curve, all of whose components are rational.

A smooth projective variety X is *rationally chain-connected* if any two general points are contained in a chain of rational curves.

A *tree* of rational curves is a connected projective curve C such that the components of C can be labeled as $C_1, \ldots, C_m$ in such a way that each C_{i+1} meets $C_1 \cup \cdots \cup C_i$ transversely in a single smooth point of $C_1 \cup \cdots \cup C_i$. A tree of rational curves on X is a map from a tree of rational curves to X. Note that the image of such a tree in X does not have to have the same intersection-theoretic properties as the tree itself.

In fact, any two points of a rationally chain-connected variety can be joined by a chain of rational curves. This follows from the fact that families of chains of rational curves specialize to such chains. Clearly rationally connected varieties are rationally chain-connected, and rationally chain-connected varieties are uniruled.

The chief result in this section is that a rationally chain-connected variety is rationally connected. The idea of the proof is to show that a chain of rational curves can be "smoothed" into an irreducible rational curve, while keeping two points fixed. This is a vast over-simplification of the proof, however.

If two points on X can be joined by a chain of rational curves, they can be joined by a tree of rational curves on X. This follows from throwing away some of the components of the chain and making intersections in the chain transverse.

Given any rational tree, we can always deform it to be smooth rational. More precisely, there is a 1-dimensional scheme T with base point 0, and a flat morphism $\mathscr{C} \to T$ with special fiber $\mathscr{C}_0 = C$ and all other fibers smooth rational curves. This can be seen by induction on the number of components in C, using an elementary blowing up argument.

Definition 6.4. Such a flat morphism $\mathscr{C} \to T$ is a *smoothing* of the rational tree C. A morphism $f : C \to X$ is *smoothable* if there is a smoothing $\mathscr{C} \to T$ and an extension of

f to a map $\mathscr{C} \to X$. A smoothing $F : \mathscr{C} \to X$ of f *keeps* $f(p_1), \dots, f(p_r)$ *fixed* if there are sections of $\mathscr{C} \to T$ contracted by F and passing through $p_1, \dots, p_r$.

Our first result on smoothings of rational trees says that under certain freeness assumptions on the components of the tree, there exists a smoothing keeping some points fixed.

Proposition 6.5. *Let $f : C \to X$ be a rational tree in X, where $C = C_1 \cup \cdots \cup C_m$ as in the definition. Let $B = \{b_1, \dots, b_r\}$ be a set of r smooth points of C, with r_i points on C_i. If the restriction of f to C_1 is $(r_1 - 1)$-free and, for each $i \geq 2$, the restriction of f to C_i is r_i-free, then f is smoothable into an $(r-1)$-free rational curve, keeping $f(b_1), \dots, f(b_r)$ fixed.*

Proof. Let $\pi : \mathscr{C} \to T$ be a smoothing of C, and $\sigma_1, \dots, \sigma_r$ sections of π through $b_1, \dots, b_r$, respectively; we shrink T if necessary so that the σ_i have disjoint images. This defines a subvariety $\mathscr{B}$ of $\mathscr{C}$, which is a relative version of B. We let $g : \mathscr{B} \to X \times T$ be the T-morphism which maps the section through b_i to $f(b_i)$. Consider the T-scheme $\mathrm{Mor}_T(\mathscr{C}, X \times T; g)$. The fiber over 0 of this scheme is $\mathrm{Mor}(C, X; f|_B)$, and by Theorem 2.6 this scheme is smooth over T at $[f]$ when $H^1(C, f^*T_X(-B))$ vanishes.

We first aim to prove that in fact $H^1(C, f^*T_X(-B))$ does vanish, so that $\mathrm{Mor}_T(\mathscr{C}, X \times T; g)$ is smooth over T at f. We do this by induction on m, the case $m = 1$ being clear. Let $C' = C_1 \cup \cdots \cup C_{m-1}$, and put $B' = B \cap C'$, $B_m = B \cap C_m$. If q is the point where C_m meets C', then there is an exact sequence

$$0 \to \mathcal{O}_{C_m}(-q - B_m) \to \mathcal{O}_C(-B) \to \mathcal{O}_{C'}(-B') \to 0,$$

from which we obtain the exact sequence

$$H^1(C_m, f^*T_X(-q - B_m)) \to H^1(C, f^*T_X(-B)) \to H^1(C', f^*T_X(-B')).$$

Our original hypothesis on C_m and the induction hypothesis ensure that the outer two groups vanish, so $H^1(C, f^*T_X(-B)) = 0$. Thus $\mathrm{Mor}_T(\mathscr{C}, X \times T; g)$ is smooth over T at $[f]$. Hence f can be deformed to an irreducible curve in the general fiber, keeping $f(b_1), \dots, f(b_r)$ fixed, which is to say that after a base change we can smooth f into a curve keeping $f(b_1), \dots, f(b_r)$ fixed.

All that remains is to show that the smoothed curves are $(r-1)$-free. But as we have seen before, $(r-1)$-freeness for a curve $f : \mathbb{P}^1 \to X$ is equivalent to the vanishing of the cohomology group $H^1(f^*T_X(-r))$. Since in our present case the cohomology group $H^1(C, f^*T_X(-r))$ vanishes, the upper semi-continuity of dimensions of cohomology groups gives the required vanishing for the general fiber. □

Another smoothing result is also needed, for a specific class of rational trees. A *rational comb* is a rational tree where all the components $C_2, \dots, C_m$ are joined to the component $D = C_1$. We call D the *handle* of the rational comb, and $C_2, \dots, C_m$ the *teeth.* This result essentially says that a rational comb in X with lots of free teeth has a smoothable rational subcomb, fixing some points on the handle.

Lemma 6.6. *Let C be a rational comb with m teeth and let $p_1, \dots, p_r$ be points on the handle D which are smooth on C. Let $f : C \to X$ be a morphism whose restriction to each tooth is free. Assume*

$$m > K_X \cdot f_* D + (r-1) \dim X + \dim_{[f|_D]} \mathrm{Mor}(\mathbb{P}^1, X; f|_{p_1,\dots,p_r}).$$

Then there exists a subcomb C' of C having at least one tooth, such that $f|_{C'}$ is smoothable, keeping $f(p_1), \dots, f(p_r)$ fixed.

Sketch of Proof. The proof relies on constructing an m-parameter family of smoothings of the comb C, via a blowing-up construction. This is not a smoothing in the 1-parameter sense, but all the nearby smoothed curves are either smooth rational curves or rational combs with fewer than m teeth. As in the proof of Proposition 6.5, we consider the relative space of morphisms associated to this family of smoothings.

We look at a neighborhood of $[f]$ in this parameter space. It can be shown that this neighborhood cannot be contracted to a point by the structure map of the parameter space; this is proved by a dimension count, making use of the above displayed inequality and the relative version of Corollary 2.7. This is where the freeness of the teeth is used; it implies that certain H^1 groups vanish.

With that done, f can be deformed along a 1-parameter subfamily of the original m-dimensional family. If the general fiber of this family is $\mathbb{P}^1$, we are done since f itself can be smoothed. Otherwise, all the fibers have at least some number m' of teeth. In this case the restriction of the parameter space to this 1-parameter subfamily decomposes into the corresponding m'-parameter family of smoothings of an m'-toothed comb and some components which are contracted by the deformations of f. By the assumption that all fibers have at least m' teeth, the nontrivial part of this decomposition has general fiber $\mathbb{P}^1$. Choosing one of the m'-teethed subcombs in the 1-parameter subfamily, we get a subcomb which is smoothable keeping $f(p_1), \dots, f(p_r)$ fixed. □

Theorem 6.7. *A rationally chain-connected smooth projective variety X over $\mathbb{C}$ is rationally connected. In addition, any pair of points in a rationally connected variety can be joined by a very free curve.*

Proof. Recall from Lemma 5.11 that the locus X_{free} of X contains a very general point of X (since X is uniruled), and has the property that any rational curve meeting X_{free}

is free. Let $x \in X_{\text{free}}$, and let $y \in X$ be arbitrary. There is a chain of rational curves $f_0, \ldots, f_s : \mathbb{P}^1 \to X$ such that $f_0(0) = x$, $f_i(\infty) = f_{i+1}(0)$, and $f_s(\infty) = y$; denote by C_i the image of f_i. We first show that there are rational curves $g_0, \ldots, g_s : \mathbb{P}^1 \to X$ meeting X_{free} such that $g_i(0) = f_i(0)$ and $g_i(\infty) = f_i(\infty)$ for all i.

We do this by induction on s. We can put $g_0 = f_0$, since by hypothesis f_0 meets X_{free}. So suppose that $g_1, \ldots, g_{i-1}$ have been constructed already. Since g_{i-1} is free, the evaluation map

$$\begin{aligned} \mathrm{ev}_\infty : \mathrm{Mor}(\mathbb{P}^1, X) &\to X \\ [g] &\mapsto g(\infty) \end{aligned}$$

is smooth at $[g_{i-1}]$; the space $\mathrm{Mor}(\mathbb{P}^1, X)$ is smooth at $[g_{i-1}]$ since $H^1(g_{i-1}^* T_X) = 0$, and the differential is obviously surjective. Let T be a component of $\mathrm{ev}_\infty^{-1}(C_i)$ containing $[g_{i-1}]$, so that T parameterizes a subset of those rational curves in X whose parameterizations send ∞ into C_i. The restriction of ev_∞ to T dominates C_i since ev_∞ is smooth at $[g_{i-1}]$.

Let $U_i \subset X$ be the dense open locus defined in Lemma 5.11, so that $X_{\text{free}} = \bigcap U_i$. Let $V_i \subset T$ be the locus of those curves meeting U_i. The claim is that V_i is open. Consider the natural flat family of curves $\pi : \mathscr{C} \to T$. Under the map $g : \mathscr{C} \to X$, the preimage $g^{-1}(U_i)$ is an open subset. It is nonempty since g_{i-1} meets X_{free}. The locus of $t \in T$ corresponding to curves meeting U_i is exactly the projection $\pi(g^{-1}(U_i))$. But π, being flat, is an open map, so this is a dense open subset of T. Thus the locus of $t \in T$ corresponding to curves meeting every U_i, hence meeting X_{free}, is a countable intersection of dense open subsets. Therefore a very general $t \in T$ corresponds to a curve meeting both C_i and X_{free}. In particular, this means that for very general $p \in C_i$, there is a curve meeting both p and X_{free}.

We now make C_i into the handle of a comb with lots and lots of teeth meeting X_{free}. We want to deform C_i while keeping its endpoints fixed, so we would like to apply Lemma 6.6 in the case $r = 2$. The lemma tells us that so long as we add enough free teeth, there will be a subcomb that is smoothable, keeping the endpoints fixed. The smoothable subcomb still has a tooth meeting X_{free}. Now by the logic of the last paragraph, a very general smoothing (with fixed endpoints) of this smoothable subcomb will be an irreducible rational curve $g_i : \mathbb{P}^1 \to X$ meeting X_{free} with $g_i(0) = f_i(0)$ and $g_i(\infty) = f_i(\infty)$.

By induction, a point in $x \in X_{\text{free}}$ can always be joined to an arbitrary point $y \in X$ by a rational chain with free components. Applying Proposition 6.5, this chain can be smoothed leaving y fixed. If $M = \mathrm{Mor}(\mathbb{P}^1, X; 0 \mapsto y)$, then this means that x is in the closure of the image of the evaluation map $\mathbb{P}^1 \times M \to X$. But x could have been any point of X_{free}, so this evaluation map is dominant. Generic smoothness and Proposition 4.3 then provide a very free rational curve through x.

Moreover, very free rational curves through x cover a general subset of X. So given two arbitrary points $x, y \in X$, we can find a third general point $z \in X$ such that x and y are both connected to z by very free rational curves. By Proposition 6.5, this chain can be smoothed fixing x and y, into a very free rational curve. □

The next corollary gives some more equivalent definitions to rational connectivity.

Corollary 6.8. *The following conditions are equivalent.*

(1) X is rationally chain-connected.
(2) X is rationally connected.
(3) There is a very free rational curve through any finite subset of X.

Proof. All but the implication (2)⇒(3) is clear. This point is proved easily by induction on the size of the finite set and the trick of precomposing very free curves with high degree maps $\mathbb{P}^1 \to \mathbb{P}^1$ to artificially inflate freeness, together with Proposition 6.5. □

6.3. **Examples and consequences of rational connectivity.** As before with uniruledness, we explore the basic examples and results with respect to rational connectivity. Many of the results have a similar flavor to those for uniruledness.

Example 6.9. Any variety dominated by a rationally connected variety is rationally connected. Hence unirational varieties are rationally connected. It is unknown whether rationally connected varieties are necessarily unirational; this is a hard open problem, largely for the reason that it is very difficult to show a variety is not unirational.

Example 6.10. A curve is rationally connected if and only if it is rational. Since a uniruled surface is ruled (Example 5.9), and since a ruled surface $\mathbb{P}^1 \times C$ has no rational curves other than the fibers of the projection $\mathbb{P}^1 \times C \to C$ unless $C \cong \mathbb{P}^1$, a surface is rationally connected if and only if it is rational.

Example 6.11. Fano varieties are rationally connected, as we shall see in Section 8.

Example 6.12. A smooth complete intersection in $\mathbb{P}^n$ is rationally connected if and only if it is uniruled (cf. Example 5.7).

The next result is quite analogous to Proposition 5.8.

Proposition 6.13. *If X is rationally connected, then $H^0(X, (\Omega_X^p)^{\otimes m}) = 0$ for all $p, m > 0$.*

Proof. If $f : \mathbb{P}^1 \to X$ is a very free rational curve, then $f^*\Omega^1_X$ decomposes as a direct sum of negative line bundles, and so has no nonzero global sections. It follows that the same holds for $f^*((\Omega^p_X)^{\otimes m})$. But very free rational curves cover X, so any global section of $(\Omega^p_X)^{\otimes m}$ vanishes on all of X. □

As with the corresponding result for uniruledness, the converse of this result is an open question, known as Mumford's conjecture. We shall see in Corollary 9.4 that Mori's conjecture implies Mumford's conjecture.

Another consequence of rational connectivity with regards to the topology of the underlying manifold is the following.

Proposition 6.14. *A smooth projective rationally connected variety over $\mathbb{C}$ is simply connected.*

Proof. By Proposition 6.13, we have $\chi(X, \mathcal{O}_X) = 1$. A finite étale cover $\pi : \tilde{X} \to X$ of X is again rationally connected, essentially since $\mathbb{P}^1$ is simply connected, so $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 1$. But $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \deg(\pi) \cdot \chi(X, \mathcal{O}_X)$, so $\deg \pi = 1$ and π is an isomorphism. Therefore X has no nontrivial finite covers.

Since X is rationally connected, we can find a very free rational curve $f : \mathbb{P}^1 \to X$ and a smooth quasi-projective subvariety M of $\mathrm{Mor}(\mathbb{P}^1, X; 0 \mapsto f(0))$ such that the evaluation $\mathrm{ev} : \mathbb{P}^1 \times M \to X$ is dominant. A general result [3, Lemma 4.19] implies that the image of $\pi_1(\mathbb{P}^1 \times M)$ in $\pi_1(X)$ has finite index. But in fact $\pi_1(\mathrm{ev}) = 0$, for if $M \to \mathbb{P}^1 \times M$ is the map $[f] \mapsto (0, [f])$ then the composition $M \to \mathbb{P}^1 \times M \to X$ is constant, while $M \to \mathbb{P}^1 \times M$ induces an isomorphism on π_1 since $\mathbb{P}^1$ is simply connected. Thus $\pi_1(X)$ is finite, hence must be zero since X has no nontrivial finite covers. □

For one last elementary result, we note that rational connectivity is both a closed and an open property in smooth proper families. The proof is identical to the one for uniruledness, once one replaces "free" by "very free" and makes use of the fact that rationally chain-connected varieties are rationally connected.

Proposition 6.15. *If $\mathscr{X} \to B$ is a smooth, proper morphism, then the locus* $\{b \in B : X_b$ is uniruled$\}$ *is both open and closed in B.* □

7. The MRC quotient

In algebraic topology, the set $\pi_0(X)$ parameterizes the path-components of a topological space X. We would like a similar construction, which parameterizes the "rationally connected components" of a given variety. Moreover, since these components might be expected to fit into families in some way, it would be nice if these components fit together into a variety. It is the chief goal of this section to outline such a construction.

In precise terms, what we are looking for is a map $X \to Z$ such that the fibers X_z are all rationally connected, and such that any rational curve in X is contained in some fiber. Unfortunately, this is too strong a requirement in general. For instance, a $K3$ surface contains countably many lines; the variety Z for such a surface would have to look truly bizarre.

The solution to this problem is to only require that the fibers of $X \to Z$ be rationally connected, and that the very general fiber be a rationally connected component. We also must only require that $X \to Z$ be a rational map, and not a morphism. This remedies the case of the $K3$ surface: if we let Z be the surface itself, the identity map $X \to Z$ works.

Definition 7.1. Let X be a smooth projective variety. A normal variety $R(X)$ is the *maximal rationally connected (MRC) quotient* of X if there is a rational map $\rho : X \dashrightarrow R(X)$ such that

(1) ρ is defined and proper on a dense open subset X' of X,
(2) the fibers of $\rho|_{X'}$ are rationally connected, and
(3) if Z is a normal variety and $\psi : X \dashrightarrow Z$ is a rational map satisfying properties (1), and (2), there is a unique rational map $\pi : Z \dashrightarrow R(X)$ such that $\psi = \pi \circ \rho$.

For another example, if $X = \mathbb{P}^1 \times C$ is an irrational ruled surface, the MRC quotient of X is just C.

Clearly, if X has an MRC quotient it is unique up to birational equivalence. We shall see that smooth projective varieties always have MRC quotients, and that their very general fibers are rationally connected components.

7.1. **Quotients by algebraic relations.** The construction of the MRC quotient is quite long and involved. We will start from a more general notion of a quotient of X by an algebraic relation. The setup is as follows. Suppose we have reduced quasi-projective schemes T and C, and a diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & X \\ \Big\downarrow{\scriptstyle \pi} & & \\ T & & \end{array}$$

We use this to define an equivalence relation, called $\mathscr{C}$-equivalence, on X, as follows. Two points of X are $\mathscr{C}$-equivalent if there is some finite subset $\{t_1, \ldots, t_m\} \subset T$ such that some connected component of $F(\mathscr{C}_{t_1}) \cup \cdots \cup F(\mathscr{C}_{t_m})$ contains the two points. We will denote the $\mathscr{C}$-equivalence class of a point $x \in X$ by $[x]_{\mathscr{C}}$, or by $[x]$ when there is no possibility of confusion.

For instance, we could let $T = \mathrm{Mor}_d(\mathbb{P}^1, X)$, $\mathscr{C} = \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)$, and take F to be the evaluation map. In this case, two points of X are $\mathscr{C}$-equivalent if they can be joined by a chain of rational curves, all of whose components have degree d. The closure of a $\mathscr{C}$-equivalence class is the set of all points that can be joined to a fixed point by rational curves of degree $\leq d$.

The first step in the construction of the MRC quotient roughly involves showing that if F and π are both *flat* morphisms, then there is a variety Z and a rational map $X \dashrightarrow Z$ whose fibers approximate the $\mathscr{C}$-equivalence classes. For some basic technical relations between constructible sets and flat morphisms which will be used in the course of the proof, we refer the reader to [3, Section 5.1].

Proposition 7.2. *Assume that π is flat with irreducible fibers, and that F is flat. Then there is a dense open subset X' of X, a variety Y, and a morphism $\tau : X' \to Y$ such that*

(1) $\tau(x_1) = \tau(x_2)$ if and only if $[x_1]_{\mathscr{C}}$ and $[x_2]_{\mathscr{C}}$ have equal closure.
(2) if $x \in X_0$, then the general point of $\tau^{-1}(\tau(x))$ can be connected to x by a chain in $\mathscr{C}$ of length $\dim X - \dim Y$.

Proof. The key idea of the proof is that if we let $V_m(x)$ be the set of points in X that can be joined to x by a $\mathscr{C}$-chain of length m, then the *closures* of the $V_m(x)$ eventually stabilize to the closure of $[x]_{\mathscr{C}}$. It will then be relatively easy to construct Y as a subvariety of the Hilbert scheme of X.

Let $\delta(x) = \lim_{m\to\infty} \dim V_m(x)$. We separate the proof into three steps.

Step one: if $m \geq \delta(x)$, then $\overline{V_m(x)} = \overline{V_{m+1}(x)}$.

If x and x' are two points of X, then $\pi(F^{-1}(x))$ meets $\pi(F^{-1}(x'))$ if and only if x and x' are joined by a $\mathscr{C}$-chain of length 1. Starting with $V_0(x) = x$, it follows that

$$V_{m+1}(x) = F(\pi^{-1}(\pi(F^{-1}(V_m(x))))).$$

This implies that the $V_m(x)$ are constructible subsets of X. Let V_m^i be the irreducible components of $V_m(x)$, and let $W_m^{i,j}$ be the irreducible components of $F^{-1}(V_m^i)$. Since F is flat and these sets are all constructible, it follows that

$$F^{-1}(\overline{V_m^i}) = \overline{F^{-1}(V_m^i)} = \bigcup_j \overline{W_{m^{i,j}}} \qquad \text{and} \qquad F(\overline{W_m^{i,j}}) = \overline{V_m^i}.$$

Also since π is flat with irreducible fibers, $\tilde{W}_m^{i,j} = \pi^{-1}(\pi(W_{m^{i,j}}))$ is irreducible. We conclude that

$$\overline{V_{m+1}(x)} = \bigcup_{i,j} \overline{F(\tilde{W}_m^{i,j})},$$

and each $\overline{F(\tilde{W}_m^{i,j})}$ contains $\overline{V_m^i}$.

Now we call a component $\overline{V_m^i}$ of $\overline{V_m(x)}$ *stable* if actually $\overline{F(\tilde{W}_m^{i,j})} = \overline{V_m^i}$ for each j, and *unstable* otherwise. Notice that $\overline{V_{m+1}(x)} = \overline{V_m(x)}$ if and only if each component of $\overline{V_m(x)}$ is stable. If $\overline{V_m^i}$ is unstable, then $\overline{V}_m^i$ is not an irreducible component of $\overline{V_{m+1}(x)}$; on the other hand, if $\overline{V_m^i}$ is stable and an irreducible component of $\overline{V_{m+1}(x)}$, then it is a stable component of $\overline{V_{m+1}(x)}$.

Suppose that $\overline{V_{m+1}(x)}$ has an unstable component. By our above decomposition of $\overline{V_{m+1}(x)}$ into a union of irreducible subsets, this component has the form $\overline{F(\tilde{W}_m^{i,j})}$. Consider the corresponding component $\overline{V_m^i}$ of $\overline{V_m(x)}$. It cannot be stable, since $\overline{F(\tilde{W}_m^{i,j})}$ is not stable. Hence $\overline{V_m^i}$ is a proper irreducible subset of $\overline{F(\tilde{W}_m^{i,j})}$, and it is an unstable component of $\overline{V_m(x)}$. Inductively, there is a chain of such irreducible sets going all the way back to the point $x \in V_0(x)$. From this we conclude that the original unstable component of $\overline{V_{m+1}(x)}$ has dimension at least $m+1$. Therefore if $m \geq \delta(x)$, all components of $\overline{V_m(x)}$ are stable, and $\overline{V_m(x)} = \overline{V_{m+1}(x)}$.

Step two: If $\dim X = n$, then there is an open dense subset X' of X such that if $x, x' \in X'$, then $x' \in \overline{V_n(x)}$ if and only if $\overline{V_n(x)} = \overline{V_n(x')}$.

To see this, let

$$V = \bigcup_{x \in X} \{x\} \times V_n(x) \subset X \times X.$$

The same trick we used to show that $V_m(x)$ is constructible applies to show that V is constructible. Let $q : \overline{V} \to X$ be the first projection.

Since we are in characteristic zero, generic flatness and general results on constructible sets implies that there is a dense open subset X' of X such that q is flat over X', with reduced fibers, and $V_n(x)$ is dense in $q^{-1}(x)$ for all $x \in X'$.

Now suppose that $x \in X'$. If $x' \in V_n(x)$, then any point in $V_n(x')$ can be connected to x by a $\mathscr{C}$-chain of length $2n$, by going through x'. Thus $V_n(x') \subset V_{2n}(x)$, and taking closures and applying step one we see that $\overline{V_n(x')} \subset \overline{V_n(x)}$. By symmetry, if $x, x' \in X'$ and $x' \in V_n(x)$, then $\overline{V_n(x)} = \overline{V_n(x')}$. This implies that the second projection of $q^{-1}(V_n(x) \cap X')$ is just $\overline{V_n(x)}$. Since q is flat and $V_n(x)$ is constructible, $q^{-1}(V_n(x) \cap X')$ is dense in $q^{-1}(\overline{V_n(x)} \cap X')$, so $q^{-1}(\overline{V_n(x)} \cap X')$ also has second projection $\overline{V_n(x)}$. But this says exactly that if $x' \in \overline{V_n(x)} \cap X'$ then $\overline{V_n(x)} = \overline{V_n(x')}$.

Step three: construction of the quotient.

The flat map $\overline{V} \times_X X' \to X'$ induces a map $\phi : X' \to \operatorname{Hilb} X$ sending $x \in X'$ to its (reduced) fiber $\overline{V_n(x)}$. The closure of the image of ϕ is irreducible, and since the image of ϕ is constructible it follows that there is a dense open subset $Y \subset \overline{\operatorname{im} \phi}$ contained in the image of ϕ. Since the scheme-theoretic image of a map from a reduced scheme is the closure of the set-theoretic image with the induced reduced structure, we can factor $\phi|_{\phi^{-1}(Y)}$ as $\phi^{-1}(Y) \to Y \to \operatorname{Hilb} X$. Shrinking X' to $\phi^{-1}(Y)$ and putting $\tau = \phi|_{\phi^{-1}(Y)}$, we claim that $\tau : X' \to Y$ is the desired quotient.

It is clear that $\tau : X' \to Y$ satisfies the first property in the statement of the proposition. For the second, we observe that since q is flat its fibers have the same dimension, namely $\dim X - \dim Y$. It follows that δ, defined for the first step of the proof, equals $\dim X - \dim Y$. Thus the general point of $\overline{V_n(x)} \cap X' = \overline{V_\delta(x)} \cap X'$ (which is the fiber containing x) is joined to x by a $\mathscr{C}$-chain of length δ. □

Unfortunately, for the construction of the MRC quotient we actually need a stronger result than this, which allows the construction of a quotient in case F and π are only proper maps. This construction requires the previous proposition in an essential way. Its proof is very technical, and far from enlightening, so we omit it.

Proposition 7.3. *With the same setup as before, assume F and π are proper. There exists a dense open subset X' of X, a variety Y, and a proper morphism $\rho : X' \to Y$ such that each fiber of ρ is a $\mathscr{C}$-equivalence class.* □

7.2. **Construction of the MRC quotient.** We now have the necessary technical tools to construct the MRC quotient. The basic idea of the proof is to show that there is a positive integer m and a dense open subset X' of X where the rationally connected component containing a very general point $x \in X'$ coincides with the set of points in X' that can be joined to x by a rational chain of degree at most m.

Theorem 7.4. *A smooth projective variety X has an MRC quotient $\rho : X \dashrightarrow R(X)$. Very general fibers of ρ are rationally connected components.*

Proof. Recall that $\operatorname{Rat}_m X$ is the union of the components of the Hilbert scheme $\operatorname{Hilb} X$ whose points correspond to curves of degree at most m, with all components rational. It is a projective scheme. Let $\mathscr{C}_m$ be the universal family over $\operatorname{Rat}_m X$, so that we have a diagram

$$\begin{array}{ccc} \mathscr{C}_m & \xrightarrow{F_m} & X \\ \big\downarrow{\scriptstyle \pi_m} & & \\ \operatorname{Rat}_m X & & \end{array}$$

with F_m and π_m projective. Applying Proposition 7.3, we get for each m a dense open subset X'_m of X and a proper map $\rho_m : X'_m \to Z_m$ whose fibers are $\mathscr{C}_m$-equivalence classes. Since $\mathscr{C}_m$-equivalence classes get larger with m, it follows that $\dim Z_m$ is nonincreasing in m. Thus the sequence $\dim Z_m$ eventually stabilizes, say at $\dim Z_{m_0}$.

The fibers of ρ_m are connected, from which it follows that if $m \geq m_0$ then the general fiber of ρ_m is irreducible of dimension $\dim X - \dim Z_{m_0}$. Let Z'_m be a dense open subset of Z_m where this property holds, and put

$$X''_m = \rho_{m_0}^{-1}(Z'_{m_0}) \cap \rho_m^{-1}(Z'_m).$$

By the construction of Z'_m, we see that two points of X''_m are $\mathscr{C}_{m_0}$-equivalent if and only if they are $\mathscr{C}_m$-equivalent. Hence two points of $\bigcap_{m\geq m_0} X''_m$ are $\mathscr{C}_{m_0}$-equivalent if and only if they can be joined by a chain of rational curves. It also follows from the fact that $\mathscr{C}_m$-equivalence classes are increasing in m that X''_m is a union of fibers of ρ_{m_0}. Putting $R(X) = Z'_{m_0}$ and $\tau = \rho_{m_0}$, we see that the very general fiber of $\tau : X'_{m_0} \to R(X)$ is a rationally connected component.

Showing that τ has the required universal property is straightforward. $\square$

8. Fano varieties

Recall that a smooth projective variety X is a *Fano variety* if its anticanonical bundle $-K_X$ is ample. As we have mentioned before, Fano varieties provide a large class of examples of rationally connected varieties. In this section, we will prove that Fano varieties are uniruled, and sketch a proof that they are rationally connected. The full proof that Fano varieties are rationally connected is similar in spirit to the proof that they are uniruled, but gets somewhat bogged down by technical details.

In this section, we will need to allow our ground field to have positive characteristic. It is precisely for this reason that we did not make the assumption $k = \mathbb{C}$ in Section 2.

The primary technical tool for proving that Fano varieties are uniruled is Mori's "bend-and-break" lemma. This result states that if a curve $f : C \to X$ on a projective variety X can be deformed while keeping the point $f(c)$ fixed, then f can be deformed so as to become reducible, with its components meeting at $f(c)$, and with one of the components rational.

Lemma 8.1 (Bend-and-break). *If $f : C \to X$ is a curve on a projective variety X, $c \in C$, and $\dim_{[f]} \mathrm{Mor}(C, X; f|_{\{c\}}) \geq 1$, there exists a rational curve on X through $f(c)$.*

Proof. We let T be the normalization of a 1-dimensional subvariety of $\mathrm{Mor}(C, X; f|_{\{c\}})$, and let $\overline{T}$ be a smooth compactification of T. We consider the rational evaluation map $\mathrm{ev} : C \times \overline{T} \dashrightarrow X$.

We claim that ev cannot be defined at every point of $\{c\} \times \overline{T}$. If it can be, then $\mathrm{ev}(\{c\} \times \overline{T}) = f(c)$. Letting U be an affine open subset of X containing $f(c)$, there is an open neighborhood V of $c \in C$ such that $\mathrm{ev}(V \times \overline{T}) \subset U$. Now for any $x \in V$, ev maps $\{x\} \times \overline{T}$ to a complete variety contained in an affine subset of X; this implies that it maps it to a point, which is of course $f(x)$. As V is dense in C, it follows that actually $\mathrm{ev}(x, \bar{t}) = f(x)$ for all $x \in C$ and $\bar{t} \in \overline{T}$. But this is absurd, since it implies that all the elements of T correspond to the same curve f.

Thus there is some $t_0 \in \overline{T}$ such that ev is not defined at (c, t_0). This indeterminacy can be resolved by a sequence of blowups at points, which produces a surface S whose fiber over (c, t_0) is a connected chain of rational curves E. At least one of these curves is not contracted by the composition $S \to C \times \overline{T} \to X$, for otherwise the indeterminacy would not possibly be resolved. Moreover, since the strict transform of $\{c\} \times \overline{T}$ meets E, we know that there is some non-contracted rational curve in S whose image in X meets $f(c)$. □

In case C is rational, this result obviously tells us nothing. We will need a similar result for the case $C = \mathbb{P}^1$, which allows us to break up a rational curve into a reducible rational curve, both of whose components have smaller degree with respect to a fixed ample divisor. The proof of the next result is fairly similar to that of the previous, so we omit it.

Proposition 8.2. *If $f : \mathbb{P}^1 \to X$ is a rational curve on a projective variety X and $\dim_{[f]} \mathrm{Mor}(\mathbb{P}^1, X; f|_{\{0,\infty\}}) \geq 2$, then $f_*\mathbb{P}^1$ is numerically equivalent to a connected reducible effective rational 1-cycle passing through $f(0)$ and $f(\infty)$.* □

We are now in position to prove that Fano varieties are uniruled. Perhaps somewhat surprisingly, the method of proof is to prove the result first for varieties over a field of characteristic p. From there we deduce the result in characteristic zero. Apparently, there is no known proof of this result using only analytic techniques.

The key thing that makes the argument work in the characteristic p case is the *Frobenius map.* Given a curve C, the Frobenius map is a map $C \to C$ which is the identity on topological spaces, but acts by the Frobenius morphism $x \mapsto x^p$ on local rings. Note that the Frobenius morphism is *not* a morphism of k-schemes. This can be fixed, however, by giving C a new k-structure; call this new curve C'. Then C and C' are isomorphic as abstract schemes (but generally not as schemes over k), so in particular they have the same genus $H^1(X, \mathcal{O}_X)$, as this number is independent of the k-structure. However, the Frobenius map $C' \to C$ has degree p, so if $C \to X$ is a curve on X of degree d then composing with the Frobenius gives us a curve on X of degree pd and the same genus.

No doubt this seems like a cheat and a bit foreign to anybody who deals primarily with characteristic zero. After all, the fact that the degrees of $C' \to X$ and $C \to X$ are different is completely undetectable by the maps on topological spaces. But then one must realize that the intuitive notion of the degree of a map as being the size of a general (more precisely, unramified) fiber fails completely in characteristic p: the Frobenius map is ramified *everywhere*, so there are no fibers nice enough for the cardinality to give a handle on the degree.

Theorem 8.3. *Any Fano variety X is uniruled. More precisely, if $n = \dim X$, there is through any point $x \in X$ a rational curve C with $(-K_X)$-degree at most $n+1$.*

Proof. Fist we prove the result in characteristic p. Let $x \in X$, and choose an arbitrary curve $f : C \to X$ with $f(c) = x$. Recall the dimension estimate

$$\dim_{[f]} \mathrm{Mor}(C, X; f|_{\{c\}}) \geq -K_X \cdot f_*C - n \cdot g(C)$$

of Corollary 2.7. Since $-K_X \cdot f_*C \geq 1$, we can, by repeated application of the "Frobenius trick" discussed above, ensure that this estimate is ≥ 1. Then by the bend-and-break Lemma 8.1, we can produce a rational curve $g : \mathbb{P}^1 \to X$ on X passing through c.

Next we potentially need to decrease the $(-K_X)$-degree of g. We do this by applying Proposition 8.2 repeatedly. Applying Corollary 2.7 again, Proposition 8.2 will apply so long as

$$-K_X \cdot g_*\mathbb{P}^1 + n(g(\mathbb{P}^1) - 1) \geq 2,$$

i.e. so long as the $(-K_X)$-degree of g is at least $n+2$. Thus we obtain a rational curve through x of degree at most $n+1$, when the characteristic of k is positive.

Now assume that k has characteristic zero. Embed X in a projective space $\mathbb{P}^n_k$, and assemble a finite set of defining equations for X. Let R be the $\mathbb{Z}$-algebra generated by the coefficients of the defining equations of X, together with the coordinates of the point x under the embedding in $\mathbb{P}^n_k$. Then we can construct the obvious scheme $\mathscr{X} \subset \mathbb{P}^n_{\mathrm{Spec}\, R}$. It is by definition a projective scheme over $\mathrm{Spec}\, R$, and has an R-point x_R (i.e. a section of the structure map $\mathscr{X} \to \mathrm{Spec}\, R$). Moreover, X is recovered from $\mathscr{X}$ by looking at the generic fiber of $\mathscr{X} \to \mathrm{Spec}\, R$, and then making a base change from Spec of the field of fractions of R to $\operatorname{Spec} k$. The point $x \in X$ is recovered from the R-point x_R by taking the value at the generic point of $\operatorname{Spec} R$.

Since we are in characteristic 0 and $\mathscr{X} \to \operatorname{Spec} R$ is projective, there is a dense open subset $U \subset \operatorname{Spec} R$ such that $\mathscr{X}_U \to U$ is smooth. The geometric generic fiber of $\mathscr{X}_U \to U$ is a Fano variety, so the restriction of the relative antidualizing sheaf $\omega^*_{\mathscr{X}_U/U}$ to the geometric generic fiber is ample. Ampleness is an open property, so after shrinking U we may assume that all the fibers of $\mathscr{X}_U \to U$ are Fano varieties.

Consider the quasi-projective Noetherian scheme

$$\rho : \coprod_{i=1}^{n+1} \operatorname{Mor}_{\operatorname{Spec} R, i}(\mathbb{P}^1_R, \mathscr{X}; 0 \mapsto x_R) \to \operatorname{Spec} R$$

parameterizing rational curves of $(-K_X)$-degree at most $n+1$ in fibers of $\operatorname{Spec} R$ passing through x_R. Since R is a finitely generated $\mathbb{Z}$-algebra, if $\mathfrak{m}$ is a maximal ideal of R then $R/\mathfrak{m}$ is a finite field. Thus the geometric fiber over a closed point in U is a Fano variety over a field of positive characteristic, and hence the image of ρ contains the closed points of U. But maximal ideals are dense in $\operatorname{Spec} R$ (again since R is a finitely generated $\mathbb{Z}$-algebra), so the image of ρ is dense in $\operatorname{Spec} R$. The image of ρ is a constructible set, and therefore it contains the generic point. This implies that the theorem holds for X. □

To prove that Fano varieties are rationally connected, we will need the following result which guarantees that there are rational curves transverse to the fibers of any given rational map. Once again, the proof works by reduction to the case of characteristic p. In this case, however, the proof in the characteristic p case is not nearly so simple. It makes use of a more sophisticated version of the bend-and-break lemma to produce curves transverse to the fibers of a map, and a handful of nasty estimates of intersection numbers.

Lemma 8.4. *Let X be a Fano variety and let Y be a quasi-projective variety. Let X' be a dense open subset of X and let $\pi : X' \to Y$ be a proper nonconstant morphism. There exists, for any point y of $\pi(X')$, a rational curve on X that meets $\pi^{-1}(y)$ but is not contracted by π.* □

Given the lemma, we can now use the existence of MRC quotients to prove that Fano varieties are rationally connected.

Theorem 8.5. *A Fano variety X defined over $\mathbb{C}$ is rationally connected.*

Proof. Let X' be the locus where the MRC quotient map $\rho : X \dashrightarrow R(X)$ is defined. Choose a very general $x \in X'$, so that the fiber $\rho^{-1}(\rho(x))$ is a rationally connected component of X'. If $\rho : X' \to R(X)$ is nonconstant, the lemma gives us a rational curve in X meeting $\rho^{-1}(\rho(x))$ which is not contracted by ρ. This is absurd, since $\rho^{-1}(\rho(x))$ is a rationally connected component. Thus ρ is constant, $R(X)$ is a point by its universal property, and X is rationally connected. □

9. Fibrations

In this section, we discuss the recent result of Graber, Harris, and Starr [4] concerning fibrations of varieties whose general fibers are rationally connected. We will again fix $k = \mathbb{C}$ for this section.

Theorem 9.1. *Let $\pi : X \to B$ be a proper morphism of varieties, with B a smooth curve. If the general fiber of π is rationally connected, then π has a section.*

Given this theorem, the following corollary is a straightforward application of Chow's lemma and the fact that rational chain-connectedness implies rational connectedness on a smooth projective variety.

Corollary 9.2. *Let $\pi : X \to Y$ be a map of varieties, with Y rationally connected. If the general fiber of π is rationally connected, then X is rationally connected.* □

Another consequence of Theorem 9.1 is that the MRC quotient $R(X)$ of a variety X is *never* uniruled. For if it were uniruled, picking a rational curve through a very general point of $R(X)$ and lifting to a rational curve in X shows that no fibers of $X \dashrightarrow R(X)$ are rationally connected components of X.

Corollary 9.3. *The MRC quotient of a smooth projective variety X is not uniruled.* □

For one more consequence, recall that in Propositions 5.8 and 6.13 we showed that a uniruled variety has vanishing plurigenera and a rationally connected variety has no global covariant tensor fields. We remarked there that the converses of these statements are known as Mori's conjecture and Mumford's conjecture, respectively.

Corollary 9.4. *Mori's conjecture implies Mumford's conjecture: if every smooth projective variety with vanishing plurigenera is uniruled (with the exception of a point, of course), then every smooth projective variety with no global covariant tensor fields is rationally connected.*

Proof. Let $X \dashrightarrow R(X)$ be the MRC fibration of the smooth projective variety X; we may take $R(X)$ to be proper. If X is not rationally connected, then $\dim R(X) \geq 1$. Since $R(X)$ is not uniruled, there is by Mori's conjecture a nonzero global section of $K_{R(X)}^{\otimes m}$. Since the indeterminacy locus of $X \dashrightarrow R(X)$ has codimension at least 2 in X, and since X is normal, we can view the pullback of this section as a nonzero global section of $(\Omega_X^1)^{\otimes nm}$, where n is the dimension of $R(X)$. □

In the remainder of the section, we will give a sketch of the proof of Theorem 9.1. First, we show that it is enough to prove the theorem in the case $B \cong \mathbb{P}^1$.

Lemma 9.5. *If Theorem 9.1 holds whenever $B \cong \mathbb{P}^1$, then it holds for arbitrary smooth curves B.*

Proof. Suppose that $\pi : X \to B$ satisfies the hypotheses of Theorem 9.1. Choose a branched cover $g : B \to \mathbb{P}^1$. We form a variety $Y \to \mathbb{P}^1$ (the *norm* of X) whose fiber over $p \in \mathbb{P}^1$ is the product of the fibers X_q, as q ranges over $g^{-1}(p)$. The product of rationally connected varieties is rationally connected, so $Y \to \mathbb{P}^1$ admits a section by the case $B \cong \mathbb{P}^1$ of Theorem 9.1. This clearly implies that $X \to B$ admits a section. □

From here on out, we will therefore assume that $B \cong \mathbb{P}^1$. Moreover, it is easy to see by Chow's lemma that we may assume X is a smooth projective variety. So the situation is that we have a map $\pi : X \to B \cong \mathbb{P}^1$ with X a smooth projective variety, such that X_b is rationally connected for general $b \in B$.

Recall the definition of the coarse moduli space of unmarked stable maps $\overline{M}_{g,0}(X, \beta)$ from Subsection 2.1. If $\pi_*\beta = d \in N_1(\mathbb{P}^1)$, then composition with π gives a natural map

$$\varphi : \overline{M}_{g,0}(X, \beta) \to \overline{M}_{g,0}(B, d).$$

Definition 9.6. A stable map $f : C \to X$ with C irreducible and nodal of genus g with $f_*[C] = \beta$ is *flexible* relative to π if φ is locally dominant at $[f]$.

Suppose for the moment that X admits a stable map $[f]$ which is flexible relative to π. Normalizing C if necessary, we may assume that C is smooth. Since $\overline{M}_{g,0}(\mathbb{P}^1, d)$ has a unique component whose general member corresponds to a stable map from a smooth curve, it follows from the properness of φ that φ maps $\overline{M}_{g,0}(X, \beta)$ onto this component of $\overline{M}_{g,0}(B, d)$. This component contains stable maps from chains of rational curves with every component either mapping isomorphically to B or mapping to a point; a stable map to X mapping to such a stable map to B then allows us to construct a section of π. Hence it will be our goal to construct a flexible stable map.

Lemma 9.7. *If $f : C \to X$ admits a flexible stable map, then π has a section.*

At this point, we perform a construction that is analogous to the comb construction in the proof that rational chain-connectedness implies rational connectedness. The idea is that, given an unmarked stable map from a smooth curve such that $\mu = \pi \circ f$ is surjective, we can glue on very free rational curves to increase the ampleness of the normal bundle of the original stable map. Increasing the ampleness of the normal bundle increases how many deformations the stable map has, and hence should allow us to produce a flexible stable map.

Proposition 9.8. *Let $f : C \to X$ be a stable map from a smooth unmarked curve to X, with μ surjective, and fix an integer n. Provided the image of f meets a smooth rationally connected fiber of π, there is another stable map $\tilde{f} : \tilde{C} \to X$ from a smooth unmarked curve to X, with the following properties.*

(1) $\tilde{\mu} = \pi \circ \tilde{f}$ is surjective, and has the same degree as μ. The branch divisor of $\tilde{\mu}$ is a small deformation of the branch divisor of μ.

(2) $\tilde{C}$ and C have the same genus.

(3) For any n points $q_1, \ldots, q_n \in \tilde{C}$, we have $H^1(\tilde{C}, N_{\tilde{f}}(-q_1 - \cdots - q_n)) = 0$.

Sketch of proof. Since μ is surjective and the image of f meets a smooth rationally connected fiber of π, we can find lots of rational curves contained in fibers of π which are very free in their fibers. Gluing these curves on to the stable map f gets us a stable map satisfying the vanishing condition on H^1. Deformations of this stable curve are not obstructed, so smoothings exist and a general such smoothing still has the H^1 vanishing property. The other points are clear. □

Now assume that we have found an unmarked stable map $f : C \to X$ from a smooth curve C with μ surjective, such that μ is simply branched and all ramification points p of μ map via f to smooth points of the fiber $X_{\mu(p)}$ of π containing them. Apply the proposition to f, taking $n = 2d+2g-2$ (this is the number of branch/ramification points of μ). The stable map $\tilde{f} : \tilde{C} \to X$ can be assumed to still satisfy all these properties, in addition to satisfying the ampleness property on the normal bundle guaranteed by the proposition. This ampleness property, together with the assumption that ramification points of $\tilde{\mu}$ map to smooth points in fibers of π via $\tilde{f}$, implies that deformations of $\tilde{f}$ move the branch divisor of $\tilde{\mu}$ in any specified direction in the variety $B_{2d+2g-2}$. Since a smooth simply branched covering of $\mathbb{P}^1$ can be reconstructed from the set of branch points alone, it follows that general deformations of $\tilde{f}$ yield general deformations of $\tilde{\mu}$. This is to say that φ is dominant near $\tilde{f}$, or $\tilde{f}$ is flexible.

Definition 9.9. An unmarked stable map $f : C \to X$ from a smooth curve C with μ simply branched and such that the ramification points of μ map via f to smooth points of fibers of π is *pre-flexible.*

Proposition 9.10. *If $\pi : X \to B$ admits a pre-flexible stable map, then π has a section.*

Thus we are reduced to showing that π admits a pre-flexible stable map. To do this, we first outline one more basic construction. This construction is more or less analogous to the construction of Proposition 9.8. The difference is that in one of the fibers X_p, we choose a very free rational curve which joins *two* of the points of the curve $f : C \to X$, and glue that in to obtain a new stable map. Adding lots of very free teeth as in Proposition 9.8, we can smooth the stable map. Doing so increases the genus of the curve by one, and adds two new simple branch points near p. The monodromy around each branch point is a transposition of the two sheets of points nearby to the two points joined by the rational curve in X_p.

Proposition 9.11. *Let $f : C \to X$ be a stable map from a smooth unmarked curve to X, with μ surjective, and fix an integer n. Provided the image of f meets a smooth rationally connected fiber of π, there is another stable map $\tilde{f} : \tilde{C} \to X$ from a smooth unmarked curve to X, with the following properties.*

(1) $\tilde{\mu} = \pi \circ \tilde{f}$ is surjective, and has the same degree as μ. The branch divisor of $\tilde{\mu}$ is a small deformation of the branch divisor of μ, plus two new branch points whose monodromy equal any given transposition.
(2) The genus of $\tilde{C}$ is one larger than the genus of C.
(3) For any n points $q_1, \dots, q_n \in \tilde{C}$, we have $H^1(\tilde{C}, N_{\tilde{f}}(-q_1 - \cdots - q_n)) = 0$. □

This construction is the chief tool in producing a pre-flexible stable map.

Theorem 9.12. *The morphism $\pi : X \to B$ admits a pre-flexible stable map.*

Sketch of Proof. We start by embedding X in projective space, and taking the right number of hyperplane sections of X so as to get a curve. This can be done in such a way that the curve we get C is smooth, and such that if $f : C \to X$ is the inclusion, then ramification points of $\mu = \pi \circ f$ occur either at smooth points of fibers of π (by a dimension count) or in multiple fibers of π. It is these multiple fibers which are particularly problematic: if $\pi : X \to B$ is to have a section, it better not have any multiple fibers which are contained in the smooth locus of X.

Let M be the set of points of B over which fibers of π have any multiple components. We first fix a base point p_0, so as to keep track of monodromy data, and we draw disjoint real arcs on B from p_0 to each branch point of μ. Write the monodromy around a point $b \in M$ as a product of transpositions. Using Proposition 9.11, we construct a new stable map with pairs of extra branch points, with each branch point in a pair having monodromy equal to the monodromy of a transposition in the product of transpositions representing the monodromy of b. These new branch points can be moved independently, by the ampleness condition on the normal bundle of the new stable map. Then we take one of the new branch points in each pair, and specialize them all to b. We do this simultaneously for all $b \in M$. Take some irreducible component where the limiting stable map is nonconstant, and normalize it. Then the monodromy around each point $b \in M$ vanishes by construction, so the limiting stable map is unramified at fibers with multiple components. But this construction can be performed in such a way that all the unused new branch points and all the old branch points remain in smooth points of their fibers, so this stable map is pre-flexible. □

This completes the proof of Theorem 9.1. □

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