THE NOETHER-LEFSCHETZ THEOREM

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Contents

1. Introduction

In this paper, we collect and flesh out results from Carlson et al. [1] to give a relatively self-contained proof of the Noether-Lefschetz theorem via Hodge theory and deformation theory. The principal result is the following theorem.

Theorem (Noether-Lefschetz)**.** *Assume that $d \geq 4$. Outside a countable union of proper subvarieties in the parameter space of degree d surfaces in $\mathbb{P}^3$, the Picard group of the corresponding surface X is isomorphic to $\mathbb{Z}$, generated by $\mathcal{O}_X(1)$. Hence every curve on X is a complete intersection.*

The first six sections, and first half of this paper, comprise the results fundamentally related to the proof of the theorem, culminating with the proof in Section 6.

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The second half of the paper is an appendix on the theory of spectral sequences, in which we develop the algebraic techniques necessary for the proof of the theorem. The appendix culminates with a theorem that allows the construction of a spectral sequence of abelian groups associated to a filtered complex of sheaves. The Hodge to de Rham spectral sequence is seen to be a special case of this general construction. Another example of this spectral sequence is crucial to our proof of the Noether-Lefschetz theorem.

2. Residue and tube over cycle maps

Our goal in this section is to try and understand the cohomology of X in terms of the cohomology of the complement $U = \mathbb{P}^{n+1} \setminus X$. There are a couple of tools for doing this. The most fundamental is the residue map, which takes a p-form on $\mathbb{P}^{n+1}$ with at most a simple pole along X and gives a holomorphic $(p-1)$-form on X. The other tool is the "tube over cycle map" $\tau : H_{k-2}(X;\mathbb{Q}) \to H_{k-1}(U;\mathbb{Q})$. This map acts by taking a homology class in X and taking a sufficiently small circle bundle over the homology class, viewed as a subset of the boundary of the disk bundle over X in $\mathbb{P}^{n+1}$.

Consider the long exact sequence in cohomology of the pair $(\mathbb{P}^{n+1}, U)$. This is by definition the long exact sequence

$$\cdots \to H^{k-1}(U) \to H^k(\mathbb{P}^{n+1}, U) \to H^k(\mathbb{P}^{n+1}) \to H^k(U) \to \cdots .$$

Using the Thom isomorphism theorem and excision, we can replace $H^k(\mathbb{P}^{n+1}, U)$ with $H^{k-2}(X)$. When we do so, the resulting long exact sequence is

$$\cdots \to H^{k-1}(U) \xrightarrow{r} H^{k-2}(X) \xrightarrow{i_*} H^k(\mathbb{P}^{n+1}) \xrightarrow{j^*} H^k(U) \to \cdots .$$

Here r is the adjoint of the aforementioned tube over cycle map τ, i_* is the Gysin map in cohomology associated to the inclusion $i : X \hookrightarrow \mathbb{P}^{n+1}$, and j^* is the induced map in cohomology of the inclusion $j : U \hookrightarrow P$.

The tube over cycle map and its transpose r helps us to compute the primitive middle cohomology $H^n_{\text{var}}(X)$ of X.

Proposition 2.1. *The map $r : H^{n+1}(U) \to H^n(X)$ is injective, with image $H^n_{\text{prim}}(X)$. In short, $r : H^{n+1}(U) \to H^n_{\text{prim}}(X)$ is an isomorphism. Moreover, the cohomology $H^n(X)$ splits as*

$$H^n(X) = H^n_{\text{prim}}(X) \oplus H^n_{\text{fix}}(X),$$

where $H^n_{\text{fix}}(X) = i^* H^n(\mathbb{P}^{n+1})$. This splitting is orthogonal.

Proof. We examine the Gysin sequence

$$H^{n+1}(\mathbb{P}^{n+1}) \to H^{n+1}(U) \xrightarrow{r} H^n(X) \to H^{n+2}(\mathbb{P}^{n+1}).$$

First suppose n is odd. Then $H^{n+2}(\mathbb{P}^{n+1})$ vanishes and so r is surjective. Its image is $H^n_{\text{prim}}(X)$, since all the cohomology of X is primitive. For injectivity, it is equivalent to see that the map $H^{n+1}(\mathbb{P}^{n+1}) \to H^{n+1}(U)$ is zero. Then we can instead show $H^{n-1}(X) \to H^{n+1}(P)$ is surjective. This map is Poincaré dual to the isomorphism $H_{n-1}(X) \to H_{n-1}(\mathbb{P}^{n+1})$, so is in fact surjective.

If instead n is even, then r is injective. Its image is the kernel of i_*. So we need to show that the kernel of i_* is the same thing as primitive cohomology. Suppose that $\alpha \in H^n(X)$ is a cohomology class, with Poincaré dual γ. Write $\omega \in H^2(\mathbb{P}^{n+1})$ for the cohomology class of a hyperplane H transverse to γ such that the hyperplane section $H \cap X$ is also transverse to γ in X. Then $i^*\omega$ is the cohomology class of the hyperplane section $H \cap X$. The Gysin homomorphism $i_* : H^{n+2}(X) \to H^{n+4}(\mathbb{P}^{n+1})$ is Poincaré dual to the isomorphism $i_* : H_{n-2}(X) \to H_{n-2}(\mathbb{P}^{n+1})$, so $i_*(\alpha \smile i^*\omega) = 0$ if and only if $\alpha \smile i^*\omega = 0$.

Now we claim that

$$i_*(\alpha \smile i^*\omega) = i_*(\alpha) \smile \omega.$$

Indeed,

$$\begin{aligned} i_*(\alpha \smile i^*\omega) &= [i_*[\alpha \smile i^*\omega]] = [i_*(\gamma \cap (H \cap X))] \\ &= [i_*\gamma \cap i_*(H \cap X)] = [i_*\gamma \cap H] = i_*\alpha \smile \omega, \end{aligned}$$

where brackets denote Poincaré dual. But $i_*\alpha \smile \omega = 0$ if and only if $i_*\alpha = 0$, so $\alpha \smile i^*\omega = 0$ if and only if $i_*\alpha = 0$. Thus the kernel of i_* is the primitive cohomology $H^n_{\text{prim}}(X)$.

For the decomposition theorem, we observe that X has no primitive cohomology in dimensions other than n and 0. It follows from the Lefschetz decomposition theorem that $H^n(X) = H^n_{\text{prim}}(X) \oplus H^n_{\text{fix}}(X)$. In fact, the decomposition is orthogonal, for if α is primitive and $\beta = i^*\omega^{n/2}$ then

$$i_*(\alpha \smile \beta) = i_*\alpha \smile \omega^{n/2} = 0$$

since α is in the kernel of the Gysin homomorphism. □

For residue maps, the setup is the following. Suppose that α is a differential p-form on $\mathbb{P}^{n+1}$ with at most a logarithmic pole along X. If $V \subset \mathbb{P}^{n+1}$ is a coordinate open set where X has equation $f = 0$ and there are local coordinates $(x_0 = f, x_1, \ldots, x_n)$, we can uniquely write

$$f\alpha = df \wedge \eta + \eta'$$

where η and η' are holomorphic forms involving only $dx_1, \ldots, dx_n$. Then

$$\alpha = \frac{df}{f} \wedge \eta + \frac{\eta'}{f}.$$

When we take the exterior derivative of this expression, the first summand still has a pole of order at most 1 along X since whenever $1/f$ appears df does as well, so that no partial derivatives of $1/f$ are taken with respect to f. But we will get a second order pole along X from the second term unless if all the coefficient functions of η' are divisible by f. So in order for $f d\alpha$ to be holomorphic, it is necessary that η'/f be holomorphic. Thus any α with at most a log-pole along X can be written uniquely in the form

$$\alpha = \frac{df}{f} \wedge \eta + \eta'$$

for holomorphic $(p-1)$-forms η and η' involving only $x_1, \ldots, x_n$. We then let

$$(\operatorname{Res} \alpha)|_{V \cap X} = \eta|_{V \cap X},$$

and patch together the different expressions over an open covering of $\mathbb{P}^{n+1}$ to form $\operatorname{Res} \alpha$.

Lemma 2.2. $\operatorname{Res} \alpha$ *is a well-defined holomorphic* $(p-1)$*-form on* X*.*

Proof. Notice that η and η' are in fact uniquely determined by f, α, V, and the choice of coordinate system, since $f\alpha$ is a holomorphic p-form. First we show that $(\operatorname{Res} \alpha)|_{V \cap X}$ is independent of the choice of defining equation f. If f' also defines X on V, we also have coordinates $(f', x_1, \ldots, x_n)$ on V; there is no need to change the other n coordinates. Now there is a nowhere vanishing function u such that $f = uf'$. Then

$$\frac{df}{f} = \frac{d(uf')}{uf'} = \frac{1}{uf'}(u\, df' + du\, f') = \frac{df'}{f'} + \frac{du}{u}.$$

If η and η' correspond to the choice of equation f, then it follows that

$$\alpha = \frac{df'}{f'} \wedge \eta + \frac{du}{u} \wedge \eta + \eta' = \left(\frac{df'}{f'} + \frac{1}{u} \frac{du}{df'} df' \right) \wedge \eta + \cdots ,$$

where the terms in the $\cdots$ involve only $dx_1, \ldots, dx_n$. We can rewrite this as

$$\frac{df'}{f'} \left(1 + \frac{f'}{u} \frac{du}{df'} \right) \wedge \eta + \cdots = \frac{df'}{f'} \wedge \left(1 + \frac{f'}{u} \frac{du}{df'} \right) \eta + \cdots ,$$

so the η for f' is $(1 + f' u^{-1}(du/df'))\eta$. As $f' = 0$ on X, the two η's agree when restricted to X.

Next we show that $\operatorname{Res} \alpha$ is independent of the choice of the coordinates $(x_1, \ldots, x_n)$, having already established independence of the choice of $x_0 = f$. So suppose we choose the system differently as $y_1, \ldots, y_n$. We can write

$$\alpha = \frac{df}{f} \wedge \psi + \psi',$$

where ψ and ψ' involve only $dy_1, \ldots, dy_n$. We are to show that $\psi|_X = \eta|_X$. Then

$$dy_i = \frac{\partial y_i}{\partial f} df + \sum_j \frac{\partial y_i}{\partial x_j} dx_j = \sum_j \frac{\partial y_i}{\partial x_j} dx_j$$

since f is a coordinate in both coordinate systems. So we see that ψ and ψ' already only involve the dx_j. This forces $\psi = \eta$ and $\psi' = \eta'$ by the uniqueness remark in the first sentence of the proof.

Since the construction is independent of the coordinate system, it follows that the forms glue together on overlaps to give a globally defined holomorphic $(p-1)$-form on X. □

The fundamental result relating the tube over cycle and residue maps is the "Residue Formula."

Theorem 2.3 (Residue Formula)**.** *Let $\alpha \in \Omega^p_P(X)$ be a holomorphic p-form on P with at most a simple pole along X. For any homology class $\gamma \in H_{p-1}(X)$,*

$$\int_\gamma \operatorname{Res} \alpha = \frac{1}{2\pi i} \int_{\tau(\gamma)} \alpha.$$

Proof. First we must give a more detailed description of what we mean by $\tau(\gamma)$. This was defined to be the homology class of the boundary of a sufficiently small tubular neighborhood of γ in the restriction of the normal bundle of X in P to γ. Here, sufficiently small means that we make the neighborhood smaller and smaller until the homology class remains constant as we make it smaller.

Assume that γ is a compact submanifold of X. Cover X by finitely many coordinate open sets U_i where X is given by $f = 0$. Let $\tilde{U}_i$ be open subsets of P such that $\tilde{U}_i$ is an open neighborhood of U_i in P and $\tilde{U}_i$ lies over U_i under the identification of a neighborhood of the zero section of $N_{X/P}$ with a neighborhood of X. Let ξ_i be a partition of unity subordinate to $\{U_i\}$ on X, and let $\tilde{\xi}_i$ be the pullback to $\tilde{U}_i$ under the projection $\tilde{U}_i \to U_i$. If

$$\alpha = \frac{df}{f} \wedge \eta + \eta',$$

then

$$\tilde{\xi}_i \alpha = \frac{df}{f} \wedge \tilde{\xi}_i \eta + \tilde{\xi}_i \eta',$$

which proves that

$$\operatorname{Res}\tilde{\xi}_i\alpha = \xi_i \operatorname{Res}\alpha.$$

The upshot is that it suffices to show in local coordinates $(f, x_1, \ldots, x_n)$ that if α is compactly supported on the hyperplane $f = 0$, then

$$\int_\gamma \operatorname{Res}\alpha = \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{\tau_\varepsilon(\gamma)} \alpha,$$

where by $\tau_\varepsilon(\gamma)$ we mean the set of points over γ with $|f| = \varepsilon$.

Since we don't need to use holomorphic coordinates (we're already using a partition of unity, so they will be useless), we may in fact assume that γ is specified by the vanishing of some of the coordinate functions x_i. The disk bundle over γ is given by points over γ with $|f| < \varepsilon$. The integral of η' over $\tau(\gamma)$ vanishes because a vector in the span of $\partial/\partial f$ and $\partial/\partial\overline{f}$ is in the tangent space to $\tau(\gamma)$ at each point, so that η' restricts to 0 on $\tau(\gamma)$.

As ε becomes small, the coefficients of η becomes essentially independent of f, since they must converge to the coefficients of $\eta|_X$. Viewing $\eta|_X$ locally as a form on P by pullback from the projection onto X, we are therefore justified in assuming the coefficients of η are independent of f. But then

$$\int_{\tau(\gamma)} \alpha = \int_{\tau(\gamma)} \frac{df}{f}\wedge\eta = \int_{S^1}\frac{df}{f}\int_\gamma \eta = 2\pi i\int_\gamma \operatorname{Res}\alpha$$

by Fubini's theorem. □

Similar to the Gysin sequence is the long exact sequence in hypercohomology associated to the short exact sequence of chain complexes of sheaves

$$0 \to \Omega^\bullet_{\mathbb{P}^{n+1}} \to \Omega^\bullet_{\mathbb{P}^{n+1}}(\log X) \stackrel{\operatorname{Res}}{\to} \Omega_X^{\bullet-1} \to 0.$$

This is in fact a short exact sequence, for the kernel of Res locally consists of forms $\alpha = (df/f)\wedge\eta+\eta'$ where η vanishes along X, so that all the coefficients of η are divisible by f and α is holomorphic. The terms in the long exact sequence of hypercohomology are the same as the terms of the Gysin sequence, due to the following result.

Theorem 2.4. *(1) There is a commutative diagram*

$$\begin{array}{ccccccc}
H^{k+1}(U) & \xrightarrow{\quad r \quad} & H^k(X) & \xrightarrow{i_*} & H^{k+2}(\mathbb{P}^{n+1}) & \xrightarrow{\quad j^* \quad} & H^{k+2}(U) \\
\downarrow & & \downarrow{\scriptstyle \frac{1}{2\pi i}} & & \downarrow & & \downarrow \\
H^{k+1}(U;\mathbb{C}) & \xrightarrow{\text{Res}} & H^k(X;\mathbb{C}) & \xrightarrow{\delta} & H^{k+2}(\mathbb{P}^{n+1};\mathbb{C}) & \xrightarrow{j^*} & H^{k+2}(U;\mathbb{C}) \\
\uparrow & & \uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \sim} & & \uparrow \\
\mathbb{H}^{k+1}(\Omega^\bullet_{\mathbb{P}^{n+1}}(\log X)) & \xrightarrow{\text{Res}} & \mathbb{H}^k(\Omega^\bullet_X) & \xrightarrow{\delta} & \mathbb{H}^{k+2}(\Omega^\bullet_{\mathbb{P}^{n+1}}) & \longrightarrow & \mathbb{H}^{k+2}(\Omega^\bullet_{\mathbb{P}^{n+1}}(\log X)).
\end{array}$$

(2) There are natural isomorphisms

$$\mathbb{H}^p(\Omega^\bullet_{\mathbb{P}^{n+1}}(\log X)) \xrightarrow{\sim} \mathbb{H}^p(\Omega^\bullet_{\mathbb{P}^{n+1}}(*X)) \xrightarrow{\sim} \mathbb{H}^p(\Omega^\bullet_U) \xrightarrow{\sim} H^p(U;\mathbb{C}).$$

In fact, all four of the vertical maps in the bottom row of the above diagram are isomorphisms.

Proof. (1) Commutativity of the upper left square follows from Theorem 2.3 since this formula shows that Res is the transpose of τ up to a factor of $2\pi i$. The connecting morphism in the middle is defined by the connecting morphism on the bottom, but morally should be the same as the connecting morphism in the exact sequence of the pair (P, U), up to the Thom isomorphism. The top i_* was also defined by this property, although it happens to be the Gysin homomorphism. The lower right square is the other slightly nontrivial one, but the top map is defined by restriction of forms, while the bottom is just an inclusion. It pretty clearly commutes. The vertical maps that are noted to be isomorphisms are in fact isomorphisms by the Abstract de Rham Theorem A.3.1, as shown in the appendix on spectral sequences.

(2) We first note that $\mathbb{H}^p(\Omega^\bullet_U)$ is isomorphic to $H^p(U;\mathbb{C})$ whether we view $\Omega^\bullet_U$ as a sheaf on $\mathbb{P}^{n+1}$ or as a sheaf on U. This is because $\Omega^\bullet_U(V) = \Omega^\bullet_U(V \setminus X)$ for any open $V \subset \mathbb{P}^{n+1}$, so that if we fix a good cover of $\mathbb{P}^{n+1}$ whose intersections with U are simultaneously good for U, then all the terms in the Čech-de Rham double complex for $\Omega^\bullet_U$ are equal, while it follows from the abstract de Rham theorem that $\mathbb{H}^p(U, \Omega^\bullet_U) \cong H^p(U;\mathbb{C})$.

By applying the five-lemma to the diagram of the previous part, we see that the composition $\mathbb{H}^p(\Omega^\bullet_{\mathbb{P}^{n+1}}(\log X)) \to H^p(U;\mathbb{C})$ is an isomorphism. Also the final map $\mathbb{H}^p(\Omega^\bullet_U) \to H^p(U;\mathbb{C})$ is an isomorphism. To prove that the other two maps are isomorphisms, it is good enough to prove that one of them is an isomorphism. So we will prove that there is an isomorphism $\mathbb{H}^p(\Omega^\bullet_{\mathbb{P}^{n+1}}(\log X)) \to \mathbb{H}^p(\Omega^\bullet_{\mathbb{P}^{n+1}}(*X))$.

First give $\Omega^\bullet_{\mathbb{P}^{n+1}}(*X)$ the order of pole filtration

$$F^p_{\text{pole}}(\Omega^\bullet_{\mathbb{P}^{n+1}}(*X)) = (0 \to 0 \to \cdots \to 0 \to \Omega^p_{\mathbb{P}^{n+1}}(X) \to \Omega^{p+1}_{\mathbb{P}^{n+1}}(2X) \to \cdots)$$

and give $\Omega^\bullet_{\mathbb{P}^{n+1}}(\log X)$ the trivial filtration. The inclusion $\Omega^\bullet_{\mathbb{P}^{n+1}}(\log X) \hookrightarrow \Omega^\bullet_{\mathbb{P}^{n+1}}(*X)$ induces a filtered quasi-isomorphism. That is, the sequence

$$0 \to \Omega^p_{\mathbb{P}^{n+1}}(\log X) \to \Omega^p_{\mathbb{P}^{n+1}}(X) \to \Omega^{p+1}_{\mathbb{P}^{n+1}}(2X)/\Omega^{p+1}_{\mathbb{P}^{n+1}}(X) \to \cdots$$

is exact. Exactness at the first two stages is clear. For exactness at the higher stages, we work in local coordinates. Let $\omega \in \Omega^{p+k}_{\mathbb{P}^{n+1}}((k+1)X)$, and suppose $d\omega \in \Omega^{p+k+1}_{\mathbb{P}^{n+1}}((k+1)X)$. Choose our coordinate system to be $(x_0 = f, x_1, \ldots, x_n)$. Then we can write

$$\omega = \sum_{|I|=p+k} \frac{g_I}{x_0^{k+1}} dx_I$$

Modulo $\Omega^{p+k}_{\mathbb{P}^{n+1}}(kX)$, we can assume that the g_I's are independent of x_0 for all I, and it still follows that $dw \in \Omega^{p+k+1}_{\mathbb{P}^{n+1}}((k+1)X)$. When we explicitly differentiate ω, the only terms with poles of order greater than $k+1$ come from the partial derivative with respect to x_0. So the condition that $d\omega = 0$ in the quotient is exactly the condition that g_I is divisible by x_0 if I does not contain 0. But g_I is independent of x_0, so then must be zero. Therefore every term of ω contains dx_0, and we can write $\omega = \frac{dx_0}{x_0^{k+1}} \wedge \psi$ for some holomorphic form ψ with no dx_0's and whose coefficient functions are independent of x_0. Now consider the form $-\frac{1}{k}x_0^{-k}\psi$. Clearly

$$d\left(-\frac{1}{k}x_0^{-k}\psi\right) = \frac{dx_0}{x_0^{k+1}} \wedge \psi - \frac{1}{k}x_0^{-k}d\psi.$$

Since ψ is holomorphic, we conclude that $d(-\frac{1}{k}x_0^{-k}\psi) = \omega \bmod \Omega^{p+k}_{\mathbb{P}^{n+1}}(kX)$, the sequence is exact, and the inclusion $\Omega^\bullet_{\mathbb{P}^{n+1}}(\log X) \hookrightarrow \Omega^\bullet_{\mathbb{P}^{n+1}}(*X)$ is a filtered quasi-isomorphism. But filtered quasi-isomorphisms are quasi-isomorphisms by Lemma A.4.1, so the inclusion is a quasi-isomorphism, and therefore induces isomorphisms in hypercohomology by Lemma A.2.2. □

3. Hodge structures on the complement

We can also consider the exact sequence

$$0 \to \Omega^\bullet_{\mathbb{P}^{n+1}} \to \Omega^\bullet_{\mathbb{P}^{n+1}}(\log X) \to \Omega^{\bullet-1}_X \to 0$$

degree by degree. Taking the long exact sequence in cohomology gives us an exact sequence

$$H^{q-1}(\Omega^{p-1}_X) \xrightarrow{\delta} H^q(\Omega^p_{\mathbb{P}^{n+1}}) \to H^q(\Omega^p_{\mathbb{P}^{n+1}}(\log X)) \xrightarrow{\text{Res}} H^q(\Omega^{p-1}_X) \xrightarrow{\delta} H^{q+1}(\Omega^p_{\mathbb{P}^{n+1}}).$$

We are primarily interested in the case where $p+q=n+1$, since then the middle group is related to $H^{n+1}(U;\mathbb{C})$. We claim that the map Res induces an isomorphism

$$\mathrm{Res}: H^q(\Omega^p_{\mathbb{P}^{n+1}}(\log X)) \to H^q(\Omega^{p-1}_X) \cap H^n_{\mathrm{prim}}(X).$$

First, the image of Res is the kernel of δ, which is a Hodge component of the kernel of the Gysin homomorphism. We have already seen the kernel of the Gysin homomorphism is primitive cohomology, so it follows that the image of Res is $H^q(\Omega^{p-1}_X)\cap H^n_{\mathrm{prim}}(X)$. On the other hand, the first connecting map δ is the Hodge component of the Gysin homomorphism, and in this case the Gysin homomorphism is Poincaré dual to an isomorphism by the Lefschetz hyperplane theorem, so is an isomorphism. Hence the first δ is surjective, the map $H^q(\Omega^p_{\mathbb{P}^{n+1}}) \to H^q(\Omega^p_{\mathbb{P}^{n+1}}(\log X))$ is the zero map, and Res is injective. So in fact Res induces the claimed isomorphism.

Because the map $\mathrm{Res}: H^{n+1}(U) \to H^n_{\mathrm{prim}}(X)$ is an isomorphism and

$$H^n_{\mathrm{prim}}(X) = \bigoplus_{\substack{p+q=n+1\\ p\geq 1}} H^q(\Omega^{p-1}_X) \cap H^n_{\mathrm{prim}}(X),$$

we deduce

$$H^{n+1}(U;\mathbb{C}) = \bigoplus_{\substack{p+q=n+1\\ p\geq 1}} H^q(\Omega^p_{\mathbb{P}^{n+1}}(\log X)) = \bigoplus_{\substack{p+q=n+2\\ p,q\geq 1}} H^{q-1}(\Omega^p_{\mathbb{P}^{n+1}}(\log X)).$$

We give $H^{n+1}(U;\mathbb{C})$ a Hodge structure of weight $n+2$ according to the decomposition on the right. This respects conjugation because the isomorphism $H^n_{\mathrm{prim}}(X) \cong H^{n+1}(U)$ holds at the rational level, so commutes with conjugation, and hence

$$\overline{H^{q-1}(\Omega^p_{\mathbb{P}^{n+1}}(\log X))} \cong \overline{H^{p-1,q-1}_{\mathrm{prim}}(X)} \cong H^{q-1,p-1}_{\mathrm{prim}}(X) \cong H^{p-1}(\Omega^q_X).$$

The map Res then becomes an isomorphism of Hodge structures of type $(-1,-1)$.

Since the hypercohomology of the polar complex $\Omega^\bullet_{\mathbb{P}^{n+1}}(*X)$ also computes the cohomology of U, we can also try to put a Hodge structure on $H^{n+1}(U)$ by putting a filtration on the hypercohomology $\mathbb{H}^{n+1}(\Omega^\bullet_{\mathbb{P}^{n+1}}(*X))$. The pole order filtration on $\Omega^\bullet_{\mathbb{P}^{n+1}}(*X)$ induces a filtration on this hypercohomology group. Namely, we let $F^p_{\mathrm{pole}}\mathbb{H}^{n+1}(\Omega^\bullet_{\mathbb{P}^{n+1}}(*X))$ be the image of the map in hypercohomology induced by the inclusion of complexes $F^p_{\mathrm{pole}}\Omega^\bullet_{\mathbb{P}^{n+1}}(*X) \hookrightarrow \Omega^\bullet_{\mathbb{P}^{n+1}}(*X)$.

Theorem 3.1. *The pole order filtration on $H^{n+1}(U)$ coincides with the filtration on $H^{n+1}(U)$ induced by the residue map. Thus the residue map induces isomorphisms*

$$\frac{H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}((n-p+1)X))}{H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}((n-p)X)) + dH^0(\Omega^n_{\mathbb{P}^{n+1}}((n-p)X)} \xrightarrow{\sim} H^{p,n-p}_{\mathrm{prim}}(X).$$

Proof. We have already seen that the inclusion $(\Omega^\bullet_{\mathbb{P}^{n+1}}(\log X), F_{\text{triv}}) \hookrightarrow (\Omega^\bullet_{\mathbb{P}^{n+1}}(*X), F_{\text{pole}})$ is a filtered quasi-isomorphism. Bott's vanishing theorem implies that

$$H^i(\mathbb{P}^{n+1}, \Omega^j_{\mathbb{P}^{n+1}}(kX)) = 0$$

for all $i, k \geq 1$, $j \geq 0$, from which it follows that the higher cohomology groups of $\operatorname{Gr}^p_{F_{\text{pole}}} \Omega^j_{\mathbb{P}^{n+1}}(*X)$ all vanish. Then by Theorem A.4.2 there is a spectral sequence

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, \operatorname{Gr}^p_F \Omega^\bullet_{\mathbb{P}^{n+1}}(*X)) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_{\mathbb{P}^{n+1}}(*X)) = H^{p+q}(U; \mathbb{C}).$$

Since the log-pole complex is filtered quasi-isomorphic to the finite pole complex, there is an analogous isomorphic spectral sequence for log-poles. In the case $p + q = n + 1$, the filtration defined on $H^{p+q}(U; \mathbb{C})$ by the log-pole complex has graded pieces given by the $E_1^{p,q}$ entries. Now the entries of the E_∞ page are subquotients of these entries, hence have no larger vector space dimension. It follows that the entries $E_1^{p,q}$ of the finite pole spectral sequence give the graded quotients of the cohomology of $H^{n+1}(U)$ with the pole filtration. This is the spectral sequence associated to the filtered complex of groups $(H^0(\Omega^\bullet_{\mathbb{P}^{n+1}}(*X)), F_{\text{pole}})$, so

$$\frac{H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}((q+1)X))}{H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}(qX)) + dH^0(\Omega^n_{\mathbb{P}^{n+1}}(qX)} = H^q(\Omega^p_{\mathbb{P}^{n+1}}(\log X)) = H^{p-1,n-(p-1)}_{\text{prim}}(X).$$

Then we get a Hodge structure of weight $n + 2$ by letting the (p, q)-part of $H^{n+1}(U)$ be

$$\frac{H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}(qX))}{H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}((q-1)X)) + dH^0(\Omega^n_{\mathbb{P}^{n+1}}((q-1)X))}.$$

This definition respects conjugation since the inclusion of the log complex into the finite pole complex commutes with conjugation and identifies the Hodge components of the log complex with our proposed Hodge components of the finite pole complex. □

Straightforward computations prove the following theorem.

Theorem 3.2. *Any rational $(n + 1)$-form φ on $\mathbb{P}^{n+1}$ with a pole of order q along the hypersurface X given by $F = 0$ has pullback to $\mathbb{C}^{n+2} \setminus \{0\}$ given by a total degree zero form $\frac{A}{F^q}\omega_{n+1}$, where ω_{n+1} is the contraction of the standard volume form on $\mathbb{C}^{n+2}$ with the Euler vector field $E = \sum z_i \frac{\partial}{\partial z_i}$. Thus* $\deg A = qd - n - 2$. *The form φ is in*

$$H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}((q-1)X)) + dH^0(\Omega^n_{\mathbb{P}^{n+1}}((q-1)X))$$

if and only if A lies in the Jacobian ideal

$$\mathfrak{j}_F = (\partial f/\partial z_0, \ldots, \partial f/\partial z_{n+1}) \subset \mathbb{C}[z_0, \ldots, z_{n+1}].$$

Letting $p+q=n+1$*, and putting* $t(p)=(n-p+1)d-n-2=\deg A$*, we thus have an isomorphism*

$$R^{t(p)} \cong H^{p,n-p}_{\text{prim}}(X),$$

where $R=\mathbb{C}[z_0,\dots,z_{n+1}]/\mathfrak{j}_F$ *is the Jacobian ring.*

4. Period maps and their derivatives

Suppose we have a family $X \to S$ of compact Kähler manifolds over a smooth complex simply connected base, with special fiber X_o. The local system consisting of the primitive cohomology $H^n_{\text{prim}}(X_s)$ is then locally trivial; a path in the base from o to s determines an isomorphism $H^n_{\text{prim}}(X_s) \cong H^n_{\text{prim}}(X_o)$ which only depends on the homotopy class of the path relative to the endpoints. The Hodge decomposition (resp. filtration) of the fiber X_s then gives a decomposition (resp. filtration) on the special fiber. The Hodge numbers of the fibers are constant because they depend on s in an upper-semicontinuous fashion and their sum is constant. Now write f^p for the dimension of F^pX_o.

Definition 4.1. The *period map* of the family $X \to S$ is the map

$$\mathscr{P} : S \to G(f^1, H^n_{\text{prim}}(X_o)) \times \cdots \times G(f^n, H^n_{\text{prim}}(X_o))$$

whose pth coordinate function sends s to the isomorphic image of $F^pH^n_{\text{prim}}(X_s)$ in $H^n_{\text{prim}}(X_o)$.

A fundamental result is that the period map is holomorphic. Another important property is Griffiths transversality, which tells us that the derivative $d\mathscr{P}_o$ of the period map at the base point o has a very special form.

Recall that the tangent space to the Grassmannian $G(f^p, H^n_{\text{prim}}(X_o))$ at the point $F^pH^n_{\text{prim}}(X_o)$ is

$$\operatorname{Hom}_{\mathbb{C}}(F^pH^n_{\text{prim}}(X_o), H^n_{\text{prim}}(X_o)/F^pH^n_{\text{prim}}(X_o).$$

Griffiths transversality tells us that in fact pth coordinate of the image of the holomorphic tangent space of S at o lands in

$$\operatorname{Hom}_{\mathbb{C}}(F^pH^n_{\text{prim}}(X_o), F^{p-1}H^n_{\text{prim}}(X_o)/F^pH^n_{\text{prim}}(X_o)),$$

which is the same thing as

$$\operatorname{Hom}_{\mathbb{C}}(F^pH^n_{\text{prim}}(X_o), H^{p-1,q}_{\text{prim}}(X_o)) \qquad (p-1+q=n).$$

Additionally, an element in the pth coordinate of the image of $d\mathscr{P}_0$ restricts to the element in the $(p+1)$st coordinate, so these maps all vanish on $F^{p+1}H^n_{\text{prim}}(X_o)$. Thus the image lies in

$$\operatorname{Hom}_{\mathbb{C}}(H^{p,q-1}_{\text{prim}}(X_o), H^{p-1,q}_{\text{prim}}(X_o)).$$

Taken together, the components of the derivative of the period map therefore define a map

$$d\mathscr{P}_0 : T_oS \to \operatorname{End}^{-1,1} H^n_{\text{prim}}(X_o).$$

Hence the derivative of the period map induces an action of T_oS on $H^n_{\text{prim}}(X_o)$ by endomorphisms of type $(-1, 1)$.

Another way to describe this action is in terms of the Kodaira-Spencer map. This map associates to a tangent vector $v \in T_oS$ a cohomology class $\kappa(v) \in H^1(X_o, \Theta_{X_o})$ as follows. Shrinking S if necessary, choose a local holomorphic vector field V with $V_o = v$. Covering the special fiber X_o with small open subsets of X, we can choose holomorphic lifts of V into each open set. Then on the pairwise intersections, the differences of the lifts project to 0 in T_oS, so the differences must themselves be in Θ_{X_o}. The differences form a Čech 1-cocycle, so we get a cohomology class $\kappa(v) \in H^1(X_o, \Theta_{X_o})$.

Now we have a cup-product mapping

$$H^1(X_o, \Theta_{X_o}) \otimes H^q_{\text{prim}}(\Omega^p_{X_o}) \to H^{q+1}(\Theta_{X_o} \otimes \Omega^p_{X_o}),$$

which when composed with the map induced by contraction $\Theta_{X_o} \otimes \Omega^p_{X_o} \to \Omega^{p-1}_{X_o}$ gives a map

$$H^1(X_o, \Theta_{X_o}) \otimes H^q_{\text{prim}}(\Omega^p_{X_o}) \to H^{q+1}_{\text{prim}}(\Omega^{p-1}_{X_o}).$$

So v induces via the Kodaira-Spencer map an endomorphism of $H^n_{\text{prim}}(X_o)$ of type $(-1, 1)$. This map corresponds with the derivative of the period map.

5. Tautological deformations of hypersurfaces

In this section, we let L be the line bundle $\mathcal{O}_{\mathbb{P}^{n+1}}(X_o)$ associated to X_o, and consider the tautological family of smooth hypersurfaces. The base S of this family is the locus of divisors (s) in the complete linear series $|L|$ such that the corresponding divisor X_s is smooth. We are only really interested in a neighborhood of the base near the base point s_o, so we do not need to worry about any monodromy issues.

The tangent space to S at o is naturally identified with

$$T = H^0(\mathbb{P}^{n+1}, L)/\mathbb{C}s_0$$

since S is just a subset of the projective space $\mathbb{P}H^0(\mathbb{P}^{n+1}, L)$. Viewing $H^0(\mathbb{P}^{n+1}, L)$ as the space of rational functions on $\mathbb{P}^{n+1}$ having at most simple poles along X_o, we consider the cup product mapping

$$H^0(\mathbb{P}^{n+1}, L) \otimes H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}(kX_o)) \to H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}((k+1)X_o)).$$

We can describe $H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}(kX_o))$ as the space of homogeneous polynomials of degree $kd - n - 2$, and if we describe the right hand side similarly then multiplication by the rational function G/F corresponds to multiplication by G. It therefore maps the Jacobian ideal into itself, and we get an induced mapping

$$H^0(\mathbb{P}^{n+1}, L) \otimes H^{p,n-p}_{\text{prim}}(X_o) \to H^{p-1,n-p+1}_{\text{prim}}(X_o).$$

The constant functions act trivially on $H^{p,n-p}_{\text{var}}(X_o)$, so we arrive at a multiplication map

$$T \otimes H^{p,n-p}_{\text{prim}}(X_o) \to H^{p-1,n-p+1}_{\text{prim}}(X_o).$$

This essentially gives us a third description of the derivative of the period map, according to the following result.

Proposition 5.1. *Up to some constant factors, the diagram*

$$\begin{array}{ccc} T \otimes H^{p,n-p}_{\text{prim}}(X_o) & \xrightarrow{\kappa \otimes 1} & H^1(X_o, \Theta_{X_o}) \otimes H^{p,n-p}_{\text{prim}}(X_o) \\ & \searrow \quad \swarrow & \\ & H^{p-1,n-p+1}_{\text{prim}} & \end{array}$$

commutes.

6. The main theorem

According to Proposition 5.1, the map $T \otimes H^{p,n-p}_{\text{prim}}(X_o) \to H^{p-1,n-p+1}_{\text{prim}}(X_o)$ given by the action of the Kodaira-Spencer class on primitive cohomology is surjective whenever the multiplication map

$$H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(X_o)) \otimes H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}((n-p+1)X_o)) \to H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}((n-p+2)X_o))$$

is surjective. We call a cohomology class $\alpha \in H^n(X_o, \mathbb{Q})$ *infinitesimally fixed* if it is annihilated by all the Kodaira-Spencer classes.

Theorem 6.1. *Suppose that the multiplication maps displayed above are surjective for $p = 1, \ldots, n$. Then infinitesimally fixed classes are fixed. That is, infinitesimally fixed classes are all pullbacks of cohomology classes in $\mathbb{P}^{n+1}$.*

Proof. We will show that an infinitesimally fixed element of $H^{p,n-p}_{\text{prim}}(X_o)$ must be zero. Since the action of the Kodaira-Spencer class maps Hodge components to Hodge components, it follows that if an element of primitive cohomology is infinitesimally fixed then it is zero. We can express an arbitrary cohomology class in $H^n(X_o)$ as a sum of a primitive class and a fixed class. Since fixed classes are infinitesimally fixed, an arbitrary class is infinitesimally fixed if and only if its primitive component is zero, i.e. if and only if it is fixed.

So let $\alpha \in H^{p,n-p}_{\text{prim}}$ be infinitesimally fixed, and let $\langle \cdot, \cdot \rangle$ be the intersection pairing on $H^n(X_o)$. By the nondegeneracy of the intersection pairing, to show $\alpha = 0$ it is enough to show that α is orthogonal to $H^{n-p,p}(X_o)$. We already know α is orthogonal to the fixed part of this cohomology, so it suffices to show α is orthogonal to $H^{n-p,p}_{\text{prim}}(X_o)$. So let $\beta \in H^{n-p,p}_{\text{prim}}(X_o)$. Our assumption that the multiplication maps are surjective implies that β can be written as a linear combination of terms $t \smile \gamma$, with $t \in T$ and γ a primitive class of type $(n-p+1, p-1)$. For any such term, we have

$$\langle \alpha, t \smile \gamma \rangle = -\langle t \smile \alpha, \gamma \rangle = 0$$

since α is infinitesimally fixed. Thus $\langle \alpha, \beta \rangle = 0$. □

Corollary 6.2. *Suppose n is even. For s outside a countable union of proper subvarieties of S, we have*

$$H^{m,m}(X_s; \mathbb{Q}) = \text{im}(H^{m,m}(\mathbb{P}^{n+1}; \mathbb{Q}) \to H^{2m}(X_s; \mathbb{Q})),$$

where the map is induced by the inclusion $X_s \hookrightarrow \mathbb{P}^{n+1}$.

Proof. The right hand side is clearly contained in the left hand side for all s, so we show the other inclusion. It suffices to show the primitive part of the left hand side is contained in the right hand side.

Let $\tilde{S}$ be the universal cover of S, and consider the family as being over $\tilde{S}$. The local system formed by the nth primitive cohomology groups of the fibers becomes trivial, so we can identify the cohomology groups of the different fibers with one another. For $\alpha \in H^n_{\text{prim}}(X_o)$, put

$$\tilde{S}_\alpha = \{s \in \tilde{S} : \alpha \in H^{m,m}(X_s)\}.$$

The set $\tilde{S}_\alpha$ is an analytic subvariety of $\tilde{S}$. If it equals all of $\tilde{S}$, then α is constant and hence infinitesimally fixed. Then by the theorem, α is fixed as a class on any fiber, i.e. it is the restriction of a cohomology class on $\mathbb{P}^{n+1}$. Now let $U \subset S$ be the complement of the images of all the proper subvarieties $\tilde{S}_\alpha \subset \tilde{S}$. Suppose $s \in U$, and let $\beta \in H^{m,m}_{\text{prim}}(X_s; \mathbb{Q})$. Then we conclude $\tilde{S}_\beta = \tilde{S}$, so β is fixed, and comes from $\mathbb{P}^{n+1}$. □

Theorem 6.3 (Noether-Lefschetz Theorem)**.** *Assume that $d \geq 4$. Outside a countable union of proper subvarieties in the parameter space of degree d surfaces in $\mathbb{P}^3$, the Picard group of the corresponding surface X is isomorphic to $\mathbb{Z}$, generated by $\mathcal{O}_X(1)$. Hence every curve on X is a complete intersection.*

Proof. Recall that $\Omega^3_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4)$. The multiplication map

$$H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(X_o)) \otimes H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}((n-p+1)X_o)) \to H^0(\Omega^{n+1}_{\mathbb{P}^{n+1}}((n-p+2)X_o))$$

is then the multiplication map

$$H^0(\mathcal{O}(d)) \otimes H^0(\mathcal{O}(d(2-p+1)-4)) \to H^0(\mathcal{O}(d(2-p+2)-4)).$$

We need to show this is surjective for $p = 1, 2$. In case $p = 1$ we have the map

$$H^0(\mathcal{O}(d)) \otimes H^0(\mathcal{O}(2d-4)) \to H^0(\mathcal{O}(3d-4)),$$

and $p = 2$ gives the map

$$H^0(\mathcal{O}(d)) \otimes H^0(\mathcal{O}(d-4)) \to H^0(\mathcal{O}(2d-4)).$$

So long as $d \geq 4$, so that all three spaces are the spaces of homogeneous forms of some nonnegative degree on $\mathbb{P}^3$, the maps are obviously surjective.

Thus from the corollary we conclude that there is a union of proper subvarieties of the parameter space of degree d surfaces in $\mathbb{P}^3$ outside of which all surfaces X satisfy

$$H^{1,1}(X;\mathbb{Q}) = \operatorname{im}(H^{1,1}(\mathbb{P}^3;\mathbb{Q}) \to H^2(X;\mathbb{Q})).$$

The claim is that the Picard group of any such surface is generated by $\mathcal{O}_X(1)$. Consider the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0.$$

The long exact sequence in cohomology gives us

$$0 \to H^1(\mathcal{O}_X^*) \to H^2(X;\mathbb{Z}) \to H^2(\mathcal{O}_X)$$

since $H^1(\mathcal{O}_X)$ is a Hodge component of $H^1(X)$, which is zero by the Lefschetz hyperplane theorem. So $\operatorname{Pic} X = H^1(\mathcal{O}_X^*)$ is naturally a subspace of $H^2(X,\mathbb{Z})$. This map is compatible with the description of line bundles as divisors, in the sense that a divisor maps to the Poincaré dual of its fundamental homology class. Hence the image of $\operatorname{Pic} X$ in $H^2(X;\mathbb{Z})$ is exactly the group of algebraic cycles. The map $H^2(X;\mathbb{Z}) \to H^2(\mathcal{O}_X) = H^{0,2}(X)$ is the projection, so its kernel is the integral classes of type $(2,0) + (1,1)$. All integral classes of pure type are of type $(1,1)$ since they are real, so in fact the kernel consists of integral classes of type $(1,1)$. Thus an integral class is of type $(1,1)$ if and only if it is algebraic. In other words,

$$\operatorname{Pic} X = H^2(X;\mathbb{Z}) \cap H^{1,1}(X).$$

But then every element of $\operatorname{Pic} X$ is a pullback from $\mathbb{P}^3$, so $\operatorname{Pic} X \cong \mathbb{Z} \cdot \mathcal{O}_X(1)$. $\square$

Appendix A. Spectral sequences

A.1. **Fundamental Notions.** A *spectral sequence* consists of a sequence (E_r, d_r), $r = 0, 1, \dots$ of graded groups $E_r = \bigoplus_{p,q\in\mathbb{Z}} E_r^{p,q}$ with homomorphisms

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that

(1) $d_r \circ d_r = 0$; and
(2) $E_{r+1}^{p,q} = \ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q+r-1}$.

Observe that d_r increases the total degree $n = p + q$ by one, so that for fixed r we have a complex (E_r^n, d_r), and E_{r+1} computes the cohomology of this complex.

A spectral sequence arises when one has a filtered complex $(K^\bullet, d)$ of groups, say with a decreasing filtration $F^\bullet$. Setting

$$Z_r^{p,q} = \{a \in F^p K^{p+q} : da \in F^{p+r} K^{p+q+1}\},$$

one defines the terms $E_r^{p,q}$ by

$$E_r^{p,q} = \frac{Z_r^{p,q}}{dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}}.$$

The differential d then induces d_r.

Proposition A.1.1. *$(E_r^{p,q}, d_r)$ is well defined, and is in fact a spectral sequence.*

Proof. For well-definedness, we must first show that $dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}$ is contained in $Z_r^{p,q}$. First let $a \in Z_{r-1}^{p+1,q-1}$. Then by definition $a \in F^{p+1}K^{p+q}$, and $da \in F^{p+r}K^{p+q+1}$. Since the filtration is decreasing, also $a \in F^pK^{p+q}$. This implies $a \in Z_r^{p,q}$, and $Z_{r-1}^{p+1,q-1} \subset Z_r^{p,q}$. If instead $a \in Z_{r-1}^{p-r+1,q+r-2}$, then $da \in F^pK^{p+q}$, and since $d^2a = 0 \in F^{p+r}K^{p+q+1}$ we see that $da \in Z_r^{p,q}$. Thus $dZ_{r-1}^{p-r+1,q+r-2} \subset Z_r^{p,q}$, and the quotient $E_r^{p,q}$ makes sense.

Also we must check that there is actually an induced map d_r. We start with the differential $d : K^{p+q} \to K^{p+q+1}$. Restrict it to the subgroup $Z_r^{p,q}$ of K^{p+q}. The image of this restriction is by definition contained in $F^{p+r}K^{p+q+1}$, and is annihilated by d. Thus the image of the restriction is contained in $Z_r^{p+r,q-r+1}$. Now we can compose with the quotient map $Z_r^{p+r,q-r+1} \to E_r^{p+r,q-r+1}$. We claim that the composite $Z_r^{p,q} \to E_r^{p+r,q-r+1}$ factors through $E_r^{p,q}$. Everything in $dZ_{r-1}^{p-r+1,q+r-2}$ is killed by d since $d^2 = 0$. And $Z_{r-1}^{p+1,q-1}$ maps under d to $dZ_{r-1}^{p+1,q-1} = dZ_{r-1}^{(p+r)-r+1,(q-r+1)+r-2}$, which is 0 in $E_r^{p+r,q-r+1}$. So we get a map $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

Since d_r is induced from d, we get that $d_r^2 = 0$ for free.

Finally we must check that $E_{r+1}^{p,q} = \ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q+r-1}$. To do this, we will explicitly identify the kernel and image in question. We first claim that

$$\ker d_r^{p,q} = \frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}}.$$

First of all, this quotient makes sense. Indeed, $dZ_{r-1}^{p-r+1,q+r-2} \subset Z_{r+1}^{p,q}$, for if a is in $Z_{r-1}^{p-r+1,q+r-2}$, then da is in F^pK^{p+q} and $d^2a = 0$. Next we claim that the quotient is contained in the kernel. To see this it suffices to check $d_r^{p,q}(Z_{r+1}^{p,q}) = 0$, or

$$dZ_{r+1}^{p,q} \subset dZ_{r-1}^{p+1,q-1} + Z_{r-1}^{p+r+1,q-r}.$$

So let $a \in Z_{r+1}^{p,q}$. Then $a \in F^pK^{p+q}$, and $da \in F^{p+r+1}K^{p+q+1}$. Since $d^2a = 0$, we see $da \in Z_{r-1}^{p+r+1,q-r}$. So indeed, the quotient above is contained in the kernel. For the other inclusion, assume that $\overline{a} \in \ker d_r^{p,q}$, where $a \in Z_r^{p,q}$. This means that $da \in dZ_{r-1}^{p+1,q-1} + Z_{r-1}^{p+r+1,q-r}$; write $da = db + c$ respecting this decomposition. Write $a = b + e$ for some e, so that $de = c$. Then $b \in Z_{r-1}^{p+1,q-1}$. Since $a \in F^pK^{p+q}$ and $b \in F^{p+1}K^{p+q}$, we have $e \in F^pK^{p+q}$, and $de \in Z_{r-1}^{p+r+1,q-r}$, so $de \in F^{p+r+1}K^{p+q+1}$. Thus $e \in Z_{r+1}^{p,q}$, which shows that the kernel is contained in quotient.

We claim that the image of $d_r^{p-r,q+r-1}$ can be identified with

$$\operatorname{im} d_r^{p-r,q+r-1} = \frac{dZ_r^{p-r,q+r-1} + Z_{r-1}^{p+1,q-1}}{dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}}.$$

The quotient makes sense since $Z_r^{p-r,q+r-1} \supset Z_{r-1}^{p-r+1,q+r-2}$. But $d_r^{p-r,q+r-1}$ is by definition the restriction of d on $Z_r^{p-r,q+r-1}$, composed with a quotient map and factored through a subgroup of the kernel, so its image is the extension of $dZ_r^{p-r,q+r-1}$ in $E_r^{p,q}$, which is what we claimed. Therefore

$$\frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}} = \frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{dZ_r^{p-r,q+r-1} + Z_{r-1}^{p+1,q-1}} = \frac{Z_{r+1}^{p,q}}{Z_{r+1}^{p,q} \cap (dZ_r^{p-r,q+r-1} + Z_{r-1}^{p+1,q-1})}.$$

We then have only to show that

$$Z_{r+1}^{p,q} \cap (dZ_r^{p-r,q+r-1} + Z_{r-1}^{p+1,q-1}) = dZ_r^{p-r,q+r-1} + Z_r^{p+1,q-1}.$$

But $Z_{r+1}^{p,q}$ contains $dZ_r^{p-r,q+r-1}$, and $Z_{r+1}^{p,q} \cap Z_{r-1}^{p+1,q-1} = Z_r^{p+1,q-1}$; a lies in the left hand side if and only if a is in $F^{p+1}K^{p+q}$ and da is in $F^{p+r+1}K^{p+q+1}$, which are exactly the conditions to be in $Z_r^{p+1,q-1}$. This completes the proof. $\square$

Successive quotients of the filtration F frequently occur in this setting, so we introduce the pth graded part

$$\operatorname{Gr}_F^p(K^n) = F^pK^n / F^{p+1}K^n.$$

The first term in the spectral sequence is

$$E_0^{p,q} = \mathrm{Gr}_F^p(K^{p+q}).$$

For, by definition, $Z_{-1}^{p,q} = F^pK^{p+q}$, so that in particular $Z_{-1}^{p+1,q-1} = F^{p+1}K^{p+q}$, $Z_{-1}^{p+2,q-3} = F^{p+2}K^{p+q-1}$, and $dZ_{-1}^{p+2,q-3} \subset F^{p+2}K^{p+q}$ since the differential preserves the filtration. Then

$$E_0^{p,q} = Z_0^{p,q}/Z_{-1}^{p+1,q-1} = F^pK^{p+q}/F^{p+1}K^{p+q} = \mathrm{Gr}_F^p(K^{p+q}).$$

And for fixed p, the differential gives us maps $\mathrm{Gr}_F^p(K^n) \to \mathrm{Gr}_F^p(K^{n+1})$. Thus we conclude that

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}_F^p(K^\bullet)).$$

If $d_r = 0$ for $r \geq k$ we say that the spectral sequence degenerates at $E_k = E_\infty$.

Lemma A.1.2. *If on each K^n the filtration $F^\bullet$ has finite length, for fixed (p,q), the groups $E_r^{p,q}$ remain the same from a certain index on.*

Proof. Recall that

$$E_r^{p,q} = \frac{Z_r^{p,q}}{dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}}.$$

It is enough to see that the groups $Z_r^{p,q}$, $Z_{r-1}^{p-r+1,q+r-2}$, and $Z_{r-1}^{p+1,q-1}$ stabilize for large enough r. But this is clear since these groups all consider filtrations on either K^{p+q} or K^{p+q-1}, so that there are only two filtrations in question. In particular, note that if we assume the filtration starts with $F^0K^n = K^n$ and ends with $F^rK^n = 0$ for some r, then for large enough r we have

$$E_\infty^{p,q} = \frac{\ker d \cap F^pK^{p+q}}{dZ_p^{0,p+q-1} + \ker d \cap F^{p+1}Z^{p+q}}. \quad \square$$

The following calculation then identifies the limit. If $i : F^p(K^\bullet) \hookrightarrow K^\bullet$ is the inclusion, we define a filtration F_∞ on $H^n(K^\bullet)$ by

$$F_\infty^p H^n(K^\bullet) = i_* H^n(F^p(K^\bullet))$$

Proposition A.1.3. *For a filtration F on $K^\bullet$ with finite length on each K^n, we have*

$$E_\infty^{p,q} = \mathrm{Gr}_{F_\infty}^p H^{p+q}(K^\bullet).$$

Proof. The cohomology of a cochain complex is just gotten by taking kernels mod images. Maps of cochain complexes induce maps on cohomology in a covariant fashion since the map takes coboundaries to coboundaries. Now we can consider p and q as being fixed. Then

$$\begin{aligned}\operatorname{Gr}^p_{F_\infty} H^{p+q}(K^\bullet) &= F^p_\infty H^{p+q}(K^\bullet)/F^{p+1}_\infty H^{p+q}(K^\bullet)\\ &= i_* H^{p+q}(F^p(K^\bullet))/i_* H^{p+q}(F^{p+1}(K^\bullet))\end{aligned}$$

The map $i_* : H^{p+q}(F^p(K^\bullet)) \to H^{p+q}(K^\bullet)$ can instead be regarded as a map $\ker d^{p+q} \cap F^p K^{p+q} \to H^{p+q}(K^\bullet)$. Then

$$\operatorname{Gr}^p_{F_\infty} H^{p+q}(K^\bullet) = i_*(\ker d^{p+q} \cap F^p K^{p+q})/i_*(\ker d^{p+q} \cap F^{p+1} K^{p+q}).$$

We have

$$i_*(\ker d^{p+q} \cap F^p K^{p+q}) = \frac{\ker d^{p+q} \cap F^p K^{p+q} + \operatorname{im} d^{p+q-1}}{\operatorname{im} d^{p+q-1}},$$

and similarly

$$i_*(\ker d^{p+q} \cap F^{p+1} K^{p+q}) = \frac{\ker d^{p+q} \cap F^{p+1} K^{p+q} + \operatorname{im} d^{p+q-1}}{\operatorname{im} d^{p+q-1}}.$$

Therefore

$$\operatorname{Gr}^p_{F_\infty} H^{p+q}(K^\bullet) = \frac{\ker d^{p+q} \cap F^p K^{p+q} + \operatorname{im} d^{p+q-1}}{\ker d^{p+q} \cap F^{p+1} K^{p+q} + \operatorname{im} d^{p+q-1}},$$

which equals

$$\frac{\ker d^{p+q} \cap F^p K^{p+q}}{\ker d^{p+q} \cap F^p K^{p+q} \cap \left(\ker d^{p+q} \cap F^{p+1} K^{p+q} + \operatorname{im} d^{p+q-1}\right)}.$$

Now $F^p K^{p+q} \cap F^{p+1} K^{p+q} = F^{p+1} K^{p+q}$, and $\operatorname{im} d^{p+q-1} \cap \ker d^{p+q} = \operatorname{im} d^{p+q-1}$, so this equals

$$\frac{\ker d^{p+q} \cap F^p K^{p+q}}{\ker d^{p+q} \cap F^{p+1} K^{p+q} + \operatorname{im} d^{p+q-1} \cap F^p K^{p+q}}.$$

We claim that $\operatorname{im} d^{p+q-1} \cap F^p K^{p+q} = dZ_p^{0,p+q-1}$. Suppose that da is in the intersection, where $a \in K^{p+q-1}$. Since $da \in F^p K^{p+q}$, we get that $a \in Z_p^{0,p+q-1}$. Conversely if $a \in Z_p^{0,p+q-1}$, then $da \in F^p K^{p+q}$ and da is in the image of d^{p+q-1}. But then the displayed quotient equals our earlier derived expression for $E_\infty^{p,q}$. □

Such filtrations are called *biregular filtrations*. One says that the *spectral sequence abuts* to $H^\bullet(K^\bullet, d)$. This is commonly denoted

$$E_r^{p,q} \Rightarrow H^{p+q}(K^\bullet, d).$$

The next result is used to compare spectral sequences for related filtered complexes.

Lemma A.1.4. *If $f : K^\bullet \to L^\bullet$ is a filtered homomorphism between complexes, there is an induced homomorphism $E(f_r)$ between the spectral sequences. If $E(f_r)$ is an isomorphism for $r = r_0$, it is an isomorphism for $r \geq r_0$ as well. In particular, if the filtrations are biregular, the spectral sequences abut to isomorphic groups.*

Proof. Denote by $'E$ the spectral sequence of L, by $'F$ the filtration on l, and by $'Z$ the Z-groups associated to L. To see that we get an induced map $E \to {'E}$, it is enough to check that $f(Z_r^{p,q}) \subset {'Z_r^{p,q}}$ and that

$$f(dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}) \subset d('Z_{r-1}^{p-r+1,q+r-2} + {'Z_{r-1}^{p+1,q-1}}).$$

This second property will follow from the first and the fact that f commutes with the differentials. But if $a \in Z_r^{p,q}$, then $a \in F^pK^{p+q}$. So $f(a) \in {'F^pL^{p+q}}$. Also $da \in F^{p+r}K^{p+q+1}$, so $df(a) = f(da) \in {'F^{p+r}L^{p+q+1}}$. Therefore $f(a) \in {'Z_r^{p,q}}$, and we get an induced map $E \to {'E}$ commuting with the differentials of the spectral sequence.

The correspondence $f \mapsto E(f_r)$ is obviously functorial. To prove the second statement, it is therefore enough to show that if $f : K^\bullet \to K^\bullet$ and $E(f_r)$ is the identity, then $E(f_{r+1})$ is the identity. So assume $E(f_r)$ is the identity. For every p, q, the induced self-map of

$$E_r^{p,q} = \frac{Z_r^{p,q}}{dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}}$$

is the identity. We need to prove that the induced self-map of

$$E_{r+1}^{p,q} = \frac{Z_{r+1}^{p,q}}{dZ_r^{p-r,q+r-1} + Z_r^{p+1,q-1}}$$

is the identity. So let $a \in Z_{r+1}^{p,q}$. Since $Z_{r+1}^{p,q} \subset Z_r^{p,q}$, we know that $f_r(a) = a$. This means that

$$f(a) \equiv a \pmod{dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}},$$

and there are $b \in Z_{r-1}^{p-r+1,q+r-2}$ and $c \in Z_{r-1}^{p+1,q-1}$ such that $f(a) = a + db + c$. Since $Z_r^{p-r,q+r-1} \supset Z_{r-1}^{p-r+1,q+r-2}$, we see that db vanishes in $E_{r+1}^{p,q}$. As for c, we have $dc = d(f(a) - a)$, from which we deduce $dc \in F^{p+r+1}K^{p+q+1}$ since both a and $f(a)$ are in $Z_{r+1}^{p,q}$. So actually $c \in Z_r^{p+1,q-1}$, whence $f_{r+1}(a) = a$. Thus f_{r+1} is the identity, and by induction f_r is the identity for all $r \geq r_0$. □

A first quadrant spectral sequence yields a particular 5-term long exact sequence.

Proposition A.1.5. *The spectral sequence for a biregularly filtered complex $(K^\bullet, d, F^\bullet)$ yields an exact sequence*

$$0 \to E_2^{1,0} \to H^1(K^\bullet) \to E_2^{0,1} \overset{d_2}{\to} E_2^{2,0} \to H^2(K^\bullet).$$

Proof. For this to be true we probably have to assume that the associated spectral sequence lies in the first quadrant. For this to be true, we can assume that the filtration is canonically bounded. That is, we assume that $F^0K^n = K^n$ and $F^{n+1}K^n = 0$. Then if $p < 0$, we have $p \leq 0$, so

$$Z_r^{p,q} = \{a \in K^{p+q} : da \in F^{p+r}K^{p+q+1}\} = Z_r^{p+1,q-1}$$

and $E_r^{p,q} = 0$; if instead $q < 0$ then $Z_r^{p,q} = 0$ and again $E_r^{p,q} = 0$. So assume the sequence is canonically bounded.

Since the sequence is a first quadrant sequence,

$$E_2^{1,0} = E_\infty^{1,0} = \mathrm{gr}^1_{F_\infty} H^1(K^\bullet) = F^1_\infty H^1(K^\bullet)/F^2_\infty H^1(K^\bullet).$$

But as the filtration is canonically bounded, $F^2K^1 = 0$ and hence

$$F^2_\infty H^1(K^\bullet) = 0.$$

Therefore $E_2^{1,0} = F^1_\infty H^1(K^\bullet)$, which is a subgroup of $H^1(K^\bullet)$. So we get an inclusion $E_2^{1,0} \to H^1(K^\bullet)$.

Clearly

$$E_3^{0,1} = E_\infty^{0,1} = \mathrm{gr}^0_{F_\infty} H^1(K^\bullet) = F^0_\infty H^1(K^\bullet)/F^1_\infty H^1(K^\bullet) = H^1(K^\bullet)/F^1_\infty H^1(K^\bullet).$$

Also $E_3^{0,1} = \ker d_2$, so clearly the image of $H^1(K^\bullet) \to E_2^{0,1}$ is the kernel. We also see that the image of $E_2^{1,0}$ in $H^1(K^\bullet)$ is exactly the kernel of $H^1(K^\bullet) \to E_2^{0,1}$.

At the next stage, we have $E_3^{2,0} = \operatorname{coker} d_2$, and

$$E_3^{2,0} = E_\infty^{2,0} = \mathrm{gr}^2_{F_\infty} H^2(K^\bullet) = F^2_\infty H^2(K^\bullet)$$

since $F^3_\infty H^2(K^\bullet) = 0$. So we have an injection $\operatorname{coker} d_2 \hookrightarrow H^2(K^\bullet)$, coinciding with the natural map $E_2^{2,0} \to H^2(K^\bullet)$ factored through the kernel, and we conclude that the whole sequence is exact. $\square$

The next result tells us that the spectral sequence of a filtered complex measures the failure of the differential to be compatible with the filtration.

Proposition A.1.6. *The spectral sequence for a filtered complex degenerates at E_1 if and only if the derivative is strictly compatible with the filtration, i.e., $F^pK^n \cap \operatorname{im} d = \operatorname{im} d|_{F^pK^{n-1}}$.*

Proof. Assume the derivative is strictly compatible with the filtration, and consider the quotient

$$E_r^{p,q} = \frac{Z_r^{p,q}}{dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}}.$$

We see that to show $d_r^{p,q} = 0$ amounts to showing that

$$dZ_r^{p,q} \subset dZ_{r-1}^{p+1,q-1} + Z_{r-1}^{p+r+1,q-r}.$$

So let $a \in Z_r^{p,q}$. Then $da \in F^{p+r}K^{p+q+1}$. So there is some $b \in F^{p+r}K^{p+q}$ such that $da = db$. If $r \geq 1$ we also have $b \in F^{p+1}K^{p+q}$, so $b \in Z_{r-1}^{p+1,q-1}$. Therefore $dZ_r^{p,q} \subset dZ_{r-1}^{p+1,q+1}$ for $r \geq 1$, and the spectral sequence degenerates at E_1.

Conversely assume that the sequence degenerates at E_1. Then

$$dZ_1^{p,q} \subset dZ_0^{p+1,q-1} + Z_0^{p+2,q-1}$$

for all $r \geq 1$. The inclusion $\operatorname{im} d|_{F^pK^{p+q-1}} \subset F^pK^{p+q} \cap \operatorname{im} d$ always holds, so let $da \in F^pK^{p+q}$ be an element of the intersection. Then $a \in K^{p+q-1}$; if $a \in F^pK^{p+q-1}$ then we are done. Suppose that s is the largest index such that $a \in F^sK^{p+q-1}$. If $s < p$, then $a \in Z_1^{s,p+q-s-1}$. By hypothesis we can write $da = db + c$, where $b \in Z_0^{s+1,p+q-s-2}$ and $c \in Z_0^{s+2,p+q-s-2}$. That is, $b \in F^{s+1}K^{p+q-1}$ and $c \in F^{s+2}K^{p+q}$. Then $c = d(a-b)$, and therefore $a - b \in Z_2^{s,p+q-s-1}$. Now using the fact that the E_2 page is degenerate, we see

$$dZ_2^{s,p+q-s-1} \subset dZ_1^{s+1,p+q-s-2} + Z_1^{s+3,p+q-s-2};$$

write

$$c = db' + c'$$

respecting this decomposition. Then writing $c' = d(a-b-b')$, we get $a - b - b' \in Z_3^{s,p+q-s-1}$. Continuing in this fashion, we create an element $\sum b$ of $Z_0^{s+1,p+q-s-2}$ with the property that $a - \sum b \in Z_r^{s,p+q-s-1}$ for r any large fixed number. Provided our filtration is finite, this guarantees $d(a-\sum b) = 0$, so $da = d(\sum b)$. Now we can apply this process to $\sum b$, getting elements $\sum e$, $\sum f$, $\sum g$, $\ldots$ with $da = d(\sum b) = d(\sum f) = d(\sum g) = \cdots$, each one step deeper into the filtration. The process stops when $s = p$, i.e. when we have constructed $\sum z \in Z_0^{p,q-1}$ satisfying $da = d(\sum z)$, and this demonstrates the strictness of the differential. □

Recall that a double complex (living in the first quadrant) consists of a bigraded group $K^{\bullet,\bullet} = \bigoplus_{p,q} K^{p,q}$ and differentials $d : K^{p,q} \to K^{p+1,q}$ and $\delta : K^{p,q} \to K^{p,q+1}$, satisfying $d^2 = \delta^2 = 0$, $d\delta = \delta d$. The associated total complex is $sK^n = \bigoplus_{p+q=n} K^{p,q}$ with differential $D = d + (-1)^p\delta$. It has two obvious filtrations,

$$'F^p = \bigoplus_{r \geq p} K^{r,s} \qquad ''F^q = \bigoplus_{s \geq q} K^{r,s},$$

and these then induce filtrations $'F$ and $''F$ on the associated total complex. The associated spectral sequences are denoted by $'E_r^{p,q}$ and $''E_r^{p,q}$.

Proposition A.1.7. *The* $'E_1$ *and* $'E_2$ *pages of this spectral sequence are given by*

$$'E_1^{p,q} = H^q(K^{p,\bullet}, d'') \qquad \text{and} \qquad 'E_2^{p,q} = H^p(H^q(K^{\bullet,\bullet}, d''), d').$$

Proof. For the first equality, we have

$$\begin{aligned} 'E_1^{p,q} &= H^{p+q}(\mathrm{gr}^p_{'F}(sK^\bullet), D) \\ &= H^{p+q}('F^p sK^\bullet / 'F^{p+1} sK^\bullet, D) \\ &= H^{p+q}\left(\bigoplus_{\substack{r \geq p \\ r+s=\bullet}} K^{r,s} \Big/ \bigoplus_{\substack{r \geq p+1 \\ r+s=\bullet}} K^{r,s}, D \right) \\ &= H^{p+q}(K^{p,\bullet - p}, d'') \\ &= H^q(K^{p,\bullet}, d'') \end{aligned}$$

We can also write $H^{p+q}(K^{p,\bullet}, D) = H^q(K^{p,\bullet}, d'')$ where D acts by d'' and the complex is graded by total degree $p + \bullet$ instead of just by $\bullet$.

For the second equality, we must identify the differential in the $'E_1$ page of the spectral sequence with d'. By definition,

$$'E_1^{p,q} = \frac{Z_1^{p,q}}{DZ_0^{p,q-1} + Z_0^{p+1,q-1}}.$$

Now let $a \in Z_1^{p,q}$. So $a \in {'F^p} sK^{p+q}$, and $Da \in {'F^{p+1}} sK^{p+q+1}$. Denote by πa the projection of a to an element of $K^{p,q}$ and by $\pi' a$ the projection of a on $'F^{p+1} sK^{p+q}$. Then $a = \pi a + \pi' a$, so $Da = D\pi a + D\pi' a$. But $D\pi a = d'\pi a$ since $Da \in {'F^{p+1}} sK^{p+q+1}$. Clearly $D\pi' a = 0$ in $'E_1^{p+1,q}$, since $\pi' a \in Z_0^{p+1,q-1}$. Therefore $Da = D\pi a = d'\pi a$ on $'E_1^{p,q}$. Since in fact $'E_1^{p,q}$ is a subquotient of $K^{p,q}$, this says exactly that D coincides with d' under the identification of $'E_1^{p,q}$ with $H^q(K^{p,\bullet}, d'')$. □

Summing up what we know, the $'E_0$ page of the spectral sequence is the bigraded group: $'E_0^{p,q} = K^{p,q}$. The first page $'E_1$ computes the cohomology in the vertical direction with respect to the vertical differential. The horizontal differentials descend to cohomology since they are maps of chain complexes, and the second page $'E_2$ computes the cohomology in the horizontal direction with respect to the descended horizontal differential. Recall that

$$E_\infty^{p,q} = \mathrm{Gr}^p_{F_\infty} H^{p+q}(sK^\bullet).$$

That is, the, hypercohomology of $K^{\bullet,\bullet}$ is graded, and the graded parts of hypercohomology can be identified with the p, q parts of the E_∞ page. In other words, E_∞^n is the hypercohomology of $K^{\bullet,\bullet}$, and the spectral sequence computes the hypercohomology of the double complex.

A.2. **Hypercohomology.** Recall that we defined hypercohomology of the holomorphic de Rham complex $\Omega^\bullet_M$ on a compact complex manifold M as the cohomology of the total complex associated to the Čech-de Rham complex. Suppose that we now start out with an arbitrary complex $(\mathscr{K}^\bullet, d)$ of sheaves on a topological space X, and suppose that we are given a Leray cover $\mathscr{U} = \{U_i\}_{i\in I}$ of X. Recall that this means that we can compute Čech cohomology directly from the cover. We can now form the double complex consisting of Čech cochains $C^\bullet(\mathscr{U}, \mathscr{K}^\bullet)$ with differential d coming from the complex of sheaves and Čech differential δ. In this case the cohomology of the associated total complex by definition is the hypercohomology group of the complex of sheaves we started with. Without assuming the cover is Leray, we have to take a direct limit over all refinements of the open cover $\mathscr{U}$:

$$\mathbb{H}^p(\mathscr{K}^\bullet, d) = \varinjlim_{\mathscr{U}} H^p(sC^\bullet(\mathscr{U}, \mathscr{K}^\bullet)).$$

To compute this hypercohomology, one uses the two spectral sequences of the double complex $C^\bullet(\mathscr{U}, \mathscr{K}^\bullet)$ that have E_1-terms,

$${}'E_1^{p,q} = \varinjlim_{\mathscr{U}} H^q(C^\bullet(\mathscr{K}^p), \delta) = H^q(X, \mathscr{K}^p)$$

and

$${}''E_1^{p,q} = \varinjlim_{\mathscr{U}} H^p(C^q(\mathscr{U}, \mathscr{K}^\bullet), d),$$

respectively.

In the first spectral sequence the derivative comes from d, and in the second it comes from δ. Hence, the E_2-terms are given by

$${}'E_2^{p,q} = H^p(H^q(X, \mathscr{K}^\bullet), d)$$

and

$${}''E_2^{p,q} = H^q(\varinjlim_{\mathscr{U}} H^p(C^\bullet(\mathscr{U}, \mathscr{K}^\bullet), d), \delta).$$

This last group can be better explained. For each q-fold intersection $U_{i_1} \cap \cdots \cap U_{i_q}$ of elements of $\mathscr{U}$, there is a factor $\mathscr{K}^\bullet(U_{i_1} \cap \cdots \cap U_{i_q})$ in the complex $(C^q(\mathscr{U}, \mathscr{K}^\bullet), d)$. In fact, the complex splits as a direct sum

$$(C^q(\mathscr{U}, \mathscr{K}^\bullet), d) = \bigoplus_{|I|=q} (\mathscr{K}^\bullet(U_I), d).$$

Cohomology of chain complexes of abelian groups commutes with direct sum, so

$$H^p(C^q(\mathscr{U}, \mathscr{K}^\bullet), d) = \bigoplus_{|I|=q} H^p(\mathscr{K}^\bullet(U_I), d).$$

For any open set U, there is a chain complex of groups $(\mathscr{K}^\bullet(U), d)$. Restriction maps commute with differentials, so induce restriction maps on cohomology. Therefore the sheaf $H^p(\mathscr{K}^\bullet, d)$ is well-defined. What we have shown is the relation

$$H^p(C^q(\mathscr{U}, \mathscr{K}^\bullet), d) = C^q(H^p(\mathscr{K}^\bullet, d)).$$

That is, cohomology commutes with taking the Čech complex. Since cohomology commutes with direct limits, we conclude that

$$''E_2^{p,q} = H^q(\varinjlim_{\mathscr{U}} C^\bullet(H^p(\mathscr{K}^\bullet, d)), \delta) = H^p(X, H^p(\mathscr{K}^\bullet, d)).$$

For reasons that become apparent below, we introduce the term *de Rham cohomology groups* for

$$H^p_{dR}(X, \mathscr{K}^\bullet) = {}'E^{p,0} = H^p(H^0(X, \mathscr{K}^\bullet), d).$$

Any global section of $\mathscr{K}^p$ is a 0-Čech cocycle, and thus if it is d-closed, it is a cocycle in the total complex $sC^\bullet(\mathscr{U}, \mathscr{K}^\bullet)$. Therefore one gets an associated element in hypercohomology:

$$h_{dR} : H^p_{dR}(\mathscr{K}^\bullet) \to \mathbb{H}^p(X, \mathscr{K}^\bullet).$$

Lemma A.2.1. *If the complex of sheaves is a trivial complex with $\mathscr{K}^q = 0$ for $q \neq s$, then the nth hypercohomology of this complex simply coincides with $H^{n-s}(X, \mathscr{K}^s)$.*

Proof. The double complex $C^\bullet(\mathscr{U}, \mathscr{K}^\bullet)$ is just the Čech complex of $\mathscr{K}^s$ with some extra zeroes. The total complex $sC^\bullet(\mathscr{U}, \mathscr{K}^\bullet)$ satisfies

$$(sC^\bullet(\mathscr{U}, \mathscr{K}^\bullet))^n = C^{n-s}(\mathscr{U}, \mathscr{K}^s),$$

so the hypercohomology in degree n is Čech cohomology in degree $n - s$ of $\mathscr{K}^s$. □

Proposition A.2.2. *A map $j : \mathscr{A}^\bullet \to \mathscr{B}^\bullet$ between complexes of sheaves induces maps between the hypercohomology groups of the complexes. If j induces an isomorphism on the level of cohomology sheaves (in which case we say that j is a* quasi-isomorphism*, it induces an isomorphism in hypercohomology, because the second spectral sequences are isomorphic.*

Proof. Clearly j induces a map of the total complexes, so induces a map on hypercohomology. If it induces an isomorphism on cohomology sheaves, then the induced map on $''E_2^{p,q}$ will also be an isomorphism. Hence the map on E_∞ pages is also an isomorphism by Lemma A.1.4, and the hypercohomology groups are isomorphic. □

A special case of the preceding situation arises when the complex of sheaves $(\mathscr{K}^\bullet, d)$ we started with in fact is an exact sequence, so that the cohomology sheaves vanish. So the E_2-terms of the second spectral sequence are all 0 and so it abuts to 0. This then also holds for the first spectral sequence ${}'E_1^{p,q} = H^q(X, \mathscr{K}^p)$.

Lemma A.2.3. *Suppose that $\mathscr{K}^\bullet$ is an exact complex in degrees ≥ 0. Assume that $H^i(X, \mathscr{K}^p) = 0$ for all $p > 0$ and $i = 1, \ldots, q-1$. Then the derivative d_{q+1} induces an isomorphism*

$$\ker(H^q(X, \mathscr{K}^0) \to H^q(X, \mathscr{K}^1))$$
$$\to H^{q+1}_{dR}(\mathscr{K}^\bullet, d) = \frac{\ker(d : H^0(X, \mathscr{K}^{q+1}) \to H^0(X, \mathscr{K}^{q+2}))}{\operatorname{im}(d : H^0(X, \mathscr{K}^q) \to H^0(X, \mathscr{K}^{q+1}))}.$$

Proof. The vanishing assumption on $H^i(X, \mathscr{K}^p) = {}'E_1{}^{p,i}$ implies that the differentials $d_2^{0,q}, \ldots, d_q^{0,q}$ are all zero, so that

$${}'E_2^{0,q} = {}'E_3^{0,q} = \cdots = {}'E_{q+1}^{0,q} = \ker d_1^{0,q} = \ker(H^q(X, \mathscr{K}^0) \to H^q(X, \mathscr{K}^1)).$$

Now $d_{q+1}^{0,q}$ maps ${}'E_{q+1}^{0,q}$ to ${}'E_{q+1}^{q+1,0}$. All the differentials d_i with $2 \leq i \leq q$ that map into the $(q+1, 0)$ position are zero, and all such differentials going out of the $(q+1, 0)$ position are zero. So

$${}'E_2^{q+1,0} = {}'E_3^{q+1,0} = \cdots = {}'E_{q+1}^{q+1,0} = H^{q+1}(H^0(X, \mathscr{K}^\bullet), d) = H^{q+1}_{dR}(\mathscr{K}^\bullet, d).$$

Thus the differential d_{q+1} is a mapping $\ker d_1^{0,q} \to H^{q+1}_{dR}(\mathscr{K}^\bullet, d)$. The spectral sequence has to converge in the $(0, q)$ and $(q+1, 0)$ positions after this step, since then all arrows map out of the first quadrant. But we know that the hypercohomology of $\mathscr{K}^\bullet$ vanishes, since it is an exact sequence. Therefore the cohomology of the complex $\ker d_1^{0,q} \to H^{q+1}_{dR}(\mathscr{K}^\bullet, d)$ vanishes, which says that $\ker d_{q+1} = \operatorname{coker} d_{q+1} = 0$. Therefore d_{q+1} is an isomorphism. □

A.3. **Abstract de Rham theorems.** We first formulate an abstract version of de Rham's theorem with the de Rham complex $\mathscr{A}^\bullet$ replaced by any complex.

Theorem A.3.1 (Abstract Theorem of de Rham)**.** *Let X be a topological space, let $\mathscr{K}^\bullet$ be a complex of sheaves on X and put*

$$\mathscr{F} = \ker\{d : \mathscr{K}^0 \to \mathscr{K}^1\}.$$

(1) If $\mathscr{K}^\bullet$ is exact, there is a canonical identification

$$H^p(X, \mathscr{F}) = \mathbb{H}^p(X, \mathscr{K}^\bullet).$$

(2) Suppose that $H^p(X, \mathscr{K}^q) = 0$ for all q and all $p > 0$. Then we have an isomorphism

$$\mathbb{H}^p(X, \mathscr{K}^\bullet) \xrightarrow{\sim} H^p_{dR}(X, \mathscr{K}^\bullet) = H^p(\Gamma(X, \mathscr{K}^\bullet), d).$$

Proof. Note that while it is assumed $\mathscr{K}^\bullet$ is exact and graded in nonnegative dimensions, it isn't assumed that there is a 0 appended at the beginning. So $d : \mathscr{K}^0 \to \mathscr{K}^1$ does not have to be injective, and the kernel is called $\mathscr{F}$. Moreover, $\mathscr{F} = H^0(\mathscr{K}^\bullet, d)$, and $H^i(\mathscr{K}^\bullet, d) = 0$. The only surviving terms in $''E_2^{p,q}$ therefore have $q = 0$, and there $''E_2^{p,0} = H^p(X, \mathscr{F})$. There is then a unique term along each diagonal of $''E_\infty^{p,q}$, so in fact the hypercohomology $\mathbb{H}^p(\mathscr{K}^\bullet, d)$ is canonically equal to $H^p(X, \mathscr{F})$.

For the second assertion, since $'E_2^{p,q} = H^p(H^q(X, \mathscr{K}^\bullet), d) = 0$ the first spectral sequence degenerates at $'E_2$. The only nonzero entry in each diagonal is $'E_2^{p,0} = H^p_{dR}(X, \mathscr{K}^\bullet)$, so the hypercohomology is $H^p_{dR}(X, \mathscr{K}^\bullet)$.

Note that if the conditions of both parts of the theorem are satisfied, then the theorem is the statement that the cohomology of $\mathscr{F}$ can be computed by taking the homology of the global sections of an acyclic resolution. □

Now if $X = M$ is a differentiable manifold and $\mathscr{K}^\bullet = \mathscr{A}_M^\bullet$ is the de Rham complex, the last assertion is precisely the isomorphism $H^n_{dR}(M) \cong \mathbb{H}^n(\mathscr{A}^\bullet)$, which is one half of the proof of the theorem that the hypercohomology of the holomorphic de Rham complex gives the usual de Rham cohomology. Another application is an abstract proof of the de Rham theorem.

Theorem A.3.2. *The de Rham isomorphism $H^n_{dR}(M) \cong H^n(M; \mathbb{R})$ holds.*

Proof. The kernel $\mathscr{F}$ of $d : \mathscr{A}^0 \to \mathscr{A}^1$ is given by the locally constant functions on M. So we see that

$$H^n_{dR}(M) \cong \mathbb{H}^n(\mathscr{A}^\bullet) \cong H^n(M; \mathbb{R}).$$

The missing elements of the proof are the exactness of the de Rham complex, and the vanishing of $H^p(X, \mathscr{A}_M^q)$ for all q and all $p > 0$. Exactness of the complex is the Poincaré lemma; that $H^p(X, \mathscr{A}_M^q)$ vanishes follows from the existence of smooth partitions of unity. □

If we take the holomorphic de Rham complex, this does not work, because in general $H^p(M, \Omega_M^q)$ does not vanish for $p > 0$. The holomorphic Poincaré lemma still gives us exactness of the sequence of sheaves, however, so we can still assert

$$\mathbb{H}^n(M, \Omega_M^\bullet) \cong H^n(M; \mathbb{C}).$$

One can also obtain directly an identification of $\mathbb{H}^n(M, \Omega_M^\bullet)$ with the de Rham group as follows. Consider the injection $\Omega_M^\bullet \hookrightarrow \mathscr{A}_M^\bullet \otimes \mathbb{C}$ of the holomorphic de Rham complex into the usual de Rham complex. On the level of cohomology sheaves this induces an isomorphism: in H^0 we just have the sheaf of locally constant functions, and the higher cohomology of both sheaves vanish by Poincaré. Since quasi-isomorphisms induce isomorphisms on hypercohomology,

$$\mathbb{H}^n(\Omega_M^\bullet) \cong \mathbb{H}^n(\mathscr{A}_M^\bullet \otimes \mathbb{C}) \cong \mathbb{H}^n(\mathscr{A}_M^\bullet) \otimes \mathbb{C} \cong H_{dR}^n(M;\mathbb{C}).$$

Another application of the abstract de Rham theorem is a proof of Dolbeaut's theorem.

Theorem A.3.3. *If M is a complex manifold, consider the Dolbeault complex $(\mathscr{A}^{p,\bullet}, \overline{\partial})$. It is quasi-isomorphic to the complex Ω^p placed in degree p. We also have the Dolbeault isomorphism*

$$H^q(M,\Omega^p) \cong H_{\overline{\partial}}^{p,q}(M) = H_{dR}^q(\mathscr{A}^{p,\bullet}, \overline{\partial}).$$

Proof. Consider the inclusion $\Omega^p \hookrightarrow \mathscr{A}^{p,0}$, viewed as a map of chain complexes of sheaves by appending zeroes everywhere else. Since a $(p,0)$-form is killed by $\overline{\partial}$ if and only if it is holomorphic, and since the complex $\mathscr{A}^{p,\bullet}$ is exact by $\overline{\partial}$-Poincaré, this inclusion is a quasi-isomorphism. Thus the complexes have isomorphic hypercohomology. The qth hypercohomology of the complex Ω^p is just $H^q(M,\Omega^p)$. On the other hand,

$$\mathbb{H}^q(\mathscr{A}^{p,\bullet}, \overline{\partial}) \cong H_{dR}^q(\mathscr{A}^{p,\bullet}, \overline{\partial})$$

since $\mathscr{A}^{p,q}$ admits partitions of unity. □

A.4. **The spectral sequence of a filtered complex of sheaves.** For additional applications we need spectral sequences in hypercohomology associated to a filtered complex of sheaves on a topological space.

We start with a filtered complex of sheaves on a topological space X, say $(\mathscr{K}^\bullet, d)$ with decreasing filtration $F^\bullet$. This means that $F^p\mathscr{K}^\bullet$ is a subcomplex of $\mathscr{K}^\bullet$. A morphism between complexes of sheaves $f : (\mathscr{K}^\bullet, d) \to (\mathscr{L}^\bullet, d')$ is a *filtered quasi-isomorphism* if f preserves the filtration and $\operatorname{Gr} f$ is a quasi-isomorphism. That is, for each p we can form the chain complexes $\operatorname{Gr}_F^p \mathscr{K}^\bullet = F^p\mathscr{K}^\bullet / F^{p+1}\mathscr{K}^\bullet$ and $\operatorname{Gr}_{'F}^p \mathscr{L}^\bullet$, and we demand that the induced maps $f^p : \operatorname{Gr}_F^p \mathscr{K}^\bullet \to \operatorname{Gr}_{'F}^p \mathscr{L}^\bullet$ be quasi-isomorphisms.

An important example is the *trivial filtration*

$$F_{\text{triv}}^p(\mathscr{K}^\bullet) = (0 \to 0 \to \cdots \to \mathscr{K}^p \to \mathscr{K}^{p+1} \to \cdots),$$

which is a decreasing filtration. Another example is the order of pole filtration F_{pole} on $\Omega_{\mathbb{P}^{n+1}}^\bullet(*X)$.

Lemma A.4.1. *Let $\mathscr{K}^\bullet$ and $\mathscr{L}^\bullet$ be filtered complexes of sheaves, and suppose that $i : \mathscr{K}^\bullet \to \mathscr{L}^\bullet$ is a filtered quasi-isomorphism. Then in fact i is a quasi-isomorphism.*

Proof. Since the result is local, we can work with a filtered complex of abelian groups instead of sheaves. Since i is a filtered quasi-isomorphism, the spectral sequences of the filtered complexes with $E_1^{p,q} = H^{p+q}(\mathrm{Gr}_F^p(K^\bullet))$ are isomorphic. Thus we get an isomorphism in the E_∞ page as well, showing $\mathrm{Gr}_{F_\infty}^p H^{p+q}(K^\bullet) \cong \mathrm{Gr}_{F_\infty}^p H^{p+q}(L^\bullet)$. We are thus reduced to showing that if a map of filtered groups induces isomorphisms on graded parts, then it is an isomorphism of filtered groups. But an element of the kernel of i would prevent a corresponding graded map from being injective, and it is possible to piece together preimages of any element by successive approximation. □

Theorem A.4.2. *Suppose that $\mathscr{K}^\bullet$ is a filtered complex of sheaves such that the graded quotients $\mathrm{Gr}_F^p \mathscr{K}^q$ have no higher cohomology. Also assume the filtration has finite length on each sheaf. Then there is a spectral sequence*

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, \mathrm{Gr}_F^p \mathscr{K}^\bullet) \Rightarrow \mathbb{H}^{p+q}(X, \mathscr{K}^\bullet).$$

Moreover, we can conclude the same result even if $\mathscr{K}^\bullet$ is only filtered quasi-isomorphic to a filtered complex with these properties.

Proof. The long exact sequences in cohomology associated to the short exact sequences of sheaves

$$0 \to F^{p+1}\mathscr{K}^q \to F^p\mathscr{K}^q \to \mathrm{Gr}_F^p \mathscr{K}^q \to 0$$

easily show that all of the sheaves $F^p\mathscr{K}^q$ have no higher cohomology. For $H^k(F^p\mathscr{K}^q)$ is always a quotient of $H^k(F^{p+1}\mathscr{K}^q)$ if $k \geq 1$, and $F^r\mathscr{K}^q = 0$ for r large enough. It also follows that the global section functor Γ commutes with Gr_F^p, seeing as $H^1(F^{p+1}\mathscr{K}^q)$ vanishes. Here we give $\Gamma\mathscr{K}^q$ the natural filtration from $\mathscr{K}^q$.

Now consider the filtered complex of groups $\Gamma\mathscr{K}^\bullet$. The E_1 page of the associated spectral sequence has

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}_F^p(\Gamma\mathscr{K}^\bullet)) = H^{p+q}(\Gamma(\mathrm{Gr}_F^p \mathscr{K}^\bullet)) = H_{dR}^{p+q}(X, \mathrm{Gr}_F^p \mathscr{K}^\bullet).$$

By the second statement in the abstract de Rham theorem,

$$H_{dR}^{p+q}(X, \mathrm{Gr}_F^p \mathscr{K}^\bullet) \cong \mathbb{H}^{p+q}(X, \mathrm{Gr}_F^p \mathscr{K}^\bullet).$$

On the other hand, the E_∞ page of this spectral sequence is

$$E_\infty^{p,q} = \mathrm{Gr}_{F_\infty}^p H^{p+q}(\Gamma\mathscr{K}^\bullet) = \mathrm{Gr}_{F_\infty}^p H_{dR}^{p+q}(\mathscr{K}^\bullet) = \mathrm{Gr}_{F_\infty}^p \mathbb{H}^{p+q}(\mathscr{K}^\bullet),$$

so

$$E_1^{p,q} = \mathbb{H}^{p+q}(\mathrm{Gr}_F^p \mathscr{K}^\bullet) \Rightarrow \mathbb{H}^{p+q}(X, \mathscr{K}^\bullet).$$

If instead $\mathscr{K}^\bullet$ is only filtered quasi-isomorphic to a complex with these properties, then we also know that it is quasi-isomorphic to that complex. But then all the hypercohomology groups in question are isomorphic, so we still get the spectral sequence. □

This theorem can be applied in the case of the embedding

$$\Omega_M^\bullet \xrightarrow{j} \mathscr{A}_M^\bullet \otimes \mathbb{C}$$

on a Kähler manifold M. When we give the right complex the filtration into Hodge types and the left complex the trivial filtration, j becomes a filtered quasi-isomorphism. The graded quotient $\mathrm{Gr}_F^p(\mathscr{A}_M^q \otimes \mathbb{C})$ is just $\mathscr{A}_M^{p,p-q} \otimes \mathbb{C}$, which has no higher cohomology since we have partitions of unity. On the other hand the graded quotient $\mathrm{Gr}_F^p(\Omega_M^\bullet)$ consists of a copy of Ω_M^p in degree p, and zeroes everywhere else. Thus there is a spectral sequence

$$E_1^{p,q} = \mathbb{H}^{p+q}(\mathrm{Gr}_F^p(\Omega_M^\bullet)) = H^q(\Omega_M^p) \Rightarrow \mathbb{H}^{p+q}(\Omega_M^\bullet) = H^{p+q}(M;\mathbb{C}).$$

This is the *Hodge to de Rham spectral sequence.* The induced filtration on the abutment is of course the Hodge filtration, the sequence degenerates at E_1, and the entries of the diagonals give the Hodge decomposition.

References

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