Interpolation on Surfaces in $\mathbb{P}^3$

Jack Huizenga

Basic Problem

Let S be a surface (projective, smooth, over $\mathbb{C}$) and let L be a line bundle on S. Fix general points $p_1, \ldots, p_r$ on S.

Basic Problem

Let S be a surface (projective, smooth, over $\mathbb{C}$) and let L be a line bundle on S. Fix general points $p_1, \ldots, p_r$ on S.

What is the dimension of the space of sections of L having a singularity of multiplicity m_i at p_i for each i?

Basic Problem

Let S be a surface (projective, smooth, over $\mathbb{C}$) and let L be a line bundle on S. Fix general points $p_1, \ldots, p_r$ on S.

What is the dimension of the space of sections of L having a singularity of multiplicity m_i at p_i for each i?

Expected Answer:

It is $m(m+1)/2$ linear conditions for a section of L to have a singularity of multiplicity m at a point p. Thus we expect the space of such sections has dimension

$$\max\{h^0(L) - \sum_i \frac{m_i(m_i+1)}{2}, 0\}.$$

An Example on $\mathbb{P}^2$

Take $S = \mathbb{P}^2$, and $L = \mathcal{O}_{\mathbb{P}^2}(2)$. If p, q are general points in the plane, how many conics are singular at both p and q?

An Example on $\mathbb{P}^2$

Take $S = \mathbb{P}^2$, and $L = \mathcal{O}_{\mathbb{P}^2}(2)$. If p, q are general points in the plane, how many conics are singular at both p and q?

Expected Answer:

We have $h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 6$, and it is 3 conditions to be singular at a point. So we expect that no conics are singular at each of two fixed points.

An Example on $\mathbb{P}^2$

Take $S = \mathbb{P}^2$, and $L = \mathcal{O}_{\mathbb{P}^2}(2)$. If p, q are general points in the plane, how many conics are singular at both p and q?

Expected Answer:

We have $h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 6$, and it is 3 conditions to be singular at a point. So we expect that no conics are singular at each of two fixed points.

Actual Answer:

The *double line* through p and q is a conic singular at both p and q, so the expected dimension is different from the actual dimension.

Notation

If $S \subset \mathbb{P}^3$ is a surface of degree d and $L = \mathcal{O}_S(e)$, we write

$$\mathscr{L} = \mathscr{L}_e^S(m_1, \ldots, m_r)$$

for the space of sections of $\mathcal{O}_S(e)$ having a singularity of multiplicity m_i at p_i for each i, where $p_1, \ldots, p_r$ are fixed general points of S.

Notation

If $S \subset \mathbb{P}^3$ is a surface of degree d and $L = \mathcal{O}_S(e)$, we write

$$\mathscr{L} = \mathscr{L}_e^S(m_1, \ldots, m_r)$$

for the space of sections of $\mathcal{O}_S(e)$ having a singularity of multiplicity m_i at p_i for each i, where $p_1, \ldots, p_r$ are fixed general points of S.

The series $\mathscr{L}$ is *nonspecial* if its dimension equals the expected dimension.

Notation

If $S \subset \mathbb{P}^3$ is a surface of degree d and $L = \mathcal{O}_S(e)$, we write

$$\mathscr{L} = \mathscr{L}_e^S(m_1, \ldots, m_r)$$

for the space of sections of $\mathcal{O}_S(e)$ having a singularity of multiplicity m_i at p_i for each i, where $p_1, \ldots, p_r$ are fixed general points of S.

The series $\mathscr{L}$ is *nonspecial* if its dimension equals the expected dimension.

The series $\mathscr{L}$ is *special* if its dimension exceeds the expected dimension.

Special series on $\mathbb{P}^2$

As just seen, $\mathscr{L}_2^{\mathbb{P}^2}(2,2)$ is special, because of the double line.

Special series on $\mathbb{P}^2$

As just seen, $\mathscr{L}_2^{\mathbb{P}^2}(2,2)$ is special, because of the double line.

Similarly, $\mathscr{L}_4^{\mathbb{P}^2}(2,2,2,2,2)$ is special. The expected dimension is $15 - 3 \cdot 5 = 0$, but there is a curve in this series: the double conic through 5 points.

Special series on $\mathbb{P}^2$

As just seen, $\mathscr{L}_2^{\mathbb{P}^2}(2,2)$ is special, because of the double line.

Similarly, $\mathscr{L}_4^{\mathbb{P}^2}(2,2,2,2,2)$ is special. The expected dimension is $15 - 3 \cdot 5 = 0$, but there is a curve in this series: the double conic through 5 points.

Conjecture (B. Segre (1960))

If the general member of $\mathscr{L} = \mathscr{L}_e^{\mathbb{P}^2}(m_1, \ldots, m_r)$ is reduced, then $\mathscr{L}$ is nonspecial.

Two conjectures for $\mathbb{P}^2$

Conjecture (B. Segre (1960))

If the general member of $\mathscr{L} = \mathscr{L}_e^{\mathbb{P}^2}(m_1, \ldots, m_r)$ is reduced, then $\mathscr{L}$ is nonspecial.

Two conjectures for $\mathbb{P}^2$

Conjecture (B. Segre (1960))

If the general member of $\mathscr{L} = \mathscr{L}_e^{\mathbb{P}^2}(m_1, \ldots, m_r)$ is reduced, then $\mathscr{L}$ is nonspecial.

Conjecture (Harbourne-Gimigliano-Hirschowitz (1980's))

There is an explicit, infinite list of all special series $\mathscr{L}$ on $\mathbb{P}^2$.

Two conjectures for $\mathbb{P}^2$

Conjecture (B. Segre (1960))

If the general member of $\mathscr{L} = \mathscr{L}_e^{\mathbb{P}^2}(m_1, \ldots, m_r)$ *is reduced, then* $\mathscr{L}$ *is nonspecial.*

Conjecture (Harbourne-Gimigliano-Hirschowitz (1980's))

There is an explicit, infinite list of all special series $\mathscr{L}$ *on* $\mathbb{P}^2$*.*

Theorem (C. Ciliberto and R. Miranda (2001))

Segre's conjecture and the HGH conjecture are equivalent.

Two conjectures for $\mathbb{P}^2$

Conjecture (B. Segre (1960))

If the general member of $\mathscr{L} = \mathscr{L}_e^{\mathbb{P}^2}(m_1, \ldots, m_r)$ *is reduced, then* $\mathscr{L}$ *is nonspecial.*

Conjecture (Harbourne-Gimigliano-Hirschowitz (1980's))

There is an explicit, infinite list of all special series $\mathscr{L}$ *on* $\mathbb{P}^2$*.*

Theorem (C. Ciliberto and R. Miranda (2001))

Segre's conjecture and the HGH conjecture are equivalent.

Theorem (M. Dumnicki and W. Jarnicki (2007))

The conjectures are true for series $\mathscr{L}$ *such that the multiplicities* m_i *are all bounded by 11.*

Special series on surfaces $S \subset \mathbb{P}^3$

If $d = \deg S$ equals 2 or 3, then S is rational, and classifying special series on S reduces to classifying special series on $\mathbb{P}^2$.

Special series on surfaces $S \subset \mathbb{P}^3$

If $d = \deg S$ equals 2 or 3, then S is rational, and classifying special series on S reduces to classifying special series on $\mathbb{P}^2$.

$d \geq 4$:

Consider the series $\mathscr{L}_2^S(4)$. We have $h^0(\mathcal{O}_S(2)) = 10$, and a quadruple point imposes 10 conditions, so we expect this series to be empty.

Special series on surfaces $S \subset \mathbb{P}^3$

If $d = \deg S$ equals 2 or 3, then S is rational, and classifying special series on S reduces to classifying special series on $\mathbb{P}^2$.

$d \geq 4$:

Consider the series $\mathscr{L}_2^S(4)$. We have $h^0(\mathcal{O}_S(2)) = 10$, and a quadruple point imposes 10 conditions, so we expect this series to be empty.

However, if H is a tangent hyperplane to S, then $2H$ meets S in a (nonreduced) curve with a singularity of multiplicity 4.

Special series on surfaces $S \subset \mathbb{P}^3$

If $d = \deg S$ equals 2 or 3, then S is rational, and classifying special series on S reduces to classifying special series on $\mathbb{P}^2$.

$d \geq 4$:

Consider the series $\mathscr{L}_2^S(4)$. We have $h^0(\mathcal{O}_S(2)) = 10$, and a quadruple point imposes 10 conditions, so we expect this series to be empty.

However, if H is a tangent hyperplane to S, then $2H$ meets S in a (nonreduced) curve with a singularity of multiplicity 4.

Conjecturally, this is the "only" way special series arise on S.

Two conjectures for $S \subset \mathbb{P}^3$

Assume $S \subset \mathbb{P}^3$ has degree $d \geq 4$ and is *very general*, so that $\operatorname{Pic} S$ is generated by $\mathcal{O}_S(1)$.

Two conjectures for $S \subset \mathbb{P}^3$

Assume $S \subset \mathbb{P}^3$ has degree $d \geq 4$ and is *very general*, so that $\operatorname{Pic} S$ is generated by $\mathcal{O}_S(1)$.

Conjecture

If the general member of $\mathscr{L} = \mathscr{L}_e^S(m_1, \ldots, m_r)$ is reduced, then $\mathscr{L}$ is nonspecial.

Two conjectures for $S \subset \mathbb{P}^3$

Assume $S \subset \mathbb{P}^3$ has degree $d \geq 4$ and is *very general*, so that $\operatorname{Pic} S$ is generated by $\mathcal{O}_S(1)$.

Conjecture

If the general member of $\mathscr{L} = \mathscr{L}_e^S(m_1, \ldots, m_r)$ is reduced, then $\mathscr{L}$ is nonspecial.

Conjecture

If $\mathscr{L}$ is special, it is of the form

1. $\mathscr{L}_e^S(2e)$, *with* $e \geq 2$ *and* $d = 4$,
2. $\mathscr{L}_e^S(2e)$, *with* $2 \leq e \leq 5$ *and* $d = 5$, *or*
3. $\mathscr{L}_e^S(2e)$, *with* $2 \leq e \leq 4$ *and* $d \geq 6$.

Results

Theorem (De Volder & Laface ($d = 4$) 2005, H. ($d \geq 5$))

The two previous conjectures are equivalent.

Results

Theorem (De Volder & Laface ($d = 4$) 2005, H. ($d \geq 5$))

The two previous conjectures are equivalent.

Theorem (H.)

If S is very general of degree $d \geq 4$, then the only special series $\mathscr{L} = \mathscr{L}_e^S(m_1, \ldots, m_r)$ with all $m_i \leq 4$ is $\mathscr{L}_2^S(4)$.

Results

Theorem (De Volder & Laface ($d = 4$) 2005, H. ($d \geq 5$))

The two previous conjectures are equivalent.

Theorem (H.)

If S is very general of degree $d \geq 4$, then the only special series $\mathscr{L} = \mathscr{L}_e^S(m_1, \ldots, m_r)$ with all $m_i \leq 4$ is $\mathscr{L}_2^S(4)$.

To get an idea for the proof of this result, let us show that the series $\mathscr{L}_3^S(4,4)$ is nonspecial when S is a general quartic surface.

Results

Theorem (De Volder & Laface ($d = 4$) 2005, H. ($d \geq 5$))

The two previous conjectures are equivalent.

Theorem (H.)

If S is very general of degree $d \geq 4$, then the only special series $\mathscr{L} = \mathscr{L}_e^S(m_1, \ldots, m_r)$ with all $m_i \leq 4$ is $\mathscr{L}_2^S(4)$.

To get an idea for the proof of this result, let us show that the series $\mathscr{L}_3^S(4,4)$ is nonspecial when S is a general quartic surface.

Note that if $\deg S = 4$, then $h^0(\mathcal{O}_S(3)) = 20$ and a quadruple point imposes 10 conditions—thus, we must show $\mathscr{L}_3^S(4,4)$ is empty when S is a general quartic surface.

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

Basic outline

Degenerate S into a union of two quadrics $U \cup V$, and choose how the two quadruple points specialize onto the reducible surface.

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

Basic outline

Degenerate S into a union of two quadrics $U \cup V$, and choose how the two quadruple points specialize onto the reducible surface.

Choose a degeneration of the line bundle $\mathcal{O}_S(3)$ over the reducible surface.

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

Basic outline

Degenerate S into a union of two quadrics $U \cup V$, and choose how the two quadruple points specialize onto the reducible surface.

Choose a degeneration of the line bundle $\mathcal{O}_S(3)$ over the reducible surface.

Show the line bundle on the reducible surface has no sections quadruple at the two points. (Quadric surfaces are easier to work with than quartic surfaces)

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

Basic outline

Degenerate S into a union of two quadrics $U \cup V$, and choose how the two quadruple points specialize onto the reducible surface.

Choose a degeneration of the line bundle $\mathcal{O}_S(3)$ over the reducible surface.

Show the line bundle on the reducible surface has no sections quadruple at the two points. (Quadric surfaces are easier to work with than quartic surfaces)

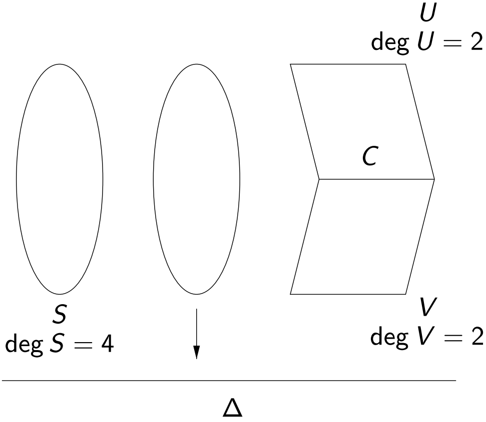
Conclude $\mathscr{L}_3^S(4,4)$ is empty for general S by semicontinuity.

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

Let S degenerate in a pencil into a union of two quadric surfaces U and V, meeting along a curve C. Call the total space of the the family X.

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

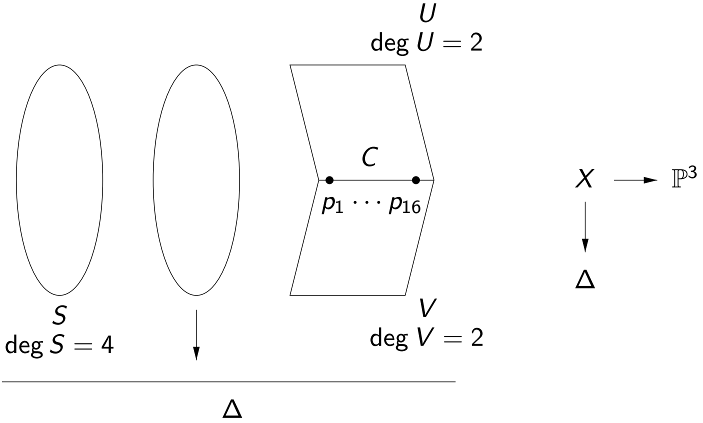
Let S degenerate in a pencil into a union of two quadric surfaces U and V, meeting along a curve C. Call the total space of the the family X.



$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^3 \\ \downarrow & & \\ \Delta & & \end{array}$$

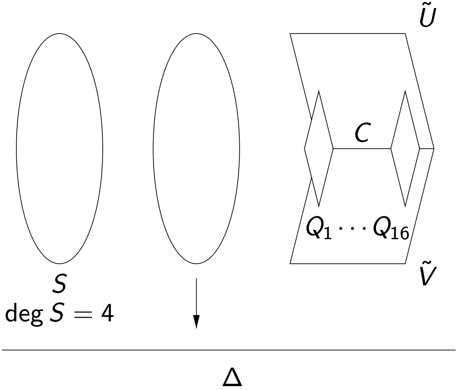
$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

The total space X fails to be smooth at the 16 points $p_1, \ldots, p_{16}$ where S meets C.



$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

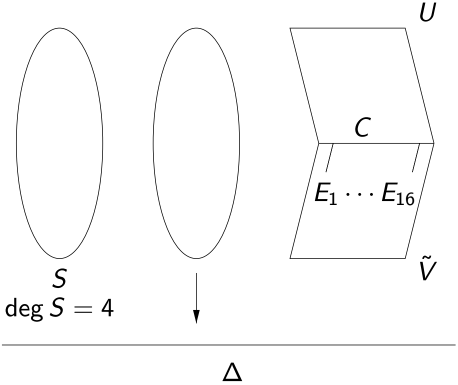
Blow up the 16 singular points. Since S, U, V are all general, the singularities were ordinary double points, and the exceptional divisors $Q_1, \ldots, Q_{16}$ are all isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

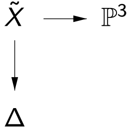


$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \mathbb{P}^3 \\ \downarrow & & \\ \Delta & & \end{array}$$

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

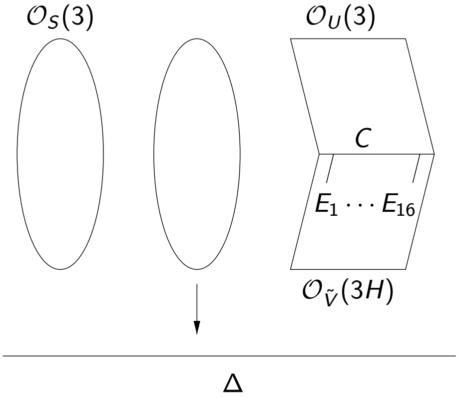
The exceptional divisors $Q_1, \ldots, Q_{16}$ meet $\tilde{U}$ and $\tilde{V}$ in lines of opposite rulings on the Q_i. We can blow down the rulings corresponding to $\tilde{U}$.

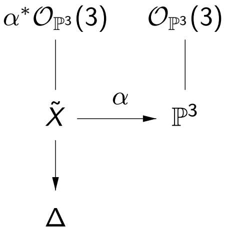




$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

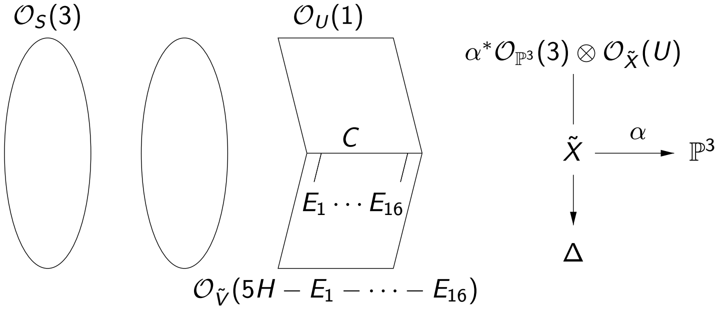
We put a line bundle on $\tilde{X}$ by pulling back $\mathcal{O}_{\mathbb{P}^3}(3)$. This line bundle restricts to $\mathcal{O}_S(3)$ on S, $\mathcal{O}_U(3)$ on U, and $\mathcal{O}_{\tilde{V}}(3H)$ on $\tilde{V} = \mathrm{Bl}_{p_1,\ldots,p_{16}}(V)$.





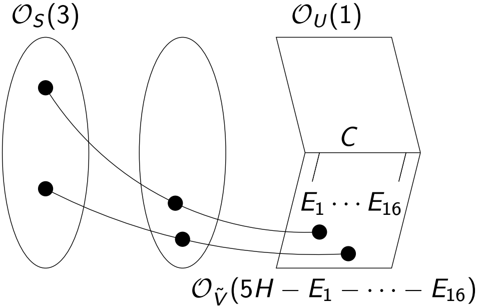
$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

If we twist this line bundle by $\mathcal{O}_{\tilde{X}}(U)$ (which we can do since $\tilde{X}$ is smooth), then the bundle still restricts to $\mathcal{O}_S(3)$ on S. But now it restricts to $\mathcal{O}_U(1)$ on U (since C has class $2H$ on U) and to $\mathcal{O}_{\tilde{V}}(5H - E_1 - \cdots - E_{16})$ on $\tilde{V}$ (since C has class $2H - E_1 - \cdots - E_{16}$ on $\tilde{V}$).



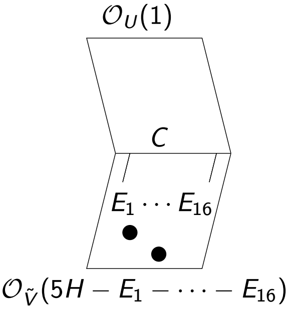
$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

Now imagine two points on S both specializing onto general points in the special fiber. If the line bundle on the special fiber has no sections which are quadruple at these two limit points, then by semicontinuity the line bundle on the general quartic surface S has no sections quadruple at two points—that is, $\mathscr{L}_3^S(4,4)$ will be empty.



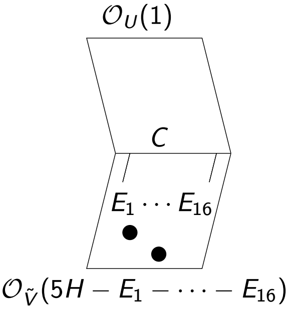
$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

We must show the line bundle on $U \cup \tilde{V}$ has no sections quadruple at each of two general points in $\tilde{V}$.



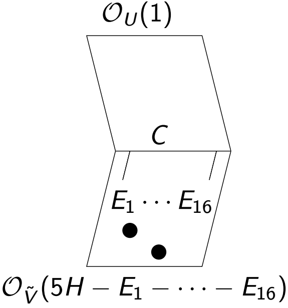
$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

We must show the line bundle on $U \cup \tilde{V}$ has no sections quadruple at each of two general points in $\tilde{V}$.



Sections of the line bundle on $U \cup \tilde{V}$ are obtained by taking sections on each factor which match on C.

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

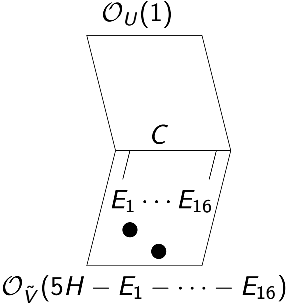


We must show the line bundle on $U \cup \tilde{V}$ has no sections quadruple at each of two general points in $\tilde{V}$.

Sections of the line bundle on $U \cup \tilde{V}$ are obtained by taking sections on each factor which match on C.

A section of $\mathcal{O}_U(1)$ vanishing on C must be identically zero on U, since C has class $2H$.

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4



We must show the line bundle on $U \cup \tilde{V}$ has no sections quadruple at each of two general points in $\tilde{V}$.

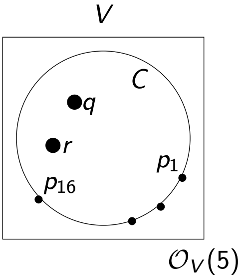
Sections of the line bundle on $U \cup \tilde{V}$ are obtained by taking sections on each factor which match on C.

A section of $\mathcal{O}_U(1)$ vanishing on C must be identically zero on U, since C has class $2H$.

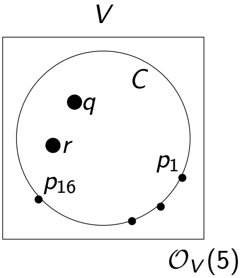
If there are no nonzero sections on $\tilde{V}$ having two quadruple points, we will be done.

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

On V, how many curves of class $5H$ contain $p_1, \ldots, p_{16} \in C$ and are quadruple at two points q, r?



$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

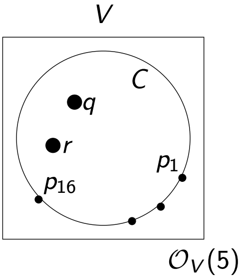


On V, how many curves of class $5H$ contain $p_1, \ldots, p_{16} \in C$ and are quadruple at two points q, r?

We have $h^0(\mathcal{O}_C(5)(-p_1 - \ldots - p_{16})) = h^0(\mathcal{O}_C(1)) = 4.$

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

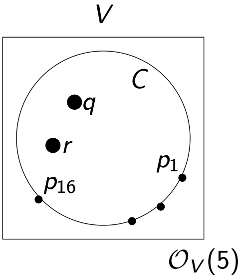
On V, how many curves of class $5H$ contain $p_1, \ldots, p_{16} \in C$ and are quadruple at two points q, r?



We have
$h^0(\mathcal{O}_C(5)(-p_1 - \ldots - p_{16})) = h^0(\mathcal{O}_C(1)) = 4.$

If we specialize q onto a general point of C, then every curve of class $5H$ containing $p_1, \ldots, p_{16}$ and quadruple at q contains C.

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4



On V, how many curves of class $5H$ contain $p_1, \ldots, p_{16} \in C$ and are quadruple at two points q, r?

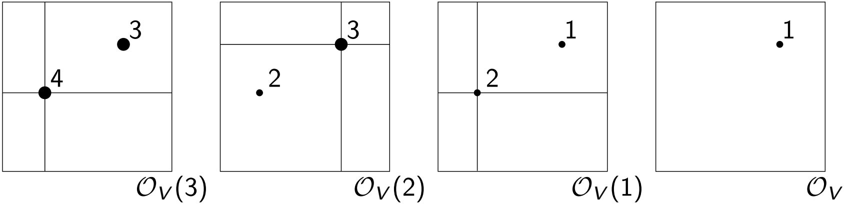
We have
$h^0(\mathcal{O}_C(5)(-p_1 - \ldots - p_{16})) = h^0(\mathcal{O}_C(1)) = 4.$

If we specialize q onto a general point of C, then every curve of class $5H$ containing $p_1, \ldots, p_{16}$ and quadruple at q contains C.

Residual to C are the curves of class $3H$ triple at q and quadruple at r.

$\mathscr{L}_3^S(4,4)$ is empty, S general of degree 4

On the quadric surface V, how many curves of class $3H$ are triple at one point and quadruple at another? Analyzing lines of the ruling shows there are none.



Thus $\mathscr{L}_3^S(4,4)$ is empty.

Acknowledgements

Thank you to Joe Harris and Marcin Dumnicki.

Selected References

C. Ciliberto & R. Miranda. *The Segre and Harbourne-Hirschowitz conjectures.* 2001.

C. De Volder & A. Laface. *Linear systems on generic K3 surfaces.* 2005.

M. Dumnicki & W. Jarnicki. *New effective bounds on the dimension of a linear system in* $\mathbb{P}^2$. 2007.

A. Gimigliano. *On linear systems of plane curves.* 1987.

B. Harbourne. *The geometry of rational surfaces and Hilbert functions of points in the plane.* 1986.

A. Hirschowitz. *Une conjecture pour la cohomologie des diviseurs sur les surfaces rationelles gènèriques.* 1989.

B. Segre. *Alcune questioni su insiemi finiti di punti in geometria algebrica.* 1960.