BRILL-NOETHER PROBLEMS, ULRICH BUNDLES AND THE COHOMOLOGY OF MODULI SPACES OF SHEAVES

IZZET COSKUN AND JACK HUIZENGA

Abstract. In this paper, we survey some recent developments on computing the cohomology of the moduli spaces of sheaves on surfaces and the Brill-Noether problem. We explain several applications to classifying stable Chern characters on Hirzebruch surfaces, classifying globally generated vector bundles on minimal rational surfaces, and constructing and classifying Ulrich bundles on surfaces. This paper grew out of the talks of the authors at the ICM Satellite Conference on Moduli Spaces in Algebraic Geometry and Applications.

1. Introduction

In this paper, we survey some recent developments on computing the cohomology of moduli spaces of sheaves on surfaces and the Brill-Noether problem following [CH18b, CH18c, CNY18, CW18]. The paper grew out of the authors' talks at the ICM Satellite Conference on Moduli Spaces in Algebraic Geometry and Applications. Except for a novel point of view and some new results on Ulrich bundles in §4, the paper is a survey of results that have been published elsewhere.

Let X be a smooth, complex projective surface and let H be an ample divisor on X. Given a Chern character $\mathbf{v}$, Gieseker [Gie77] and Maruyama [Mar78] constructed a moduli space $M_{X,H}(\mathbf{v})$ that parameterizes S-equivalence classes of H-Gieseker semistable sheaves on X. These moduli spaces play a central role in studying linear series and cycles on X, in Donaldson's theory of 4-manifolds [Don90] and in mathematical physics [Wit95]. Consequently, it is of fundamental importance to understand the geometry of $M_{X,H}(\mathbf{v})$.

In this paper, we focus on two problems concerning the geometry of $M_{X,H}(\mathbf{v})$. First, we will survey several current developments in the Brill-Noether Problem.

Problem 1.1 (Brill-Noether)**.** *Let $\mathcal{V}$ be a general sheaf in an irreducible component of $M_{X,H}(\mathbf{v})$. Compute the cohomology of $\mathcal{V}$.*

One can further ask to characterize the locus of sheaves $\mathcal{V}$ that do not have the generic cohomology and describe the dimensions of the cohomology jump loci. More generally, one can pose the interpolation problem. Given a coherent sheaf $\mathcal{W}$ and a sheaf $\mathcal{V} \in M_{X,H}(\mathbf{v})$, compute the cohomology of $\mathcal{W} \otimes \mathcal{V}$ (see [CH14, CH15]). We will restrict ourselves to the most basic form of the Brill-Noether problem and in §3 describe the solution of the problem for Hirzebruch surfaces following [CH18c]. We will discuss the solution of the Brill-Noether problem for more general rational surfaces under some restrictions on $\mathbf{v}$ following [CH18b] and describe some partial results and examples in the case of K3 surfaces discussed in [CNY18].

The Brill-Noether problem underlies many of the other open problems concerning the geometry of $M_{X,H}(\mathbf{v})$. Despite the efforts of many mathematicians in the last four decades, we do not know when

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$M_{X,H}(\mathbf{v})$ is nonempty in general. Bogomolov's inequality says that the discriminant Δ of a semistable sheaf is nonnegative and places strong restrictions on the Chern characters of semistable sheaves. On the other hand, O'Grady's Theorem [O'G96] guarantees the existence of slope stable sheaves provided that $\Delta \gg 0$. However, a complete classification of Chern characters $\mathbf{v}$ of semistable sheaves is known only for a handful of special surfaces such as $\mathbb{P}^2$ [DLP85], K3 surfaces and abelian surfaces (see [HL10]). Using our solution of the Brill-Noether problem for Hirzebruch surfaces, we can give an algorithm for classifying the stable Chern characters on a Hirzebruch surface for any polarization [CH18e]. We will briefly outline the algorithm in §3.

The Brill-Noether problem and more generally the interpolation problem play a central role in constructing theta and Brill-Noether divisors. These in turn play a central role in describing ample and effective cones of moduli spaces of sheaves (see [ABCH13, BM14, BC13, CH16, CH18a, CHW17]). We will not discuss the birational geometry of moduli spaces in this paper, but refer the reader to the survey [CH15].

The solution of the Brill-Noether problem on Hirzebruch surfaces allows one to obtain a Gaeta-type resolution on Hirzebruch surfaces. This resolution in turn allows one to classify Chern characters of moduli spaces whose general member is globally generated. In §3, we will briefly recall the classification.

Finally, the solution of the Brill-Noether problem has strong implications for the construction and classification of Ulrich bundles. In §4, we will give a quick proof of the classification of Ulrich bundles on Hirzebruch surfaces for every polarization. Using the Brill-Noether theorems for rational surfaces, we will construct Ulrich bundles on rational surfaces in some new cases. Finally, for general surfaces, we will show that the asymptotic Brill-Noether theorem easily implies the following existence theorem.

Theorem 1.2. *Let X be a smooth, complex, projective surface and let H be an ample divisor. Then there exists Ulrich bundles of rank two on (X, mH) if m is sufficiently large.*

This theorem has many consequences for the cohomology tables on (X, mH) and imply that their Chow forms are linear (see [ESW03, ES11]). The presentation and some of the results in this section are new.

In the final section §5, we will discuss the following fundamental open problem.

Problem 1.3. *Compute the singular cohomology of $M_{X,H}(\mathbf{v})$.*

As with many of the open problems concerning $M_{X,H}(\mathbf{v})$, the cohomology of $M_{X,H}(\mathbf{v})$ is known in some cases. The cohomology has been computed by Göttsche [Got90] when the rank of $\mathbf{v}$ is one. The Betti numbers of $M_{X,H}(\mathbf{v})$ have been computed when X is $\mathbb{P}^2$ or a ruled surface and the rank is two [Got96, Got99, Yos94, Yos95, Yos96a, Yos96b]. The Betti numbers are also known when $M_{X,H}(\mathbf{v})$ is smooth and X is a K3 surface or abelian surface [Huy03, Muk84, Yos99, Yos01]. In [CW18], the authors conjecture that the Betti numbers of $M_{X,H}(\mathbf{v})$ stabilize as Δ tends to infinity and that the stable Betti numbers do not depend on the rank, c_1 or the ample H. Consequently, while the Betti numbers of $M_{X,H}(\mathbf{v})$ are hard to compute, conjecturally the stable Betti numbers have already been computed by Göttsche. In §5, we will survey this conjecture and a motivic proof of it for rational surfaces X and polarizations H satisfying $K_X \cdot H < 0$ following [CW18].

Organization of the paper. In §2, we recall basic definitions concerning moduli spaces of sheaves. In §3, we survey Brill-Noether type theorems on rational surfaces, concentrating primarily on Hirzebruch surfaces following [CH18b, CH18c]. In §4, we present applications of Brill-Noether theorems to the existence and classification of Ulrich bundles. This section contains several new results and new proofs of results on Ulrich bundles. In §5, we survey a conjecture on the cohomology of $M_{X,H}(\mathbf{v})$ and its motivic proof in the case of rational surfaces following [CW18].

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2. Preliminaries

In this section, we summarize preliminary facts concerning rational surfaces, semistable and prioritary sheaves and moduli spaces of sheaves that we will freely quote in the rest of the paper

Rational surfaces. We refer the reader to [Bea83, Cos06a, Cos06b, Har77] for more detailed discussions on rational surfaces. For a nonnegative integer e, let $\mathbb{F}_e$ denote the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ and let $\pi : \mathbb{F}_e \to \mathbb{P}^1$ be the natural projection. The Picard group of $\mathbb{F}_e$ is

$$\operatorname{Pic}(\mathbb{F}_e) = \mathbb{Z}E \oplus \mathbb{Z}F,$$

where E is the class of a section of self-intersection $-e$ and F is the class of a fiber of π. The intersection form is determined by

$$E^2 = -e, \quad E \cdot F = 1, \quad F^2 = 0.$$

The effective cone of $\mathbb{F}_e$ is spanned by E and F and the nef cone is spanned by $E + eF$ and F. The canonical class of $\mathbb{F}_e$ is

$$K_{\mathbb{F}_e} = -2E - (e+2)F.$$

The cohomology of a line bundle $L = \mathcal{O}_{\mathbb{F}_e}(aE + bF)$ on $\mathbb{F}_e$ is easy to compute. We have

$$\chi(L) = (a+1)(b+1) - \frac{e}{2}a(a+1).$$

By Serre duality we may assume that $L \cdot F \geq -1$, in which case, $H^2(\mathbb{F}_e, L) = 0$. If $L \cdot F = -1$, then L has no cohomology. If $L \cdot F > -1$ and $L \cdot E \geq -1$, then $H^1(\mathbb{F}_e, L) = 0$ and $h^0(\mathbb{F}_e, L) = \chi(L)$. If $L \cdot E < -1$, then $h^0(\mathbb{F}_e, L) = h^0(\mathbb{F}_e, L(-E))$ and the cohomology is inductively determined (see [CH18c, Theorem 2.1]) .

The minimal rational surfaces are $\mathbb{P}^2$ and $\mathbb{F}_e$ with $e \neq 1$. Every other rational surface can be obtained from these surfaces by a sequence of blowups.

If X is a smooth, complex projective surface and $p \in X$ is a point, let $\phi : \hat{X} \to X$ denote the blowup of X at p. The Picard group of $\hat{X}$ is isomorphic to $\operatorname{Pic}(X) \oplus \mathbb{Z}E_p$, where E_p is the exceptional divisor over p. The canonical class of $\hat{X}$ is given by $\phi^* K_X + E_p$. In particular, if H is a polarization on X with $H \cdot K_X < 0$, then $\phi^*(H)$ is a big and nef divisor that has $\phi^*(H) \cdot K_{\hat{X}} < 0$. Since there are ample divisors arbitrarily close to $\phi^*(H)$, there are polarizations $\hat{H}$ on $\hat{X}$ such that $\hat{H} \cdot K_{\hat{X}} < 0$. Hence, all rational surfaces X contain polarizations H such that $H \cdot K_X < 0$.

Semistable sheaves. Let X be a smooth, complex projective surface. Let H be an ample divisor on X. In this paper, all the sheaves we consider will be coherent of pure-dimension. We refer the reader to [CH15, HL10] and [LeP97] for more detailed information on Gieseker (semi)stability and moduli spaces of stable sheaves.

Let $\mathbf{v}$ denote a positive rank Chern character on X and define the H-*slope* $\mu_H(\mathbf{v})$, the *total slope* $\nu(\mathbf{v})$ and *discriminant* $\Delta(\mathbf{v})$ by the formulae

$$\mu_H(\mathbf{v}) = \frac{c_1(\mathbf{v}) \cdot H}{r(\mathbf{v})}, \quad \nu(\mathbf{v}) = \frac{c_1(\mathbf{v})}{r(\mathbf{v})}, \quad \Delta(\mathbf{v}) = \frac{1}{2}\nu(\mathbf{v})^2 - \frac{\operatorname{ch}_2(\mathbf{v})}{r(\mathbf{v})},$$

respectively. The H-slope, total slope and discriminant of a sheaf $\mathcal{V}$ of positive rank is defined to be the H-slope, total slope and discriminant of its Chern character. The Chern character $(r, \operatorname{ch}_1, \operatorname{ch}_2)$ of a positive rank sheaf can be recovered from (r, ν, Δ). The advantage is that the slope and the discriminant are additive on tensor products

$$\nu(\mathcal{V} \otimes \mathcal{W}) = \nu(\mathcal{V}) + \nu(\mathcal{W})$$
$$\Delta(\mathcal{V} \otimes \mathcal{W}) = \Delta(\mathcal{V}) + \Delta(\mathcal{W}).$$

If L is a line bundle on X, then $\Delta(L) = 0$. Consequently, tensoring a sheaf with a line bundle preserves the discriminant. Set

$$P(\nu) = \chi(\mathcal{O}_X) + \frac{1}{2}\nu \cdot (\nu - K_X).$$

The Riemann-Roch formula in terms of these invariants reads

$$\chi(\mathcal{V}) = r(\mathcal{V})(P(\nu(\mathcal{V})) - \Delta(\mathcal{V})).$$

Definition 2.1. A torsion-free coherent sheaf $\mathcal{V}$ is called *μ_H-(semi)stable* if for every nonzero subsheaf $\mathcal{W}$ of smaller rank, we have

$$\mu_H(\mathcal{W}) \underset{(-)}{<} \mu_H(\mathcal{V}).$$

The Hilbert polynomial $P_{H,\mathcal{V}}$ and the reduced Hilbert polynomials $p_{H,\mathcal{V}}$ of a pure d-dimensional, coherent sheaf $\mathcal{V}$ with respect to H are defined by

$$P_{H,\mathcal{V}}(m) = \chi(\mathcal{V}(mH)) = a_d \frac{m^d}{d!} + \text{l.o.t}, \qquad p_{H,\mathcal{V}} = \frac{P_{H,\mathcal{V}}}{a_d}.$$

The sheaf $\mathcal{V}$ is *H-Gieseker (semi)stable* if for every proper subsheaf $\mathcal{W}$,

$$p_{H,\mathcal{W}}(m) \underset{(-)}{<} p_{H,\mathcal{V}}(m)$$

for $m \gg 0$.

Two semistable sheaves $\mathcal{V}$ and $\mathcal{W}$ are *S-equivalent* with respect to a notion of stability if they have the same Jordan-Hölder factors with respect to that notion of stability. Gieseker [Gie77] and Maruyama [Mar78] constructed moduli spaces $M_{X,H}(\mathbf{v})$ parameterizing S-equivalence classes of H-Gieseker semistable sheaves on X with Chern character $\mathbf{v}$.

Let $\mathcal{M}_{X,H}(\mathbf{v})$ denote the stack of H-Gieseker semistable sheaves. This stack has several open substacks that play important roles in calculations such as the stack of μ_H-stable sheaves $\mathcal{M}^{\mu s}_{X,H}(\mathbf{v})$ and the stack of μ_H-stable locally free sheaves $\mathcal{M}^{\mu s \circ}_{X,H}(\mathbf{v})$. The stack is an open substack of the stack of all μ_H-semistable sheaves $\mathcal{M}^{\mu ss}_{X,H}(\mathbf{v})$. We observe that these stacks are all defined as open substacks of the algebraic stack of coherent sheaves on X (see [St18, Tag 93.5] for a detailed discussion) satisfying further properties. Since a stable sheaf $\mathcal{V}$ has $\mathrm{Hom}(\mathcal{V}, \mathcal{V}) \cong \mathbb{C}$, these stacks are not Deligne-Mumford stacks, but only Artin stacks.

Prioritary sheaves. Let D be a divisor on X. A torsion-free coherent sheaf $\mathcal{V}$ is *D-prioritary* if $\mathrm{Ext}^2(\mathcal{V}, \mathcal{V}(-D)) = 0$. Prioritary sheaves are easier to work with than semistable sheaves. We denote the stack of D-prioritary sheaves on X with Chern character $\mathbf{v}$ by $\mathcal{P}_{X,D}(\mathbf{v})$. If H is an ample divisor such that $H \cdot (K_X + D) < 0$, then μ_H-semistable sheaves are D-prioritary because

$$\mathrm{Ext}^2(\mathcal{V}, \mathcal{V}(-D)) = \mathrm{Hom}(\mathcal{V}, \mathcal{V}(K_X + D))^* = 0$$

by μ_H-semistability. Hence, $\mathcal{M}_{X,H}(\mathbf{v})$ and $\mathcal{M}^{\mu ss}_{X,H}(\mathbf{v})$ are (possibly empty) open substacks of $\mathcal{P}_{X,D}(\mathbf{v})$.

If X is a birationally ruled surface and F is the fiber class, then a theorem of Walter [Wal98] asserts that $\mathcal{P}_{X,F}(\mathbf{v})$ is an irreducible stack whenever nonempty. Walter's theorem generalizes an earlier theorem of Hirschowitz and Laszlo [HL93] that asserts that the stack $\mathcal{P}_{\mathbb{P}^2,L}(\mathbf{v})$, where L is the class of a line on $\mathbb{P}^2$, is irreducible whenever nonempty. These results imply the corresponding irreducibility of the moduli spaces of sheaves and are very useful for cohomology computations because they allow us to show vanishing of the cohomology of a general sheaf by exhibiting a prioritary sheaf with vanishing cohomology.

Elementary modifications. Given a torsion free sheaf $\mathcal{V}$, a point $p \in X$ and a surjection $\phi : \mathcal{V} \twoheadrightarrow \mathcal{O}_p$, the kernel $\mathcal{V}'$ defined by the sequence

$$0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{O}_p \to 0$$

is called an *elementary modification* of $\mathcal{V}$. The reader can find the basic properties of elementary modifications in [CH18c, §2.3]. We have that

$$r(\mathcal{V}') = r(\mathcal{V}), \quad \nu(\mathcal{V}') = \nu(\mathcal{V}), \quad \Delta(\mathcal{V}') = \Delta(\mathcal{V}) + \frac{1}{r}.$$

If $\mathcal{V}$ is μ_H-(semi)stable, then $\mathcal{V}'$ is μ_H-(semi)stable since if $\mathcal{W}$ were a destabilizing subsheaf of $\mathcal{V}'$, it would also destabilize $\mathcal{V}$ under the natural inclusion. Since $H^i(X, \mathcal{O}_p) = 0$ for $i \geq 1$, the long exact sequence of cohomology implies that $H^2(X, \mathcal{V}') = H^2(X, \mathcal{V})$. Furthermore, as long as p and ϕ are general and $h^0(X, \mathcal{V}) > 0$, the map $H^0(X, \mathcal{V}) \to H^0(X, \mathcal{O}_p)$ is surjective. Hence, if $h^0(X, \mathcal{V}) > 0$, then the long exact sequence of cohomology implies

$$h^0(X, \mathcal{V}') = h^0(X, \mathcal{V}) - 1, \quad h^1(X, \mathcal{V}') = h^1(X, \mathcal{V}).$$

Similarly, if $h^0(X, \mathcal{V}) = 0$, then

$$h^0(X, \mathcal{V}') = 0, \quad h^1(X, \mathcal{V}') = h^1(X, \mathcal{V}) + 1.$$

Given a D-prioritary sheaf $\mathcal{V}$, a general elementary modification of $\mathcal{V}$ is also D-prioritary [CH18c, Lemma 2.7]. By the integrality of the Euler characteristic, the discriminants of different integral Chern classes differ by an integer multiple of $\frac{1}{r}$. Hence, once there is a μ_H-(semi)stable or D-prioritary sheaf with a given r, μ and Δ_0, elementary modifications construct a μ_H-(semi)stable or D-prioritary sheaves with r, μ and any possible $\Delta \geq \Delta_0$. Note, however, that elementary modifications do not need to preserve Gieseker (semi)stability. For example, $\mathcal{V} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ on $\mathbb{P}^2$ is Gieseker semistable, but there are no Gieseker semistable sheaves with the same Chern character as one elementary modification of $\mathcal{V}$ (see [CH15] or [DLP85]).

3. The cohomology of the general sheaf

In this section, we discuss the problem of determining the cohomology of a general sheaf in the moduli space following [CH18b, CH18c]. Given a general sheaf $\mathcal{V} \in M_{X,H}(\mathbf{v})$, we would like to have criteria under which the Euler characteristic $\chi(\mathcal{V})$ and $c_1(\mathcal{V})$ determine the cohomology groups of $\mathcal{V}$. Since the general member of $M_{X,H}(\mathbf{v})$ is not locally free when $\mathrm{rk}(\mathbf{v}) = 1$, the problem behaves differently depending on whether $\mathrm{rk}(\mathbf{v}) = 1$ or $\mathrm{rk}(\mathbf{v}) > 1$.

Rank one sheaves. Let $\mathcal{V}$ be a rank one sheaf on a smooth, complex projective surface X. Then $\mathcal{V} \cong I_Z \otimes L$, where I_Z is the ideal sheaf of a zero-dimensional scheme and L is a line bundle on X. If Z is a general set of points on X, then the standard exact sequence

$$0 \to L \otimes I_Z \to L \to \mathcal{O}_Z \to 0$$

implies that

$$h^2(X, L) = h^2(X, L \otimes I_Z).$$

Furthermore, the map

$$H^0(X, L) \to H^0(X, \mathcal{O}_Z)$$

has maximal rank by the assumption that Z is a general set of points. We conclude that $L \otimes I_Z$ has no global sections if and only if the length of Z satisfies $|Z| \geq h^0(X, L)$ and $\mathcal{V}$ has no higher cohomology if and only if L does not have any higher cohomology and $|Z| \leq h^0(X, L)$. Consequently, computing the cohomology of $\mathcal{V}$ reduces to computing the cohomology of the line bundle $\mathcal{V}^{**}$. Of course, computing the cohomology of a line bundle on a surface can be a challenging problem. For example, if X is a very general blowup of $\mathbb{P}^2$ along $k \geq 10$ points, then the effective cone of X is not known.

Higher rank sheaves. The ideal situation occurs for $\mathbb{P}^2$. Let L denote the class of a line on $\mathbb{P}^2$ and let $\mathcal{V} \in M_{\mathbb{P}^2,L}(\mathbf{v})$ be a general sheaf. Since $\mathrm{rk}(\mathbf{v}) \geq 2$, $\mathcal{V}$ is locally free. By replacing $\mathcal{V}$ with its Serre dual, we may assume that $\mu_L(\mathcal{V}) \geq -\frac{3}{2}$. If $k \leq \mu_L(\mathbf{v}) < k+1$, then there exists a direct sum of line bundles

$$\mathcal{W} = \mathcal{O}_{\mathbb{P}^2}(k)^m \oplus \mathcal{O}_{\mathbb{P}^2}(k+1)^{r-m}$$

that has the same rank and c_1 as $\mathcal{V}$. Furthermore, $\Delta(\mathcal{W}) \leq 0$ and since $k \geq -2$, $\mathcal{W}$ has no higher cohomology. Consequently, by a sequence of elementary modifications, we obtain an L-prioritary sheaf that has at most one nonzero cohomology group. Since the stack of L-prioritary sheaves on $\mathbb{P}^2$ is irreducible and contains the semistable sheaves, we conclude the following theorem of Göttsche and Hirschowitz.

Theorem 3.1. [GH94] *Let $r \geq 2$. Then the general sheaf in $M_{\mathbb{P}^2,L}(\mathbf{v})$ has at most one nonzero cohomology group.*

In particular, $\chi(\mathbf{v})$ and $\mu_L(\mathbf{v})$ determine the cohomology of the general sheaf in $M_{\mathbb{P}^2,L}(\mathbf{v})$. If $\chi(\mathbf{v}) < 0$, then $\mathcal{V}$ has $h^1(\mathbb{P}^2, \mathcal{V}) = -\chi(\mathbf{v})$ and all other cohomology groups vanish. If $\chi(\mathbf{v}) > 0$ and $\mu_L(\mathbf{v}) \geq -\frac{3}{2}$, then $h^0(\mathbb{P}^2, \mathcal{V}) = \chi(\mathbf{v})$ and all other cohomology groups vanish. Finally, if $\chi(\mathbf{v}) > 0$ and $\mu_L(\mathbf{v}) < -\frac{3}{2}$, then $h^2(\mathbb{P}^2, \mathcal{V}) = \chi(\mathbf{v})$ and all other cohomology groups vanish.

In general, the presence of negative self-intersection curves give an obstruction for the vanishing of cohomology. For example, on a Hirzebruch surface, it may happen that while $\chi(\mathcal{V}) \leq 0$, $\chi(\mathcal{V}(-E)) > 0$. Assuming that $H^2(\mathbb{F}_e, \mathcal{V}(-E)) = 0$, we conclude that $H^0(\mathbb{F}_e, \mathcal{V}(-E)) \neq 0$. Since this is a subset of $H^0(\mathbb{F}_2, \mathcal{V})$, this cohomology group cannot be zero. In [CH18c], we show that this is the only obstruction for the vanishing of cohomology.

Theorem 3.2 ([CH18c], Theorem 3.1). *Let $\mathbf{v}$ be an integral Chern character on $\mathbb{F}_e$ with positive rank r and $\Delta \geq 0$. Then the stack $\mathcal{P}_{\mathbb{F}_e,F}(\mathbf{v})$ of F-prioritary sheaves is nonempty and irreducible. Let $\mathcal{V} \in \mathcal{P}_{\mathbb{F}_e,F}(\mathbf{v})$ be a general sheaf.*

(1) *If $\nu(\mathbf{v}) \cdot F \geq -1$, then $h^2(\mathbb{F}_e, \mathcal{V}) = 0$. If $\nu(\mathbf{v}) \cdot F \leq -1$, then $h^0(\mathbb{F}_e, \mathcal{V}) = 0$. In particular, if $\nu(\mathbf{v}) \cdot F = -1$, then both h^0 and h^2 vanish and $h^1(\mathbb{F}_e, \mathcal{V}) = -\chi(\mathbf{v})$.*
(2) *If $\nu(\mathbf{v}) \cdot F > -1$ and $\nu(\mathbf{v}) \cdot E \geq -1$, then $\mathcal{V}$ has at most one nonzero cohomology group. Thus if $\chi(\mathbf{v}) \geq 0$, then $h^0(\mathbb{F}_e, \mathcal{V}) = \chi(\mathbf{v})$, and if $\chi(\mathbf{v}) \leq 0$, then $h^1(\mathbb{F}_e, \mathcal{V}) = -\chi(\mathbf{v})$.*
(3) *If $\nu(\mathbf{v}) \cdot F > -1$ and $\nu(\mathbf{v}) \cdot E < -1$, then $H^0(\mathbb{F}_e, \mathcal{V}) = H^0(\mathbb{F}_e, \mathcal{V}(-E))$, hence the Betti numbers of $\mathcal{V}$ are inductively determined using the previous two parts.*
(4) *If $\nu(\mathbf{v}) \cdot F < -1$ and $\mathrm{rk}(\mathbf{v}) \geq 2$, then Serre duality determines the Betti numbers of $\mathcal{V}$.*

Just like in the case of $\mathbb{P}^2$, Theorem 3.2 can be proved by exhibiting an elementary modification of a direct sum of line bundles with the desired cohomology. One can obtain any rank and total slope satisfying $\nu \cdot F \geq -1$ and $\nu \cdot E \geq -1$ by taking direct sums of line bundles of the form

$$\mathcal{V} = L(-E-(e+1)F)^a \oplus L(-F)^b \oplus L^c \quad \text{or} \quad \mathcal{V} = L(-E-(e+1)F)^a \oplus L(-E-eF)^b \oplus L,$$

for a nef line bundle L on $\mathbb{F}_e$. Taking elementary modifications yields the theorem in view of Walter's irreducibility theorem.

The same technique can be applied to more general rational surfaces X. As the Picard rank of X increases, our knowledge of the cohomology of line bundles on X becomes less complete. Correspondingly, our knowledge of the cohomology of higher rank bundles becomes less sharp. Nevertheless, as long as we impose some positivity conditions on $\nu(\mathbf{v})$, then we obtain a Brill-Noether type theorem. For example, let X be the blowup of $\mathbb{P}^2$ at k distinct points. Let $\mathbf{v}$ be a Chern character of rank r and let the total slope be $\nu(\mathbf{v}) = \delta L - \sum_{i=1}^k \alpha_i E_i$. Then we have that

$$\delta = d + \frac{q}{r}, \quad \alpha_i = a_i + \frac{q_i}{r}$$

for some integers d, q, a_i and q_i with $0 \le q < r$ and $0 \le q_i < r$. Set

$$\gamma(\mathbf{v}) = \frac{q^2}{2r^2} - \frac{q}{2r} + \sum_{i=1}^{k} \left(\frac{q_i}{2r} - \frac{q_i^2}{2r^2} \right).$$

Theorem 3.3 ([CH18b], Theorem 4.5)**.** *Let X be the blowup of $\mathbb{P}^2$ at k distinct points. Let $\mathbf{v}$ be a positive rank Chern character on X with total slope*

$$\nu(\mathbf{v}) = \delta L - \alpha_1 E_1 \cdots - \alpha_k E_k$$

with $\delta \ge 0$ and $\alpha_i \ge 0$. Suppose that the line bundle

$$\lfloor \delta \rfloor L - \lceil \alpha_1 \rceil E_1 \cdots - \lceil \alpha_k \rceil E_k$$

does not have higher cohomology. Assume that $\Delta(\mathbf{v}) \ge \gamma(\mathbf{v})$. Then $\mathcal{P}_{X,L-E_1}(\mathbf{v})$ is nonempty and the general sheaf in $\mathcal{P}_{X,L-E_1}(\mathbf{v})$ has at most one nonzero cohomology group.

A similar theorem holds for blowups of $\mathbb{F}_e$ (see [CH18b]). For del Pezzo surfaces, toric surfaces or surfaces where the blown-up points are in certain special positions, one can often be more explicit about the ranges where Brill-Noether type theorems hold.

Non-rational surfaces. For more general surfaces, there are other obstructions to vanishing of cohomology. For example, higher cohomology of line bundles can contribute to the cohomology of higher rank sheaves.

Example 3.4. [CNY18] Let X be a K3 surface of Picard rank 1 generated by H with $H^2 = 2n$. Then there is a stable bundle on X with resolution

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(H)^{n+2} \to \mathcal{V} \to 0.$$

One can think of $\mathcal{V}$ as the pullback of the tangent bundle $T_{\mathbb{P}^{n+1}}$ under the map induced by $\mathcal{O}_X(H)$. The bundle $\mathcal{V}$ is the unique point of its moduli space and has

$$h^0(X, \mathcal{V}) = (n+2)^2 - 1, \quad h^1(X, \mathcal{V}) = 1.$$

By the long exact sequence of cohomology, $H^2(X, \mathcal{O}_X) \cong \mathbb{C} \cong H^1(X, \mathcal{V})$.

The set of Chern characters where the cohomology is not as expected can exhibit fairly complicated behavior depending on arithmetic conditions.

Example 3.5. [CNY18] Let X be a K3 surface of Picard rank 1 obtained as a double cover of $\mathbb{P}^2$. Then the pullback of exceptional bundles on $\mathbb{P}^2$ are stable bundles on the K3 surface. Let f_k be the k-th Fibonacci number where $f_1 = f_2 = 1$. For $k \ge 2$, there is a bundle on X with resolution

$$0 \to \mathcal{O}_X^{f_{2k-2}} \to \mathcal{O}_X^{f_{2k}}(H) \to \mathcal{V}_k \to 0.$$

The bundle $\mathcal{V}_k$ has rank f_{2k-1} and is the unique point in its moduli space. Furthermore, $h^0(X, \mathcal{V}_k) = 2f_{2k} + f_{2k-1}$ and $h^1(X, \mathcal{V}_k) = f_{2k-2}$. When $k = 2$, this example specializes to the previous example when $n = 1$.

A complete computation of the cohomology of the general sheaf on an arbitrary surface is unlikely to be ever attained. Nevertheless, an asymptotic theorem holds on every smooth, complex projective surface. A consequence of O'Grady's irreducibility theorem [O'G96] and Serre vanishing is that if $r \ge 2$ and $\Delta \gg 0$, then the general sheaf in $M_{X,H}(r, c, \Delta)$ has only h^1.

Proposition 3.6. [CH18d, Proposition 7]. *Let $\mathbf{v}$ be a Chern character such that $\Delta(\mathbf{v}) \gg 0$ (depending on $X, H, r,$ and ν) and let $\mathcal{V} \in M_{X,H}(\mathbf{v})$ be a general sheaf. Then the only nonvanishing cohomology group of $\mathcal{V}$ is h^1.*

Similarly, if $\nu(\mathbf{v})$ is sufficiently ample, then the general slope stable sheaf has only one nonvanishing cohomology group. To make this more precise, pick a $\mathbb{Z}$-basis $C_1, \dots, C_k$ for $\mathrm{Pic}(X)$. Given any Chern character $\mathbf{v}$, we can twist the character by a line bundle L so that

$$\nu(\mathbf{v}(L)) = \sum_{i=1}^{k} a_i C_i, \quad \text{with } 0 \le a_i < 1.$$

We call the resulting Chern character $\mathbf{v}_0$ the *normalized Chern character* with respect to the basis $C_1, \dots, C_k$ and its slope $\nu(\mathbf{v}_0)$ the *normalized slope.*

Theorem 3.7. *Let X be a smooth, projective surface and let R be a positive integer. Let $\mathbf{v}_0$ be a normalized Chern character of positive rank $r \le R$ on X. Given any ample divisor A on X, there exists an integer m_0 (depending only on X, H, A, R and the basis of* $\mathrm{Pic}(X)$*) such that if $m \ge m_0$ and $\mathbf{v} = \mathbf{v}_0(mA)$, then the general μ_H-stable sheaf $\mathcal{V}$ has at most one nonzero cohomology group.*

Proof. Let S be the set of total slopes ν_0 of normalized Chern characters $\mathbf{v}_0$ of rank at most R. Then S is finite. Hence, if we can find an integer m_0 that works for each such slope ν_0, by taking the maximum over S, we get an m_0 that works for all normalized Chern characters of rank at most R. Similarly, there are only finitely many ranks $0 < r \le R$, hence, we can also fix the rank $r \le R$. By O'Grady's Theorem [O'G96], there exists Δ_0 (depending on X, H and R) such that if $\Delta \ge \Delta_0$, all the moduli spaces $M_{X,H}(r, \nu, \Delta)$ are irreducible and contain slope stable elements. In particular, the open subset $M^{\mu s}_{X,H}(r, \nu, \Delta)$ is irreducible. Tensoring by a line bundle preserves μ_H-stability (though it does not in general preserve Gieseker (semi)stability). Since tensoring by a line bundle does not change the discriminant, the same discriminant bounds guarantee that the moduli spaces $M^{\mu s}_{X,H}(r, \nu + mA, \Delta)$ are irreducible.

By Bogomolov's inequality $\Delta \ge 0$. Since the rank is bounded by R, there are only finitely many (nonempty) moduli spaces $M^{\mu s}_{X,H}(r, \nu, \Delta)$ with $r \le R$, $\Delta < \Delta_0$ and normalized slope. Furthermore, each of these moduli spaces have only finitely many irreducible components. Choose a general $\mathcal{V}$ in such a component. By Serre vanishing, given any ample A, there exists an integer m_0 such that for all $m \ge m_0$, $\mathcal{V}(mA)$ has no higher cohomology. Since there are only finitely many moduli spaces of rank at most R, $\Delta \le \Delta_0$ and normalized slope and these moduli spaces each have finitely many components, we can find one m_0 that works for all these moduli spaces simultaneously. If $\Delta > \Delta_0$, by taking general elementary modifications of a slope-stable sheaf with no higher cohomology and discriminant Δ_0, we obtain a slope-stable sheaf of discriminant Δ and at most one nonzero cohomology group. Since these moduli spaces are irreducible, the vanishing of cohomology holds for the general sheaf in these moduli spaces by semicontinuity. This concludes the proof of the theorem. $\square$

For specific classes of surfaces, it is interesting to give effective bounds on Δ and ν that guarantee the vanishing of cohomology. For example, on K3 surfaces of Picard rank one [CNY18] proves the following theorem.

Theorem 3.8. [CNY18] *Let X_n be a K3 surface of Picard rank 1 generated by H with $H^2 = 2n$. Let $\mathbf{v} = (r, \nu, \Delta)$ be a Chern character such that $r \ge 2$ and $\nu = aH$ with $a > 0$. Let $\mathcal{V}$ be a general sheaf in $M_{X_n,H}(\mathbf{v})$.*

(1) *If $a \ge r + 1$, then $\mathcal{V}$ has at most one nonzero cohomology group.*
(2) *If $\chi(\mathbf{v}) \le r$, then $\mathcal{V}$ has at most one nonzero cohomology group.*
(3) *Given a positive integer r_0, there are only finitely many moduli spaces $M_{X_n,H}(\mathbf{v})$ with the rank r of $\mathbf{v}$ satisfying $2 \le r \le r_0$ where the general sheaf has more than one nonzero cohomology group.*

In fact, one can enumerate the potential counterexamples to the Brill-Noether theorem explicitly and often compute the cohomology of the general sheaf in these examples (see [CNY18]). The techniques for proving this theorem rely on Bridgeland stability conditions and would take us too far a field to survey in this paper.

3.1. **Applications.** There are many applications of Brill-Noether theorems ranging from the classification of stable Chern characters to the construction of theta-divisors. We will mention a few of these applications here and refer the reader to the original papers for more applications. We will concentrate on Hirzebruch surfaces to illustrate the methods. In the next section, we will highlight applications to the construction and classification of Ulrich bundles.

First, Brill-Noether Theorems are extremely useful in finding resolutions of a general sheaf in the moduli space. For example, one has a Gaeta-type resolution for the general sheaf in $M_{\mathbb{F}_e,H}(\mathbf{v})$.

Theorem 3.9 ([CH18c], Theorem 4.1)**.** *Let* $\mathbf{v}$ *be an integral Chern character on* $\mathbb{F}_e$ *of positive rank and assume that*

$$\Delta(\mathbf{v}) \geq \frac{1}{4} \text{ if } e=0, \quad \Delta(\mathbf{v}) \geq \frac{1}{8} \text{ if } e=1, \quad \Delta(\mathbf{v}) \geq 0 \text{ if } e \geq 2.$$

Then the general sheaf $\mathcal{V} \in \mathcal{P}_{\mathbb{F}_e,F}(\mathbf{v})$ *admits a Gaeta-type resolution*

$$0 \to L(-E-(e+1)F)^a \to L(-E-eF)^b \oplus L(-F)^c \oplus L^d \to \mathcal{V} \to 0, \tag{1}$$

for some line bundle L *and nonnegative integers* a, b, c, d*.*

The Gaeta-type resolution immediately shows that the moduli space is unirational. Furthermore, having an explicit resolution of the general sheaf allows one to study properties of these sheaves. For example, one can classify moduli spaces where the general sheaf is globally generated. We warn the reader that being globally generated is not an open condition. However, if we assume the vanishing of H^1, then it becomes an open condition.

Theorem 3.10. [CH18c, Theorem 5.1] *Let* $\mathbf{v}$ *be a Chern character on* $\mathbb{F}_e$*,* $e \geq 1$ *such that* $\mathrm{rk}(\mathbf{v}) \geq 2$*,* $\Delta(\mathbf{v}) \geq 0$ *and* $\nu(\mathbf{v})$ *is nef. Then the general sheaf in* $\mathcal{P}_{\mathbb{F}_e,F}(\mathbf{v})$ *is globally generated if and only if one of the following holds:*

(1) *We have* $\nu(\mathbf{v}) \cdot F = 0$ *and* $\mathbf{v} = \mathrm{ch}(\pi^*(\mathcal{O}_{\mathbb{P}^1}(a)^m \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)^{r-m}))$ *for some* $a \geq 0$*.*
(2) *We have* $\nu(\mathbf{v}) \cdot F > 0$ *and* $\chi(\mathbf{v}(-F)) \geq 0$*.*
(3) *We have* $\nu(\mathbf{v}) \cdot F > 0$*,* $\chi(\mathbf{v}(-F)) < 0$ *and* $\chi(\mathbf{v}) \geq r+2$*.*
(4) *We have* $e=1$*,* $\nu(\mathbf{v}) \cdot F > 0$*,* $\chi(\mathbf{v}(-F)) < 0$*,* $\chi(\mathbf{v}) \geq r+1$ *and* $\mathbf{v} = (\mathrm{rk}(\mathbf{v})+1)\,\mathrm{ch}(\mathcal{O}_{\mathbb{F}_1}) - \mathrm{ch}(\mathcal{O}_{\mathbb{F}_1}(-2E-2F))$*.*

Similar theorems hold for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ (see [CH18c, Theorem 5.2, Corollary 5.3]). On $\mathbb{P}^2$, the general bundle in $M_{\mathbb{P}^2,L}(\mathbf{v})$ is globally generated as soon as $\nu(\mathbf{v})$ is effective and $\chi(\mathbf{v}) \geq r+2$. There are a few easily classifiable cases with $\chi(\mathbf{v}) = r+1$ or r where the general sheaf is globally generated. On $\mathbb{P}^1 \times \mathbb{P}^1$, there are two fiber classes, so the statement has to be slightly altered accordingly.

These theorems give a sufficient condition for ampleness of vector bundles. If $\mathcal{V}$ is globally generated, then $\mathcal{V}(H)$ is ample for any ample line bundle. However, this is far from necessary. There are ample bundles that have no sections. The following problem remains very much open.

Problem 3.11. *Given a surface* X*, classify the Chern characters of ample bundles on* X*.*

We do not know a solution of Problem 3.11 even for $\mathbb{P}^2$. For example, Gieseker [Gie71] shows that a general bundle $\mathcal{V}$ of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 4} \to \mathcal{V} \to 0$$

is ample provided $d \gg 0$. It is easy to see that $d \geq 7$, but we do not know an exact bound that guarantees ampleness.

The Brill-Noether theorems can be used to obtain sharp Bogomolov inequalities. For example, in [CH18e], we classify the stable Chern characters on $\mathbb{F}_e$ for any polarization. The Brill-Noether Theorem and the Gaeta-type resolution are the fundamental tools. Let $H_m = E + (m+e)F$. When $m > 0$, then H_m is ample and every ample integral divisor is an integer multiple of H_m for some positive rational number m. One first shows that an H_m-semistable sheaf is $H_{\lceil m \rceil + 1}$-prioritary. This breaks the problem into understanding the H_k-prioritary sheaves for positive integers k, and then computing the Harder-Narasimhan filtration of the general H_k-prioritary sheaf. The Gaeta-type resolution provides an answer to the first question.

Let $\nu(\mathbf{v}) = \epsilon E + \varphi F$. Then define

$$\psi := \varphi + \frac{1}{2} e(\lceil \epsilon \rceil - \epsilon) - \frac{\Delta}{1 - (\lceil \epsilon \rceil - \epsilon)},$$

and let

$$L_0 := L_{\lceil \epsilon \rceil, \lceil \psi \rceil} = \lceil \epsilon \rceil E + \lceil \psi \rceil F.$$

Then the general sheaf in $\mathcal{P}_{\mathbb{F}_e, F}(\mathbf{v})$ admits a Gaeta-type resolution with $L = L_0$. Expressing the condition that the sheaf be H_k prioritary, one sees that we need $\chi(\mathbf{v}(-L_0 - H_k)) \leq 0$. Conversely, one can construct H_k-prioritary sheaves satisfying this inequality to conclude the following theorem.

Theorem 3.12. [CH18e, Theorem 4.16] *Let $\mathbf{v}$ be an integral Chern character of positive rank on $\mathbb{F}_e$ with $\Delta(\mathbf{v}) \geq 0$ and let k be a positive integer. Then the stack $\mathcal{P}_{H_k}(\mathbf{v})$ is nonempty if and only if*

$$\chi(\mathbf{v}(-L_0 - H_k)) \leq 0.$$

Note that this theorem can be rephrased as an inequality that Δ needs to satisfy in order to have H_k-prioritary sheaves. It already significantly strengthens the Bogomolov inequality for Hirzebruch surfaces.

The second step in classifying the stable Chern characters on $\mathbb{F}_e$ is to determine the H_m-Harder-Narasimhan filtration of the general $H_{\lceil m \rceil + 1}$-prioritary sheaf. There exist H_m-semistable sheaves when the length of the filtration is one. Determining the general Harder-Narasimhan filtration reduces to a finite computational problem that can be solved recursively on the rank. We now briefly explain the process.

Suppose the general H_m-Harder-Narasimhan filtration has length ℓ and the graded pieces have Chern characters $\mathbf{v}_i = (r_i, \nu_i, \Delta_i)$. Then we must have that

$$\sum_{i=1}^{\ell} \mathbf{v}_i = \mathbf{v}.$$

Furthermore, the moduli spaces $M_{\mathbb{F}_e, H_m}(\mathbf{v}_i)$ are nonempty since the graded pieces are semistable sheaves. The fact that the sheaves are $H_{\lceil m \rceil + 1}$-prioritary and the Schatz-stratum corresponding to this Harder-Narasimhan filtration has codimension 0 lead to additional inequalities that place strong restrictions on $\mathbf{v}_i$.

First, the prioritary condition implies that the restriction of the general sheaf to a general rational curve in the class $H_{\lceil m \rceil}$ or $H_{\lfloor m \rfloor}$ has balanced splitting. This translates to the inequality that

$$|(\nu_i - \nu) \cdot H_m| < 1.$$

Next, a dimension computation shows that for the Schatz-stratum to have codimension 0, we must have the orthogonality relations

$$\chi(\mathbf{v}_i, \mathbf{v}_j) = 0 \quad \text{for} \quad i < j.$$

From these two facts we can see that the slopes ν_i have to come from a bounded region, and since the ranks r_i are bounded there are only finitely many possibilities for the ν_i. Furthermore, the orthogonality relations imply the discriminant Δ_i has to be the minimal possible discriminant of an H_m-semistable sheaf with rank r_i and total slope ν_i. Hence, there are finitely many possible $\mathbf{v}_i$ that can be the Chern characters of the graded pieces of the generic H_m-Harder-Narasimhan filtration.

Conversely, if one can find Chern characters $\mathbf{v}_i$, $1 \le i \le \ell$, that satisfy these constraints, then the general Harder-Narasimhan filtration has factors with these Chern characters. More precisely, we have the following theorem.

Theorem 3.13. [CH18e, Theorem 5.3] *Let $\mathbf{v}$ be the Chern character such that $\mathcal{P}_{H_{\lceil m \rceil}}(\mathbf{v})$ is nonempty. Let $\mathbf{v}_1, \dots, \mathbf{v}_\ell \in K(\mathbb{F}_e)$ be positive rank Chern characters satisfying the following properties:*

(1) $\sum_{i=1}^{\ell} \mathbf{v}_i = \mathbf{v}$,
(2) *The reduced Hilbert polynomials q_i of $\mathbf{v}_i$ are strictly decreasing $q_1 > \cdots > q_\ell$,*
(3) $\mu_{H_m}(\mathbf{v}_1) - \mu_{H_m}(\mathbf{v}_\ell) \le 1$,
(4) $\chi(\mathbf{v}_i, \mathbf{v}_j) = 0$ *for $i < j$,*
(5) $M_{\mathbb{F}_e, H_m}(\mathbf{v}_i)$ *is nonempty,*

Then the Harder-Narasimhan filtration of the general sheaf in $\mathcal{P}_{H_{\lceil m \rceil}}(\mathbf{v})$ has length ℓ and the factors have Chern characters $\mathbf{v}_i$.

Thus determining the Harder-Narasimhan filtration of a general $H_{\lceil m \rceil}$-prioritary sheaf becomes a finite computational problem. In particular, one obtains an algorithm for classifying Chern characters of H_m-semistable sheaves. The problem can be further simplified. For instance, using the fact that $K(\mathbb{F}_e) \cong \mathbb{Z}^4$, one can show $\ell \le 4$.

Remark 3.14. Recall that an *exceptional bundle* $\mathcal{V}$ is a simple bundle with $\mathrm{Ext}^i(\mathcal{V}, \mathcal{V}) = 0$ for $i > 0$. In the case of $\mathbb{P}^2$, exceptional bundles control the semistability of sheaves [DLP85]. More precisely, either the subbundle or the quotient in the general Harder-Narasimhan filtration is an exceptional bundle. Hence, one can give explicit inequalities for the discriminants of semistable bundles without computing the semistable bundles of lower rank, recursively. One just needs to consider exceptional bundles. Unfortunately, it may happen that neither the quotient nor the subbundle in a generic Harder-Narasimhan filtration of length 2 on $\mathbb{F}_e$ is an exceptional bundle (see [CH18e] for explicit examples). In general, it is not enough to consider the constraints given by exceptional bundles.

As a consequence of [CH18a], the classification of stable Chern characters has applications to birational geometry. For example, one can compute the extremal rays of the ample cone of $M_{\mathbb{F}_e, H_m}(\mathbf{v})$ if $\Delta \gg 0$. We will not discuss applications to birational geometry in this exposition, but refer the reader to [CH15].

4. Ulrich bundles on surfaces

Brill-Noether Theorems have immediate applications to the construction and classification of Ulrich bundles. In this section, we spell this out more explicitly.

Definition 4.1. Let $X \subset \mathbb{P}^n$ be a smooth, projective variety of dimension d. An *Ulrich bundle* V on X is a bundle that satisfies $H^i(X, E(-j)) = 0$ for $1 \le j \le d$ and all i.

Ulrich bundles play a central role in the study of Chow forms of a variety [ESW03], the minimal resolution conjecture (see [AGO17]) and Boij-Söderberg Theory (see [ES11]). For example, Eisenbud and Schreyer show that the cone of cohomology tables of X is the same as that of $\mathbb{P}^d$ if and only if X admits an Ulrich bundle [ES11]. Eisenbud and Schreyer raise the question whether every projective variety admits an Ulrich bundle. Existence is known in some cases including smooth curves [ESW03], complete intersections [BaHU91], Grassmannians [CMR15] and some two-step flag varieties [CCHMW17, CJ17], del Pezzo surfaces [CKM13], certain rational surfaces [ESW03, Kim16], K3 surfaces [AGO17], abelian surfaces [Bea16] and certain Enriques surfaces [BN18].

In this section, we would like to advocate a moduli theoretic approach for the construction and classification of Ulrich bundles. We say that a moduli space $M_{X,H}(\mathbf{v})$ satisfies *the BN property* if the general sheaf in every component of $M_{X,H}(\mathbf{v})$ has at most one nonzero cohomology group. The relation between Brill-Noether type theorems and Ulrich bundles is provided by the following proposition.

Proposition 4.2. *Let $\mathcal{V}$ be an Ulrich bundle for a polarized surface (X, H) of rank r, total slope $\nu + H$ and discriminant Δ. Then*

$$2\nu \cdot H = H^2 + H \cdot K_X \quad \text{and} \quad 2\Delta = \nu^2 - \nu \cdot K_X + 2\chi(\mathcal{O}_X).$$

Conversely, if r, ν, Δ satisfy these equalities, $M_{X,H}(r, \nu, \Delta)$ contains locally free sheaves and the moduli spaces $M_{X,H}(r, \nu, \Delta)$ and $M_{X,H}(r, \nu - H, \Delta)$ satisfy the BN property, then the general locally free sheaf in $M_{X,H}(r, \nu + H, \Delta)$ is an Ulrich bundle.

Furthermore, if $2\nu = H + K_X$, $M_{X,H}(r, \nu, \Delta)$ contains locally free sheaves and satisfies the BN property, then (X, H) admits Ulrich bundles of every rank divisible by r.

Proof. Assume that $\mathcal{V}$ is an Ulrich bundle on (X, H) of rank r and total slope $\nu + H$. Then by Riemann-Roch, we can compute the Euler characteristic of $\mathcal{V}(-H)$ and $\mathcal{V}(-2H)$ to obtain

$$\frac{1}{2}\nu^2 - \frac{1}{2}\nu \cdot K_X + \chi(\mathcal{O}_X) - \Delta = 0$$
$$\frac{1}{2}(\nu - H)^2 - \frac{1}{2}(\nu - H) \cdot K_X + \chi(\mathcal{O}_X) - \Delta = 0$$

The first equation gives the desired relation for Δ. Substituting the first equation into the second, we get that

$$2\nu \cdot H = H^2 + H \cdot K_X.$$

Assume $M_{X,H}(r, \nu, \Delta)$ contains a locally free sheaf and let $\mathcal{V}$ be a general such sheaf. Then by twisting $\mathcal{V}$ by $\mathcal{O}_X(H)$ and $\mathcal{O}_X(-H)$, we also obtain locally free sheaves in the moduli spaces $M_{X,H}(r, \nu + H, \Delta)$ and $M_{X,H}(r, \nu - H, \Delta)$, respectively. By the numerical assumptions on ν and Δ, we have that

$$\chi(\mathcal{V}) = \chi(\mathcal{V}(-H)) = 0.$$

Hence, if the BN property holds for the general sheaf in every component of $M_{X,H}(r, \nu, \Delta)$ and $M_{X,H}(r, \nu - H, \Delta)$, then $\mathcal{V}$ and $\mathcal{V}(-H)$ have no cohomology. Consequently, $\mathcal{V}(H)$ is an Ulrich bundle on X.

Observe that $2\nu = H + K_X$ always satisfies the equality $2\nu \cdot H = H^2 + H \cdot K_X$. In this case, the character $(r, \nu - H, \Delta)$ is Serre dual to (r, ν, Δ). Consequently, the vanishing of the cohomology for the general locally free sheaf in $M_{X,H}(r, \nu, \Delta)$ implies the same vanishing for $M_{X,H}(r, \nu - H, \Delta)$ by Serre duality. Hence, if $M_{X,H}(r, \nu, \Delta)$ contains locally free sheaves and satisfies the BN property, then for a general locally free sheaf $\mathcal{V} \in M_{X,H}(r, \nu, \Delta)$, $\mathcal{V}(H)$ is an Ulrich bundle. This concludes the proof of the proposition. $\square$

Asymptotic results. Proposition 4.2 combined with O'Grady's Theorem yields an asymptotic existence theorem on any smooth, complex, projective surface.

Theorem 4.3. *Let X be a complex, projective surface and let H_0 be any ample line bundle on X. Then there exists a positive integer m_0 such that for all $m \geq m_0$, (X, mH_0) admits an Ulrich bundle of every positive even rank.*

Furthermore, if K_X (resp., $K_X + H_0$) is divisible by 2 in the Picard group and $2m \geq m_0$ (resp., $2m + 1 \geq m_0$), then $(X, 2mH_0)$ (resp., $(X, (2m+1)H_0)$) admits an Ulrich bundle of every rank $r \geq 2$.

Proof. Given $r \geq 2$ and a total slope ν, by O'Grady's theorem [O'G96], there exists a constant Δ_0 such that for $\Delta \geq \Delta_0$ the moduli space $M_{X,H_0}(r, \nu, \Delta)$ is irreducible, contains locally-free, μ_{H_0}-stable sheaves and has the expected dimension. Let $\mathcal{V} \in M_{X,H_0}(r, \nu, \Delta)$ be a general sheaf. By Serre's theorem and semicontinuity of cohomology, there exists an integer $m_1(r, \nu)$ such that for $m \geq m_1(r, \nu)$,

$$h^i(X, \mathcal{V}(mH_0)) = 0 \quad \text{for} \quad i = 1, 2.$$

By applying a sequence of $h^0(X, \mathcal{V}(mH_0))$ general elementary modifications to $\mathcal{V}(mH_0)$, we obtain a new μ_{H_0}-stable sheaf $\mathcal{V}'$ that satisfies $h^i(X, \mathcal{V}') = 0$ for all i. Elementary modifications preserve μ_H-stability and only increase the discriminant. Hence, by O'Grady's theorem, the moduli space containing $\mathcal{V}'$ is irreducible and the general sheaf in the moduli space is locally free. By semicontinuity of cohomology, a general deformation $\mathcal{W}$ of $\mathcal{V}'$ is locally free and has no cohomology. Observe that

$$r(\mathcal{W}) = r, \quad \nu(\mathcal{W}) = \nu + mH_0.$$

By Serre duality, the Serre dual $\mathcal{W}^*(K_X)$ of $\mathcal{W}$ also does not have any cohomology and has

$$r(\mathcal{W}^*(K_X)) = r, \quad \nu(\mathcal{W}^*(K)) = -\nu - mH_0 + K_X.$$

First, set $\nu = \frac{K_X}{2}$, which we can do if r is even or if K_X is divisible by 2 in the Picard group. We claim that $\mathcal{W}(2mH_0)$ is an Ulrich bundle of rank r on $(X, 2mH_0)$. Observe that $\mathcal{W}(-2mH_0)$ is μ_{H_0}-stable bundle with the same rank, total slope and discriminant as the Serre dual of $\mathcal{W}$. Hence, if $\mathcal{W}$ is general, $\mathcal{W}(-2m_0H)$ has no cohomology by semicontinuity. This shows that $\mathcal{W}(2mH_0)$ is an Ulrich bundle of rank r on $(X, 2mH_0)$.

Similarly, set $\nu = \frac{K_X+H_0}{2}$, which we can do if the rank is even or if $K_X + H_0$ is divisible by 2 in the Picard group. We claim that $\mathcal{W}((2m+1)H_0)$ is an Ulrich bundle of rank r on $(X, (2m+1)H_0)$. Observe that $\mathcal{W}(-(2m+1)H_0)$ is μ_{H_0}-stable bundle with the same rank, total slope and discriminant as the Serre dual of $\mathcal{W}$. The same argument as in the previous case applies.

If we let

$$m_0 = \max\left(2m_1\left(2, \frac{K_X}{2}\right), 2m_1\left(2, \frac{K_X+H_0}{2}\right) + 1\right),$$

(where m_1 is the bound from the first paragraph required to apply Serre's theorem) we have constructed a rank 2 Ulrich bundle for (X, mH_0) if $m \geq m_0$. If we have an Ulrich bundle $\mathcal{V}$ of rank 2, then $\mathcal{V}^{\oplus j}$ is an Ulrich bundle of rank $2j$.

Similarly, if K_X (respectively, $K_X + H_0$) is divisible by 2, we can construct both rank 2 and rank 3 Ulrich bundles on $(X, 2mH_0)$ (respectively, $(X, (2m+1)H_0)$) by this construction for

$$m \geq \max\left(m_1\left(2, \frac{K_X}{2}\right), m_1\left(3, \frac{K_X}{2}\right)\right), \quad \text{resp., } m \geq \max\left(m_1\left(2, \frac{K_X+H_0}{2}\right), m_1\left(3, \frac{K_X+H_0}{2}\right)\right).$$

Once we have Ulrich bundles $\mathcal{V}$ and $\mathcal{W}$ of ranks 2 and 3, by taking appropriate direct sums, we obtain Ulrich bundles of every rank at least 2. This concludes the proof of the theorem. ☐

Definition 4.4. A polarized variety (X, H) is called of *Ulrich wild representation type* if one can find arbitrarily large dimensional families of indecomposable Ulrich bundles on X for H.

Corollary 4.5. *Let X be a complex, projective surface and let H_0 be any ample line bundle on X. Then there exists a positive integer m_0 such that for $m \geq m_0$ (X, mH_0) is of wild-representation type.*

Proof. The proof of Theorem 4.3 constructs a moduli space $M_{X,H_0}(\mathbf{v})$ of μ_{H_0}-stable, rank 2 Ulrich bundles for (X, mH_0). We can make $\Delta(\mathbf{v})$ arbitrarily large by making m large, but for our purposes it suffices to choose m so that the dimension of $M_{X,H_0}(\mathbf{v})$ is at least 1 and $\chi(\mathbf{v}, \mathbf{v}) < 0$. We can now construct indecomposable bundles by taking extensions of these rank 2 bundles. To aid the proof it will be convenient to assume that the rank 2 stable factors are all distinct. Suppose that we have constructed a family $\mathcal{F}$ of indecomposable μ_{H_0}-semistable Ulrich bundles of rank $2j$ of dimension at least $jd(\Delta)$ whose Jordan-Hölder factors belong to the moduli space $M_{X,H_0}(\mathbf{v})$ and are all distinct. Let $\mathcal{V} \in \mathcal{F}$ and $\mathcal{W} \in M_{X,H_0}(\mathbf{v})$ and distinct from the Jordan-Hölder factors of $\mathcal{V}$. Consider extensions of the form

$$0 \to \mathcal{W} \to \mathcal{U} \to \mathcal{V} \to 0.$$

Since $\chi(\mathcal{V}, \mathcal{W}) < 0$, there exist nontrivial such extensions.

Let $\mathcal{U}$ be any such nontrivial extension. We claim $\mathcal{U}$ is indecomposable. If $\mathcal{U} = \mathcal{V}_1 \oplus \mathcal{V}_2$, then $\mathcal{V}_1$ and $\mathcal{V}_2$ are semistable with the same Jordan-Hölder factors as $\mathcal{U}$. In particular, $\mathcal{W}$ appears as a factor in exactly one of $\mathcal{V}_1$ or $\mathcal{V}_2$. Without loss of generality, assume $\mathcal{W}$ is a factor of $\mathcal{V}_1$. Then it follows that $\mathcal{V} \cong (\mathcal{V}_1 / \mathcal{W}) \oplus \mathcal{V}_2$. Since $\mathcal{V}$ is indecomposable, $\mathcal{W} \cong \mathcal{V}_1$ contradicting the nontriviality of the extension.

The Jordan-Hölder factors of $\mathcal{U}$ all belong to $M_{X,H_0}(\mathbf{v})$ and two such bundles cannot be isomorphic unless they have the same factors. In particular, the dimension of the family of such nonisomorphic bundles is at least $(j+1)d(\Delta)$. Furthermore, $\mathcal{U}$ is Ulrich for mH_0. We conclude that (X, mH_0) is of Ulrich wild representation type. $\square$

If we have more precise Brill-Noether theorems, we can obtain more precise classifications of Ulrich bundles. We illustrate the principle in a few examples.

Hirzebruch surfaces. Since we have a complete classification of the cohomology of the general sheaf on a Hirzebruch surface, we also get a complete classification of the invariants of Ulrich bundles. A similar classification has been worked out by V. Antonelli [Ant18].

Theorem 4.6. *Let $H = aE + bF$ be an ample divisor on $\mathbb{F}_e$. Let $\mathbf{v} = (r, \nu, \Delta) = (r, \alpha E + \beta F, \Delta)$ be an integral Chern character with $r \geq 2$. There exists a locally free F-prioritary sheaf $\mathcal{V}$ with Chern character $\mathbf{v}$ satisfying*

$$H^i(\mathbb{F}_e, \mathcal{V}(-H)) = H^i(\mathbb{F}_e, \mathcal{V}(-2H)) = 0 \ \forall i$$

if and only if

$$a - 1 + \frac{ea(a-1)}{2b} \leq \alpha \leq 2a - 1 - \frac{ea(a-1)}{2b},$$
$$\beta = \left(e - \frac{b}{a}\right)(\alpha + 1) + 3b - 1 - \frac{e}{2}(3a+1)$$

and

$$\Delta(\mathcal{V}) = \left(\frac{e}{2} - \frac{b}{a}\right)(\alpha^2 + (2-3a)\alpha + 2a^2 - 3a + 1).$$

Proof. Let $\mathcal{V}$ be an F-prioritary sheaf with $\nu(\mathcal{V}) = \alpha E + \beta F$ such that

$$H^i(\mathbb{F}_e, \mathcal{V}(-H)) = H^i(\mathbb{F}_e, \mathcal{V}(-2H)) = 0$$

for all i. As in Proposition 4.2 by Riemann-Roch, we have the relations

$$(\alpha - a + 1)(\beta - b + 1) - \frac{e}{2}(\alpha - a)(\alpha - a + 1) = (\alpha - 2a + 1)(\beta - 2b + 1) - \frac{e}{2}(\alpha - 2a)(\alpha - 2a + 1).$$

Solving for β in terms of α, we obtain

$$\beta = \left(e - \frac{b}{a}\right)\alpha + 3b - 1 - \frac{b}{a} - \frac{e}{2}(3a - 1).$$

Similarly, we have

$$\Delta(\mathcal{V}) = \left(\frac{e}{2} - \frac{b}{a}\right)(\alpha^2 + (2-3a)\alpha + 2a^2 - 3a + 1).$$

Since H is ample, we have that $a, b > 0$ and $b > ae$. In particular, $\frac{ea-2b}{2a} < 0$. Since an Ulrich bundle is slope semistable [CKM13, Proposition 2.6], by the Bogomolov inequality we must have $\Delta \geq 0$. The roots of the quadratic

$$\alpha^2 + (2-3a)\alpha + 2a^2 - 3a + 1$$

are $a-1$ and $2a-1$. Consequently

$$a - 1 \leq \alpha \leq 2a - 1$$

to guarantee that $\Delta(\mathcal{V}) \geq 0$. In particular, $\alpha - a \geq -1$ and $\alpha - 2a \leq -1$. If the rank of $\mathcal{V}$ is at least 2, we can use Serre duality to compute the cohomology of $\mathcal{V}(-2H)$. The Serre dual Chern character has total slope

$$(2a - \alpha - 2)E + (2b - \beta - e - 2)F.$$

By Theorem 3.2, the general F-prioritary sheaf $\mathcal{W}$ with

$$\Delta(\mathcal{W}) \geq 0, \quad \nu(\mathcal{W}) \cdot F \geq -1, \quad \chi(\mathcal{V}) = 0$$

has vanishing cohomology if and only if $\nu(\mathcal{W})\cdot F = -1$ or $\nu(\mathcal{W})\cdot E \geq -1$. If $\alpha - a \neq -1$ and $\alpha - 2a \neq -1$, we conclude that there exists an F-prioritary sheaf with vanishing cohomology if and only if

$$E\cdot((\alpha-a)E+(\beta-b)F)\geq -1 \quad \text{and} \quad E\cdot((2a-\alpha-2)E+(2b-\beta-e-2)F)\geq -1.$$

The first inequality yields

$$-e(\alpha-a)+\beta-b\geq -1.$$

Substituting for β in terms of α and simplifying we obtain

$$\alpha \leq 2a-1-\frac{ea(a-1)}{2b}.$$

Similarly, the second inequality yields

$$-e(2a-\alpha-2)+2b-\beta-e-2\geq -1.$$

Substituting for β, we obtain

$$\alpha\geq a-1+\frac{ea(a-1)}{2b}.$$

If $\alpha = a-1$, the second inequality becomes $-\frac{e}{2}-1\geq -1$ and can only be satisfied if $e=0$. Similarly, if $\alpha = 2a-1$, then the first inequality forces e to be 0. We conclude that in all cases we must have

$$a-1+\frac{ea(a-1)}{2b}\leq \alpha\leq 2a-1-\frac{ea(a-1)}{2b}.$$

This provides necessary and sufficient conditions for the existence of Ulrich bundles on $(\mathbb{F}_e, aE+bF)$. $\square$

Corollary 4.7. *Let $H = aE+bF$ be an ample divisor on $\mathbb{F}_e$. Then there exists an Ulrich bundle of rank 2 with invariants*

$$\mathbf{v}=(r,\nu,\Delta)=\left(2,\left(\frac{3}{2}a-1\right)E+\left(\frac{3}{2}b-\frac{e}{2}-1\right)F,\frac{a}{8}(2b-ae)\right).$$

Proof. In Theorem 4.6, the discriminant Δ is maximized when $\alpha=\frac{3}{2}a-1$. In that case, the expressions for β and Δ become

$$\beta=\frac{3}{2}b-\frac{e}{2}-1,\quad \Delta=\frac{a}{8}(2b-ae).$$

We further have $\mathbf{v}(-2H)$ and $\mathbf{v}(-H)$ have Serre dual Chern characters and by Theorem 3.2 for a general F-prioritary bundle all the cohomology vanishes. $\square$

Remark 4.8. It is also easy to classify the ample classes $H=aE+bF$ that admit a rank one Ulrich sheaf. If the rank of $\mathcal{V}$ is 1, then $\mathcal{V}(-2H)=I_Z((\alpha-2a)E+(\beta-2b)F)$ for some zero-dimensional scheme Z. Given a line bundle L and a zero-dimensional scheme Z, we have that

$$h^2(\mathbb{F}_e, L\otimes I_Z)=h^2(\mathbb{F}_e,L),\quad h^1(\mathbb{F}_e,L\otimes I_Z)\geq h^1(\mathbb{F}_e,L),\quad h^0(\mathbb{F}_e,L\otimes I_Z)\leq h^0(\mathbb{F}_e,L).$$

Furthermore, if Z is nonempty, at least one of the last two inequalities is strict. Since $(\alpha-2a)E+(\beta-2b)F$ is not effective, we conclude that $h^1(\mathbb{F}_e,\mathcal{V}(-2H))\neq 0$ unless Z is empty. We conclude that $\mathcal{V}$ is a line bundle. If $e\geq 1$, the only line bundles with vanishing cohomology are $\mathcal{O}_{\mathbb{F}_e}(-F), \mathcal{O}_{\mathbb{F}_e}(-E+tF)$ and $\mathcal{O}_{\mathbb{F}_e}(-2E-(e+1)F)$. Hence, we must have $a=1$ or 2. If $a=2$, we see that $\alpha=a$, $b=e$ and $\beta=e-1$. In this case, H is not ample. Hence, we must have $a=1$. If $H=E+bF$ for $b>e$, then the line bundles $\mathcal{O}_{\mathbb{F}_e}((2b-e-1)F)$ and $\mathcal{O}_{\mathbb{F}_e}(E+(b-1)F)$ are Ulrich. If $e=0$, the line bundles with vanishing cohomology are $\mathcal{O}_{\mathbb{F}_0}(-E+tF)$ and $\mathcal{O}_{\mathbb{F}_0}(tE-F)$. In this case, given any ample line bundle $\mathcal{O}_{\mathbb{F}_0}(aE+bF)$, the two line bundles $\mathcal{O}_{\mathbb{F}_0}((2a-1)E+(b-1)F)$ and $\mathcal{O}_{\mathbb{F}_0}((a-1)E+(2b-1)F)$ are Ulrich.

Remark 4.9. Using the Beilinson spectral sequence, one can in fact show that the Ulrich bundles in these moduli spaces are given by the Gaeta resolution (see [Ant18]).

Nonminimal rational surfaces. Eisenbud and Schreyer [ESW03] and Kim [Kim16] have obtained partial results on the existence of Ulrich bundles on rational surfaces more generally. More precise results are available for special rational surfaces such as del Pezzo surfaces (see [CKM13]). Here we obtain the following general statement on arbitrary blowups of $\mathbb{P}^2$ at distinct points. The same techniques yield results on infinitely near blowups. We leave it to the reader to formulate the corresponding statements.

Theorem 4.10. *Let X be the blowup of $\mathbb{P}^2$ at k distinct points. Let $H = aL - \sum_{i=1}^k b_i E_i$ be a very ample divisor such that $a \geq 3$, $b_i \geq 1$ and $a \geq \sum_{i=1}^k b_i + (1-k)$. Then (X, H) admits an Ulrich bundle of every positive even rank. In fact, the general sheaf in $\mathcal{P}_{X,L-E_1}(\mathbf{v}(H))$ where*

$$\operatorname{rk}\mathbf{v} = 2r, \quad \nu(\mathbf{v}) = \frac{K_X + H}{2}, \quad \Delta(\mathbf{v}) = 1 + \frac{1}{2}(\nu^2 - \nu \cdot K_X)$$

is an Ulrich bundle $\mathcal{V}$ with a resolution of the form

$$0 \to \mathcal{O}_X(H-2L)^\alpha \to \mathcal{O}_X(H-L)^\beta \oplus \bigoplus_{i=1}^k \mathcal{O}_X(H-E_i)^{\gamma_i} \to \mathcal{V} \to 0, \text{ or}$$

$$0 \to \mathcal{O}_X(H-2L)^\alpha \oplus \mathcal{O}_X(H-L)^\beta \to \bigoplus_{i=1}^k \mathcal{O}_X(H-E_i)^{\gamma_i} \to \mathcal{V} \to 0, \text{ where}$$

$$\alpha = r(a - \sum_{i=1}^k b_i + k - 1), \quad \gamma_i = r(b_i - 1)$$

and β and the type of resolution is determined by the rank of $\mathcal{V}$.

Proof. By [CH18b, Theorem 3.12], let $\mathbf{v}$ be a Chern character with $\chi(\mathbf{v}) = 0$, $\operatorname{rk}(\mathbf{v}) \geq 2$ and $\nu(\mathbf{v}) = \delta L - \sum_{i=1}^k \epsilon_i E_i$ such that $\delta, \epsilon_i \geq 0$ and $\delta - \sum_{i=1}^k \epsilon_i \geq -1$, then the stack $\mathcal{P}_{X,H-E_1}(\mathbf{v})$ of $(H-E_1)$-prioritary sheaves is nonempty, contains locally-free elements and the general element of the stack has no cohomology. Furthermore, the general element of $\mathcal{P}_{X,H-E_1}(\mathbf{v})$ admits a Gaeta-type resolution of the form as in the statement of the theorem. Let $\mathbf{v}$ be a Chern character on X of rank $2r$ and

$$\nu = \frac{K_X + H}{2} = \frac{1}{2}\left((a-3)L - \sum_{i=1}^k (b_i - 1)E_i\right) \quad \Delta(\mathbf{v}) = 1 + \frac{1}{2}(\nu^2 - \nu \cdot K_X).$$

By Riemann-Roch, we have $\chi(\mathbf{v}) = 0$. Observe that

$$(a-3) - \sum_{i=1}^k (b_i - 1) = a - \sum_{i=1}^k b_i + (k-3) \geq -2$$

by assumption. Since $a \geq 3$ and $b_i \geq 1$, the conditions of [CH18b, Theorem 3.12] are satisfied and we conclude that $\mathcal{P}_{X,H-E_1}(\mathbf{v})$ is a nonempty, irreducible stack whose general member is a locally free sheaf with no cohomology. The same holds by Serre duality for the general member of the stack $\mathcal{P}_{X,H-E_1}(\mathbf{v}(-H))$. Consequently, the general member of the stack $\mathcal{P}_{X,H-E_1}(\mathbf{v}(H))$ is an Ulrich bundle on X. The desired resolution is a consequence of [CH18b, Theorem 3.12]. $\square$

An analogous theorem holds for blowups of Hirzebruch surfaces.

Theorem 4.11. *Let X be the blowup of $\mathbb{F}_e$ at k distinct points not contained on the negative self-intersection section E. Let $H = aE + bF - \sum_{i=1}^k c_i E_i$ be a very ample divisor such that*

$$a \geq 2, \quad a \geq \sum_{i=1}^k c_i - k, \quad b \geq \sum_{i=1}^k c_i + (a-1)e - k, \quad c_i \geq 1.$$

Then (X,H) admits an Ulrich bundle of every even positive rank. Moreover, the general sheaf in $\mathcal{P}_{X,F}(\mathbf{v}(H))$ where

$$\operatorname{rk}\mathbf{v} = 2r, \quad \nu(\mathbf{v}) = \frac{K_X+H}{2}, \quad \Delta(\mathbf{v}) = 1 + \frac{1}{2}(\nu^2 - \nu\cdot K_X)$$

is an Ulrich bundle $\mathcal{V}$ with a resolution of the form

$$0 \to \mathcal{O}_X(H-E-(e+1)F)^\alpha \to \mathcal{O}_X(H-E-eF)^\beta \oplus \mathcal{O}_X(H-F)^\gamma \oplus \bigoplus_{i=1}^k \mathcal{O}_X(H-E_i)^{\delta_i} \to \mathcal{V} \to 0,$$

where

$$\alpha = r(b-(e-1)a - \sum_{i=1}^k c_i + e + k - 2), \quad \beta = r(b - ea - \sum_{i=1}^k c_i + e + k),$$

$$\gamma = r(a - \sum_{i=1}^k c_i + k), \quad \delta_i = r(c_i - 1).$$

Proof. The theorem follows from [CH18b, Theorem 3.15] using the same proof as that of Theorem 4.10. □

K3 surfaces. Let X be a K3 surface of Picard rank 1 with ample generator H_0 such that $H_0^2 = 2n$. Let $\mathbf{v}$ be a Chern character and let $m(\mathbf{v}) = (r(\mathbf{v}), c_1(\mathbf{v}), \operatorname{ch}_2(\mathbf{v}) + r(\mathbf{v}))$ be the associated Mukai vector. Assume that $m(\mathbf{v})^2 \geq -2$. By Theorem 3.8, if $\chi(\mathbf{v}) = 0$, then the cohomology of the general sheaf in $M_{X,H}(\mathbf{v})$ vanishes. We obtain the following classification of Ulrich bundles on K3 surfaces of Picard rank one, recovering results of [AGO17]. There has been further work on the even rank case (see [CaG18, Fae18]).

Theorem 4.12. [AGO17, CNY18] *Let X be a K3 surface as above and let $H = kH_0$ be a very ample divisor on X. If k is even, then there exists an Ulrich bundle for every rank $r > 1$. If k is odd, there exists an Ulrich bundle for every even rank $r \geq 2$. The general bundle in the moduli space with Mukai vector*

$$\left(r, \frac{3rk}{2}, 2rnk^2 - r\right)$$

is an Ulrich bundle. Hence, there is a

$$\frac{1}{2}nr^2k^2 + 2r^2$$

dimensional family of Ulrich bundles on (X, kH_0).

4.1. **Other cohomological conditions.** There are other cohomology vanishing conditions that can be analyzed using our Brill-Noether type theorems. As an example we give a quick solution of Eisenbud and Schreyer's conjecture on the existence of bundles with natural cohomology on $\mathbb{P}^1\times\mathbb{P}^1$ as an immediate consequence of Theorem 3.2. Recall that a bundle $\mathcal{V}$ on $\mathbb{P}^1\times\mathbb{P}^1$ has *natural cohomology* if $\mathcal{V}\otimes L$ has at most one nonzero cohomology group for any line bundle L on $\mathbb{P}^1\times\mathbb{P}^1$. The conjecture was first proved by Solis [Sol18] by constructing the required bundles as monads.

Corollary 4.13. *Let F_1 and F_2 denote the two fiber classes on $\mathbb{P}^1\times\mathbb{P}^1$. Let $\mathbf{v}$ be an integral Chern character such that $\operatorname{rk}(\mathbf{v}) \geq 2$ and $\Delta \geq 0$. Then $\mathcal{P}_{\mathbb{P}^1\times\mathbb{P}^1,F_1}(\mathbf{v})$ is nonempty, and the general $\mathcal{V} \in \mathcal{P}_{\mathbb{P}^1\times\mathbb{P}^1,F_1}(\mathbf{v})$ is locally free and has at most one nonzero cohomology group. In particular, the very general member of $\mathcal{P}_{\mathbb{P}^1\times\mathbb{P}^1,F_1}(\mathbf{v})$ is a bundle with natural cohomology.*

Proof. Since $\Delta(\mathbf{v}) \geq 0$, $\mathcal{P}_{\mathbb{P}^1\times\mathbb{P}^1,F_1}(\mathbf{v})$ is nonempty by Theorem 3.2. Moreover, the general F_1-prioritary sheaf is also F_2-prioritary. Hence, by Theorem 3.2, if $\nu\cdot F_i \geq -1$ for $i = 1$ or 2, then the general F_i-prioritary sheaf does not have any h^2. Similarly, if $\nu\cdot F_i \leq -1$ for $i=1$ or 2, then the general F_i-prioritary sheaf does not have any h^0. Hence, if $F_1\cdot\nu \geq -1$ and $F_2\cdot\nu\leq -1$, then the general F_1-prioritary sheaf can only have h^1. Similarly, if $F_1 \cdot \nu \leq -1$ and $F_2 \cdot \nu \geq -1$, then the general F_1-prioritary sheaf can only have h^1.

On the other hand, by Theorem 3.2, if $\nu \cdot F_i \geq -1$ for $i = 1$ and 2, then the general F_1-prioritary sheaf can only have one nonzero cohomology group (h^0 or h^1 depending on the Euler characteristic). Similarly, if $\nu \cdot F_i \leq -1$ for $i = 1$ and 2, then the general prioritary sheaf has at most one nonzero cohomology group (h^1 or h^2 depending on the Euler characteristic). Since these four regions tile all possible slopes, we conclude that the general F_1-prioritary sheaf of rank at least 2 and $\Delta \geq 0$ has at most one nonzero cohomology group. We conclude that for the very general sheaf $\mathcal{V}$ in $\mathcal{P}_{\mathbb{P}^1 \times \mathbb{P}^1, F_1}(\mathbf{v})$ all twists $\mathcal{V}(nF_1 + mF_2)$ have at most one nonzero cohomology group. Hence, $\mathcal{V}$ has natural cohomology. □

5. The cohomology of the moduli space of sheaves

In this section, we will survey some recent work on the cohomology of the moduli spaces of sheaves on surfaces following [CW18]. The main philosophy of the paper [CW18] is that the Betti numbers of the moduli spaces $M_{X,H}(\mathbf{v})$ are hard to compute; however, for many interesting surfaces these Betti numbers stabilize as the discriminant Δ tends to infinity. In all known examples, the stable Betti numbers are independent of the rank and first Chern class, hence are easy to compute from Göttsche's calculations in rank one [Got90]. An analogous stabilization should hold for Hodge numbers. In fact, [CW18] shows that, at least for rational surfaces, the stabilization holds more fundamentally at the motivic level.

We begin by briefly recalling the Grothendieck ring of varieties and virtual Poincaré and Hodge polynomials. These notions will be necessary to formulate the main theorem of [CW18].

The Grothendieck ring of varieties. The *Grothendieck ring of varieties over the complex numbers* $\mathbb{C}$, $K_0(\mathrm{var}_{\mathbb{C}})$, is the quotient of the free abelian group on isomorphism classes varieties $[X]$ of finite type over $\mathbb{C}$ by the scissor relations,

$$[X] = [Y] + [Z]$$

if Y and Z are disjoint locally closed subvarieties of X with $X = Y \cup Z$. Define multiplication by

$$[X] \cdot [Y] = [X \times Y].$$

Let $\mathbb{L}$ be the class of the affine line $[\mathbb{A}^1]$. By Hironaka's resolution of singularities [Hir64], $K_0(\mathrm{var}_{\mathbb{C}})$ is generated by the classes of smooth projective irreducible varieties.

The Poincaré polynomial induces a map

$$P(t) : K_0(\mathrm{var}_{\mathbb{C}}) \to \mathbb{Z}[t]$$

called the *virtual Poincaré polynomial* [Joy07]. Similarly, the Hodge polynomial induces a map

$$H(x, y) : K_0(\mathrm{var}_{\mathbb{C}}) \to \mathbb{Z}[x, y]$$

called the *virtual Hodge polynomial* [Joy07].

Let $R = K_0(\mathrm{var}_{\mathbb{C}})[\mathbb{L}^{-1}]$. The ring R has a $\mathbb{Z}$-graded decreasing filtration $\mathcal{F}$ generated by

$$[X]\mathbb{L}^a \in \mathcal{F}^i \quad \text{if} \quad \dim(X) + a \leq -i.$$

Define the ring A^- as the completion of R with respect to this filtration. Note that elements of A^- have a natural notion of dimension generalizing the dimension of smooth, projective varieties. For $a \in A^-$, we will denote this dimension by $\dim(a)$. The virtual Poincaré polynomial can be extended to R (resp. A^-), where it takes values in $\mathbb{Z}[t^{\pm 1}]$ (resp. $\mathbb{Z}((t^{-1}))$). Similarly, the virtual Hodge polynomials can be extended to R (respectively, A^-), were it takes values in $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ (resp., $\mathbb{Z}((x^{-1}, y^{-1}))$).

Given a sequence of smooth projective varieties X_i of dimension d_i, we would like to have a notion of stabilization in A^- that guarantees that the low-degree Betti numbers of X_i stabilize. Consider the sequence $\mathbb{L}^{-d_i}[X_i]$. By Poincaré duality,

$$P_{[X_i]}(t) = P_{\mathbb{L}^{-d_i}[X_i]}(t^{-1}), \quad H_{[X_i]}(x, y) = H_{\mathbb{L}^{-d_i}[X_i]}(x^{-1}, y^{-1}).$$

If the sequence $\mathbb{L}^{-d_i}[X_i]$ converges in A^-, then the low-degree Betti numbers (and Hodge numbers) of the X_i stabilize. This motivates the following definition.

Definition 5.1. A sequence of elements $a_i \in A^-$ *stabilizes* to a if the sequence $\mathbb{L}^{-\dim(a_i)}a_i$ converges to a.

It is convenient to record the sequence $a_i = \mathbb{L}^{-d_i}[X_i]$ in a generating function

$$F(q) = \sum_i a_i q^i,$$

which can be thought of as an element of $A^-\{\{q\}\}$, the ring of Puiseux series in q with coefficients in A^-. We use Puiseux series (instead of power series) because it is often convenient to index the moduli spaces by the discriminant Δ or the second Chern character d. Since Δ and d can take fractional values, it is convenient to allow series in q with fractional exponents with bounded denominator. The following easy proposition relates stabilization of the sequence to the convergence of the generating function.

Proposition 5.2. [CW18, Proposition 3.6] *Let a_i be a sequence in A^-. Then a_i converges to a in A^- if and only if the series $(1-q)\sum a_i q^i$ is convergent at $q=1$ and the sum at $q=1$ is a.*

Zeta functions. Kapranov [Kap00] defined the *motivic zeta function* of a variety X by

$$Z_X(q) = \sum_{n=0}^{\infty} [X^{(n)}] q^n,$$

where $[X^{(n)}]$ denotes the class of the nth symmetric product X^n/S_n of X in the Grothendieck ring of varieties. When X is a smooth projective surface, we will be interested in the Betti realization of the zeta function. In order to distinguish this function from the motivic zeta function, we will denote it by $\zeta_X(q,t)$. By [Mac62], we have

$$\zeta_X(q,t) = \sum_{n=0}^{\infty} P_{X^{(n)}}(t) q^n = \frac{(1+qt)^{b_1(X)}(1+qt^3)^{b_3(X)}}{(1-q)(1-qt^2)^{b_2(X)}(1-qt^4)},$$

where $b_i(X)$ is the ith Betti number of X.

With these preliminaries in place, we can discuss the cohomology of $M_{X,H}(\mathbf{v})$. When $\mathrm{rk}(\mathbf{v}) = 1$, Göttsche computed the cohomology. We briefly recall his results.

The Betti numbers of rank 1 sheaves. Let $X^{[n]}$ denote the Hilbert scheme of n points on X. When $r=1$, a stable sheaf $\mathcal{F} \in M_{X,H}(1,c,\Delta)$ is isomorphic to $L \otimes I_Z$, where L is a line bundle with $\mathrm{ch}_1(L) = c$ and I_Z is an ideal sheaf of points on X with $|Z| = \Delta$. There is a natural isomorphism from $\mathrm{Pic}^c(X) \times X^{[\Delta]}$ to $M_{X,H}(1,c,\Delta)$ given by tensor product. The inverse morphism sends a rank one sheaf $\mathcal{F}$ to the pair $(\mathcal{F}^{**}, \mathcal{F} \otimes \mathcal{F}^*)$.

Göttsche [Got90] computed the Betti numbers of $X^{[n]}$. Let

$$P_n(t) = \sum_{i=0}^{2n} b_i(X^{[n]}) t^i$$

be the Poincaré polynomial of $X^{[n]}$. It is convenient to form a generating function $F(q,t)$ incorporating these polynomials. Göttsche proves the following formula

$$F(q,t) = \sum_{n=0}^{\infty} P_n(t) q^n = \prod_{m=1}^{\infty} \zeta_X(t^{2m-2}q^m, t).$$

By the Künneth formula, it follows that the Betti numbers of $M_{X,H}(1,c,\Delta)$ are given by

$$G(q,t) = \sum_{\Delta=0}^{\infty} \sum_{i=0}^{2\Delta+b_1(X)} b_i(M_{X,H}(1,c,\Delta)) t^i q^\Delta = (1+t)^{b_1(X)} \prod_{m=1}^{\infty} \zeta_X(t^{2m-2}q^m, t).$$

An important consequence of these formulae is that the Betti numbers of $M_{X,H}(1,c,\Delta)$ stabilize as Δ tends to infinity [Got90, Corollary 2.11]. By Proposition 5.2, the generating function for the stable Betti numbers $b_{i,\mathrm{Stab}}(X)$ can be computed as follows

$$\sum b_{i,\mathrm{Stab}(X)} t^i = (1+t)^{b_1(X)} ((1-q) \prod_{m=1}^{\infty} \zeta_X(t^{2m-2}q^m, t))|_{q=1}.$$

The Hodge numbers of rank 1 sheaves. More generally, Göttsche, Soergel [GS93] and Cheah [Che96] compute the generating function for the Hodge numbers of the Hilbert scheme of points $X^{[n]}$.

$$F(x,y,q) = \sum_{n=1}^{\infty} h^{a,b}(X^{[n]}) q^n = \prod_{m=1}^{\infty} \prod_{a,b=0}^{2} (1 + (-1)^{a+b+1} x^{a+m-1} y^{b+m-1} q^m)^{(-1)^{a+b+1} h^{a,b}(X)},$$

where $h^{p,q}(X)$ denotes the (p,q)-Hodge number of X. The denominator depends only on q, precisely when $a=b=0$ and $m=1$. In $(1-q)F(x,y,q)$, the coefficient of every term $x^\alpha y^\beta$ is a polynomial in q. Hence, the Hodge numbers stabilize and the stable Hodge numbers are given by

$$\prod_{\substack{a,b=0,\\ (a,b)\neq(0,0)}}^{2} (1 + (-1)^{a+b+1} x^a y^b)^{(-1)^{a+b+1} h^{a,b}(X)} \prod_{m=2}^{\infty} \prod_{a,b=0}^{2} (1 + (-1)^{a+b+1} x^{a+m-1} y^{b+m-1})^{(-1)^{a+b+1} h^{a,b}(X)}.$$

Finally, by the Künneth formula, we have

$$G(x,y,q) = \sum_{\Delta=1}^{\infty} h^{a,b}(M_{X,H}(1,c,\Delta)) q^\Delta = (1+x)^{\frac{b_1}{2}} (1+y)^{\frac{b_1}{2}} F(x,y,q).$$

Hence, these Hodge numbers stabilize to

$$(1+x)^{\frac{b_1}{2}} (1+y)^{\frac{b_1}{2}} ((1-q)F(x,y,q)|_{q=1}).$$

The motive of the Hilbert scheme. Moreover, Göttsche computes the motive of the Hilbert scheme of points on a surface. Let $\mathcal{P}(\Delta)$ denote the set of partitions of Δ. Let α be the partition $\alpha_1^{i_1}, \cdots, \alpha_j^{i_j}$, where the part α_m is repeated i_m times. Let $|\alpha|$ be the length (equivalently, the number of parts) of the partition. Let $X^{(\alpha)}$ denote the product $X^{(i_1)} \times \cdots \times X^{(i_j)}$. Göttsche [Got01] shows that

$$[X^{[\Delta]}] = \sum_{\alpha \in \mathcal{P}(\Delta)} [X^{(\alpha)} \times \mathbb{A}^{\Delta - |\alpha|}].$$

Furthermore, one has the following equality of generating functions

$$\sum_{\Delta=1}^{\infty} [X^{[\Delta]}] \mathbb{L}^{-2\Delta} q^\Delta = \prod_{m=1}^{\infty} Z_X(\mathbb{L}^{-m-1} q^m).$$

Vakil and Wood [VW15, Conjecture 1.25] conjecture that the sequence $[X^{(\Delta)}]\mathbb{L}^{-2\Delta}$ converges in A^-. By Göttsche's formula, this conjecture also implies that the sequence $[X^{[\Delta]}]\mathbb{L}^{-2\Delta}$ converges in A^-. The conjecture is known when X is a rational surface, but is open in general and seems to be a fairly subtle question.

The Betti numbers of higher rank moduli spaces. In contrast, the Betti numbers of $M_{X,H}(\mathbf{v})$ are generally unknown when the rank of $\mathbf{v}$ is at least 2. In [CW18], the authors take the point of view that while the individual Betti numbers for higher rank moduli spaces are often hard to compute, the Betti numbers should stabilize as the discriminant tends to infinity. Furthermore, in all examples when the stabilization is known, the stable Betti numbers are independent of the rank, c_1 and the ample class H. In particular, the stable numbers are computed by Göttsche's calculation. More precisely, the authors make the following conjecture.

Conjecture 5.3. [CW18, Conjecture 1.1] *Fix a rank $r > 0$ and a first Chern character c. Let $b_{i,\mathrm{Stab}}(X)$ denote the ith stable Betti number of $M_{X,H}(1, c, \Delta)$. Then the ith Betti number of $M_{X,H}(r, c, \Delta)$ stabilizes to $b_{i,\mathrm{Stab}}(X)$ as Δ tends to ∞. More precisely, given an integer k, there exists $\Delta_0(k)$ such that for $\Delta \geq \Delta_0(k)$ and $i \leq k$*

$$b_i(M_{X,H}(r, c, \Delta)) = b_{i,\mathrm{Stab}}(X).$$

Furthermore, if H is in a compact subset $\mathcal{C}$ of the ample cone of X, then $\Delta_0(k)$ can be chosen independently of $H \in \mathcal{C}$.

Conjecture 5.3 fits with the philosophy of Donaldson [Don90], Gieseker and Jun Li [GL94, LiJ93, LiJ94] that the geometry of $M_{X,H}(\gamma)$ becomes better behaved as Δ tends to ∞. A consequence of O'Grady's theorem [O'G96] shows that the zeroth Betti number is 1 if Δ is sufficiently large. Jun Li shows that the first and second Betti numbers stabilize to the expected value when $r = 2$ [LiJ97]. Yoshioka computes the Betti numbers of moduli spaces of rank 2 sheaves on $\mathbb{P}^2$ and proves the stabilization of the Betti numbers [Yos94, Corollary 6.3]. Yoshioka [Yos95, Yos96b] and Göttsche [Got96] compute the Betti and Hodge numbers of $M_{X,H}(\gamma)$ when $r = 2$ and X is a ruled surface. Yoshioka [Yos95, Yos96a] observes the stabilization of the Betti numbers for rank 2 bundles on ruled surfaces. Göttsche observes that the small Hodge numbers are independent of the ample H and gives a nice formula for them. Göttsche further extends his results to rank 2 bundles on rational surfaces for polarizations that are K_X-negative in [Got99] (see also [Yos95]). Manschot [Man11, Man14] building on the work of Mozgovoy [Moz13] gives a formula for the Betti numbers of the moduli spaces when $X = \mathbb{P}^2$. The stabilization of the Betti numbers can be observed from the tables provided in these papers.

The conjecture is known for smooth moduli spaces of sheaves on K3 and abelian surfaces. By work of Mukai [Muk84], Huybrechts [Huy03] and Yoshioka [Yos99], smooth moduli spaces of sheaves on a K3 surface X are deformations of the Hilbert scheme of points on X of the same dimension. In particular, they are diffeomorphic to $X^{[n]}$ of the same dimension. Hence, their Betti numbers agree without taking any limits.

Yoshioka [Yos01] obtains similar results on abelian surfaces. A smooth moduli space of sheaves $M_{X,H}(\gamma)$ is deformation equivalent to $X^* \times X^{[n]}$ of the same dimension, where X^* is the dual abelian surface. In this case as well the cohomology is isomorphic to the cohomology of $X^* \times X^{[n]}$ without the need to take limits.

There is also closely related gauge theory literature on the Atiyah-Jones Conjecture (see [AJ78, CW18, Tau84, Tau89] for details).

Similarly, one may conjecture an analogue of Conjecture 5.3 for Hodge numbers of $M_{X,H}(r, c, \Delta)$, at least when $M_{X,H}(r, c, \Delta)$ is smooth. These conjectures are at the level of invariants. It would be interesting to find geometric reasons underlying the stabilization of Betti numbers. Unfortunately, we do not know any algebraic maps between these moduli spaces, though it is possible to define certain correspondences.

Assuming the Vakil-Wood Conjecture is true and that the classes of the Hilbert schemes $[X^{[n]}]$ stabilize in the ring A^-, one can further speculate that the classes $[M_{X,H}(r, c, \Delta)]$ also stabilize to the same stable limit in A^-. The main theorem of [CW18] proves this for rational surfaces provided $K_X \cdot H < 0$. The stabilization of the virtual Poincaré and Hodge polynomials are immediate consequences. In particular, when the virtual polynomials agree with the Poincaré and Hodge polynomials, we obtain the stabilization of Betti and Hodge numbers.

Theorem 5.4. *Let X be a smooth, complex projective rational surface and let H be a polarization such that $H \cdot K_X < 0$. As Δ tends to ∞, the classes $[\mathcal{M}_{X,H}(r, c, \Delta)]$ of the moduli stacks of Gieseker semistable sheaves stabilize in A^- to*

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^{\chi_{\mathrm{top}}(X)}}.$$

In particular, the virtual Betti and Hodge numbers of $\mathcal{M}_{X,H}(r, c, \Delta)$ stabilize, and the generating functions for the stable virtual numbers $b_{i,\mathrm{Stab}}$ and $h^{p,q}_{\mathrm{Stab}}$ are given by

$$\sum_{i=0}^{\infty} b_{i,\mathrm{Stab}} t^i = \prod_{i=1}^{\infty} \frac{1}{(1 - t^{2i})^{\chi_{\mathrm{top}}(X)}} \quad \text{and} \quad \sum_{p,q=0}^{\infty} h^{p,q}_{\mathrm{Stab}} x^p y^q = \prod_{i=1}^{\infty} \frac{1}{(1 - (xy)^i)^{\chi_{\mathrm{top}}(X)}}.$$

When the moduli spaces $M_{X,H}(r, c, \Delta)$ are smooth, projective varieties, one obtains the following consequence, first described by Yoshioka in a remark in [Yos96a, §3.6].

Theorem 5.5. [CW18, Theorem 1.8] *Let X be a smooth, complex, projective rational surface and let H be a polarization such that $K_X \cdot H < 0$. Assume that the moduli spaces $M_{X,H}(r, c, \Delta)$ do not contain any strictly semistable sheaves. Then the Betti and Hodge numbers of $M_{X,H}(r, c, \Delta)$ stabilize to the stable Betti and Hodge numbers of the Hilbert scheme of points $X^{[\Delta]}$ as Δ tends to infinity.*

Conjecture 5.3 expresses the hope that even when the moduli spaces $M_{X,H}(\mathbf{v})$ are singular, Poincaré duality holds in larger and larger ranges and the cohomology is pure in higher and higher degrees as Δ tends to infinity.

Question 5.6. *Is the cohomology of $M_{X,H}(r, c, \Delta)$ pure in increasing degrees as Δ tends to infinity? Does Poincaré duality hold in increasing degrees as Δ tends to infinity?*

If the cohomology is pure in increasing degrees, then the virtual Poincaré polynomial would coincide with the actual Poincaré polynomial in increasing degrees and Theorem 5.5 would imply an analogue of Theorem 5.5 without smoothness assumptions on $M_{X,H}(r, c, \Delta)$.

It is natural to expect Conjecture 5.3 to apply in more general contexts. The same stabilization should hold for spaces closely related to $M_{X,H}(\mathbf{v})$ such as moduli spaces of pure one-dimensional sheaves and the Matsuki-Wentworth moduli spaces of twisted Gieseker semistable sheaves. More interestingly, it would be interesting to find an analogue of the conjecture for moduli spaces of Bridgeland semistable objects on a surface (see [CW18, §1]).

We refer the reader to [CW18] for a proof of Theorem 5.4. The proof is a detailed study of the effect of wall-crossing and blowup on $[M_{X,H}(\mathbf{v})]$ in A^- using Joyce's machinery. The steps can be summarized as follows.

(1) First, using Joyce's wall-crossing formula [Joy08, Theorem 6.21], one shows that if H_1 is an ample class and H_2 is a big and nef class on X such that $K_X \cdot H_i < 0$ for $i = 1, 2$, then $[\mathcal{M}_{X,H_1}(\mathbf{v})]$ stabilizes in A^- if and only if $[\mathcal{M}_{X,H_2}(\mathbf{v})]$ stabilizes. Furthermore, in case they stabilize, they stabilize to the same element of A^- (see [CW18, Proposition 4.7]).

(2) Second, using Mozgovoy's blowup formula [Moz13, Proposition 7.3], one shows that the classes of the moduli spaces on X stabilize in A^- if and only if the classes of corresponding moduli spaces on the blowup of X stabilize. Furthermore, in case both stabilize, the limits differ by a factor of

$$\prod_{k=1}^{\infty} \frac{1}{1 - \mathbb{L}^{-k}}$$

as expected (see [CW18, Lemma 4.10, Proposition 4.11]).

(3) Third, one uses Mozgovoy's [Moz13, Theorem 1.1] formula for F-semistable sheaves on the Hirzebruch surface $\mathbb{F}_1$ to calculate the stable classes in one explicit case. Then using wall-crossing one obtains the stable classes for moduli spaces on $\mathbb{F}_1$ for every polarization (see [CW18, §5]).
(4) Finally, using blowup and wall-crossing repeatedly one proves the theorem for all rational surfaces starting with the minimal rational surfaces $\mathbb{P}^2$ and $\mathbb{F}_e$, $e \neq 1$ (see [CW18, §5, 6]).

Currently, all our evidence for Conjecture 5.3 comes from rational surfaces and K-trivial surfaces, Jun Li's Theorem [LiJ97] and by analogy to the Atiyah-Jones conjecture. It would be especially interesting to verify the conjecture for examples of surfaces of general type, even in rank 2.

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Department of Mathematics, Statistics and CS, University of Illinois at Chicago, Chicago, IL 60607
E-mail address: `coskun@uic.edu`

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802
E-mail address: `huizenga@psu.edu`