

Chromatic capacity and graph operations

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Abstract

The *chromatic capacity* $\chi_{\text{cap}}(G)$ of a graph G is the largest k for which there exists a k -coloring of the edges of G such that, for every coloring of the vertices of G with the same colors, some edge is colored the same as both its vertices. We prove that there is an unbounded function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\chi_{\text{cap}}(G) \geq f(\chi(G))$ for almost every graph G , where χ denotes the chromatic number. We show that for any positive integers n and k with $k \leq n/2$ there exists a graph G with $\chi(G) = n$ and $\chi_{\text{cap}}(G) = n - k$, extending a result of Greene. We obtain bounds on $\chi_{\text{cap}}(K_n^r)$ that are tight as $r \rightarrow \infty$, where K_n^r is the complete n -partite graph with r vertices in each part. Finally, for any positive integers p and q we construct a graph G with $\chi_{\text{cap}}(G) + 1 = \chi(G) = p$ that contains no odd cycles of length less than q .

Key words: Chromatic capacity, Emulsive edge coloring, Compatible vertex coloring, Split coloring, Lexicographic product

1 Introduction

Let G be a simple graph. If the edges and vertices of G are colored simultaneously, we will call an edge *monochromatic* if its color is the same as the color of each of its vertices. An edge coloring is *compatible* with a vertex coloring if there is no monochromatic edge. If an edge coloring is such that there is no compatible vertex coloring using the same color set, then it is an *emulsion*, and is said to be *emulsive*.

Definition 1 *The chromatic capacity $\chi_{\text{cap}}(G)$ of a graph G is the largest integer k for which there exists an emulsion $c: E(G) \rightarrow \{1, \dots, k\} = [k]$. Should G have no edges, we define $\chi_{\text{cap}}(G) = 0$.*

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An analogous definition can be made if G is a multigraph. However, except when stated otherwise, all of our graphs will be simple graphs.

The concept behind chromatic capacity largely stems from a paper by Cochand and Duchet [4]. For an acyclic digraph D , they used a graph of large chromatic capacity to construct a graph $G(D)$ all of whose acyclic orientations contained D as an induced subgraph (the existence of such a graph was demonstrated by Rödl more than ten years earlier in [14]). Since then, emulsive edge colorings have been studied in a number of papers. Most of the research that has been done has concerned complete graphs, and it has been shown that

$$(1 - o(1))\sqrt{n} < \chi_{\text{cap}}(K_n) < 1 + \sqrt{2n}$$

for every n (see [7, Theorems 4 and 5]).

While the complete graphs have been studied in depth, much less has been achieved for general graphs. Cochand and Duchet [4] and Archer [2] independently prove the same upper bound for the chromatic capacity of a general graph. As far as lower bounds are concerned, Greene proves in [8, Theorem 2] that

$$\chi_{\text{cap}}(G)^2 \ln \chi_{\text{cap}}(G) > (1 - o(1)) \cdot \frac{\chi(G)^2}{2n}$$

for any graph G on n vertices, where χ denotes the usual chromatic number and the $o(1)$ term goes to 0 as $\chi(G) \rightarrow \infty$ (independently of n). We note that this lower bound is not particularly useful for graphs that have a large number of vertices relative to their chromatic number. One of the major open questions in the study of the chromatic capacity is whether or not this dependence on the number of vertices is necessary. Greene's Conjecture states that there exists an unbounded function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\chi_{\text{cap}}(G) \geq f(\chi(G))$ for every graph G . Should there be such an f , we would have a meaningful lower bound for the chromatic capacity in terms of the chromatic number alone. This is quite interesting, especially since there is also an upper bound on the chromatic capacity in terms of the chromatic number. Namely, for any graph G , we have $\chi_{\text{cap}}(G) \leq \chi(G) - 1$ (after all, a proper vertex coloring of G is automatically compatible with any edge coloring). Because of this upper bound on chromatic capacity, the class of graphs satisfying $\chi_{\text{cap}}(G) = \chi(G) - 1$ is of particular interest.

In Section 2, we provide evidence that suggests Greene's Conjecture may in fact be true, by proving that it holds for almost every graph (the notion of "almost every" will be properly defined in Section 2).

In Section 3, we give a generalization of a construction used by Greene in [8, Theorem 3] to demonstrate that the inequality $\chi_{\text{cap}}(G) \leq \chi(G) - 1$ is tight, and use it to show that for every n and every positive integer $k \leq n/2$ there is a graph G with $\chi(G) = n$ and $\chi_{\text{cap}}(G) = n - k$. We also use this generalization to obtain a lower bound on $\chi_{\text{cap}}(K_n^r)$ that is tight as $r \rightarrow \infty$, where K_n^r denotes the complete n -partite graph with r vertices in each part. This improves on the bounds

$$(1 - o(1))\sqrt{n} < \chi_{\text{cap}}(K_n^r) < \min\{n, \sqrt{2er(n-1)}\}$$

originally stated in Greene [8], where the $o(1)$ term goes to 0 as $n \rightarrow \infty$ (independently of r), which reduce to

$$(1 - o(1))\sqrt{n} < \chi_{\text{cap}}(K_n^r) \leq n - 1$$

when r is large. We prove that $\chi_{\text{cap}}(K_n^r) = n - 1$ for sufficiently large r .

In Section 4, we study the chromatic capacity of the product of two graphs for three different types of graph products. We are able to use a result on the chromatic capacity of lexicographic products to show that for any positive integers p and q there exists a graph G with $\chi_{\text{cap}}(G) + 1 = \chi(G) = p$ that contains no odd cycles of length less than q . This partially answers a question of Greene in [8], which asks if there exist graphs with arbitrarily large chromatic number and girth that satisfy the equality $\chi_{\text{cap}}(G) = \chi(G) - 1$ (the girth of a graph is the length of its shortest cycle). Furthermore, by using another theorem about the chromatic capacity of lexicographic products we are able to prove that determining whether a graph has chromatic capacity at most k is NP-hard whenever $k \geq 2$.

In Section 5, we study a number of local graph operations, including vertex deletion, vertex identification, and the Hajós construction. Using these tools we demonstrate that any graph G must contain a subgraph G^* such that $\chi_{\text{cap}}(G^*) = \chi_{\text{cap}}(G)$, $\chi(G^*) = \chi(G)$, and $\delta(G^*) \geq \chi_{\text{cap}}(G)$, where δ denotes the minimum degree.

We conclude with a large number of open problems regarding chromatic capacity.

2 Greene's Conjecture

In [8, Corollary 7], Greene shows that any graph with chromatic number at least 4 must have chromatic capacity at least 2. After making this observation,

he made the following conjecture:

Conjecture 2 (Greene) *There exists an unbounded $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\chi_{\text{cap}}(G) \geq f(\chi(G))$ for every graph G .*

Essentially, Greene's conjecture states that a graph with high chromatic number must necessarily have a high chromatic capacity. Greene's proof that a chromatic number of 4 necessitates a chromatic capacity of at least 2 does not easily generalize to prove this conjecture, as his proof follows from a complete characterization of the graphs with chromatic capacity 1.

While the conjecture remains open, we prove a result that provides some evidence that the conjecture may in fact be true. For $p \in [0, 1]$, we define $\mathcal{G}(n, p)$ to be the probability space whose elements are the graphs on n vertices, where the probability of selecting a fixed graph G with m edges is $p^m(1 - p)^{\binom{n}{2} - m}$. Intuitively, a random graph $G \in \mathcal{G}(n, p)$ on n vertices is chosen by joining two vertices by an edge randomly and independently with probability p . For a graph property \mathcal{P} and a fixed probability p , we say that G has property \mathcal{P} for *almost every* $G \in \mathcal{G}(n, p)$ or that the property \mathcal{P} holds *almost surely* if the probability that a randomly selected $G \in \mathcal{G}(n, p)$ has property \mathcal{P} tends to 1 as $n \rightarrow \infty$. We will prove that Greene's conjecture holds for almost every $G \in \mathcal{G}(n, p)$ whenever $p \in (0, 1)$.

Before proceeding, we recall a well known theorem in the study of random graphs.

Theorem 3 ([6, p. 240]) *Let $p \in (0, 1)$, and let $\varepsilon > 0$. Almost every $G \in \mathcal{G}(n, p)$ satisfies*

$$\chi(G) > \frac{\ln(1/(1-p))}{2+\varepsilon} \cdot \frac{n}{\ln n}.$$

By using this theorem and an inequality in [8], we are able to prove the following theorem:

Theorem 4 *Let $p \in (0, 1)$. There exists an unbounded $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\chi_{\text{cap}}(G) \geq f(\chi(G))$ for almost every $G \in \mathcal{G}(n, p)$.*

PROOF. We recall the result of Greene in [8] that states that

$$\chi_{\text{cap}}(G)^2 \ln \chi_{\text{cap}}(G) > (1 - o(1)) \frac{\chi(G)^2}{2n} \tag{1}$$

for every graph G on n vertices, where the $o(1)$ term goes to zero as $\chi(G) \rightarrow \infty$. Substituting the result of Theorem 3 into this inequality, we find that

$$\chi_{\text{cap}}(G)^2 \ln \chi_{\text{cap}}(G) > (1 - o(1)) \frac{(\ln(1/(1-p)))^2}{8 + \varepsilon'} \cdot \frac{n}{(\ln n)^2}$$

for almost every $G \in \mathcal{G}(n, p)$, where $\varepsilon' > 0$ is arbitrary. The function $x/(\ln x)^2$ is increasing for $x > e^2$, so it follows that if $\chi(G) \geq 8$ then

$$\chi_{\text{cap}}(G)^2 \ln \chi_{\text{cap}}(G) > (1 - o(1)) \frac{(\ln(1/(1-p)))^2}{9} \cdot \frac{\chi(G)}{(\ln \chi(G))^2} \quad (2)$$

almost surely since $\chi(G) \leq n$ for every graph G . As a graph $G \in \mathcal{G}(n, p)$ almost surely has $\chi(G) \geq 8$ (by Theorem 3) and since the left side of equation (2) is increasing in $\chi_{\text{cap}}(G)$ while the right side is unbounded, this implies the existence of such an f . \square

We note that if we approximate the $\ln \chi_{\text{cap}}(G)$ term on the left side of equation (2) from above by $\ln \chi(G)$ then we obtain that

$$\chi_{\text{cap}}(G)^2 > (1 - o(1)) \frac{(\ln(1/(1-p)))^2}{9} \cdot \frac{\chi(G)}{(\ln \chi(G))^3},$$

and therefore that we can pick $f \in \Omega(\sqrt{x/(\ln x)^3})$. Since $\chi_{\text{cap}}(K_n) < 1 + \sqrt{2n}$ and $\chi(K_n) = n$, it follows that any function satisfying the conclusion of Conjecture 2 must be bounded above by $1 + \sqrt{2x}$. Therefore it is somewhat surprising that the f in the conclusion of Theorem 4 is of such a large order considering the way in which it was found. While the lower bound for $\chi(G)$ in Lemma 3 is tight in the sense that if we replace the ε term with $-\varepsilon$ then the lower bound becomes an upper bound (see [6, p. 240]), we do not believe that the inequality (1) used to relate $\chi_{\text{cap}}(G)$ to $\chi(G)$ and n is particularly tight.

3 Pinwheels

In this section, we define a graph operation called the pinwheel, which is a generalization of a construction considered in Greene [8]. The pinwheel construction behaves extremely nicely with respect to both chromatic capacity and chromatic number, and will be very useful later in this section in proving Corollary 8 and Theorem 9.

Definition 5 *For graphs G_1 and G_2 , the join $G_1 + G_2$ is the graph obtained*

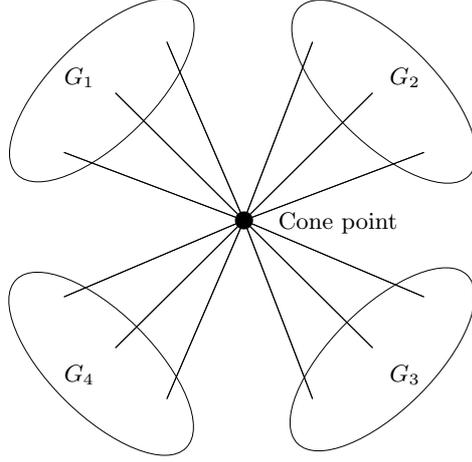


Fig. 1. The pinwheel $\mathcal{P}G$ of a graph G with $\chi_{\text{cap}}(G) = 3$.

from the disjoint union $G_1 \amalg G_2$ by connecting every vertex in G_1 with every vertex in G_2 by an edge.

The cone on G (also called the one-point suspension), denoted by $\mathcal{C}G$, is the graph $G + K_1$. The new vertex in $\mathcal{C}G$ is called the cone point.

The pinwheel on G , $\mathcal{P}G$, is defined by

$$\mathcal{P}G = \mathcal{C} \left(\prod_{i=1}^{\chi_{\text{cap}}(G)+1} G \right).$$

The i th copy of G lying in $\mathcal{P}G$ is called the i th vane, and an edge e joining the cone point to the i th vane is a spoke to the i th vane (see Figure 1).

In [8], Greene defines a sequence of graphs $\{G_n\}$ recursively by letting $G_0 = K_1$ and $G_{n+1} = \mathcal{P}G_n$, and demonstrates that $\chi_{\text{cap}}(G_n) = \chi(G_n) - 1 = n$ for every n . In doing so, he shows that $\chi_{\text{cap}}(\mathcal{P}G) \geq \chi_{\text{cap}}(G) + 1$ for an infinite family of graphs. As the following theorem demonstrates, this inequality holds for every graph, and is in fact tight for every graph G . In addition, for every graph G we have the equality $\chi(\mathcal{P}G) = \chi(G) + 1$. Considering that one of the main questions regarding the chromatic capacity involves its relation with the chromatic number, this is quite useful.

Theorem 6 For any graph G , the pinwheel $\mathcal{P}G$ satisfies $\chi_{\text{cap}}(\mathcal{P}G) = \chi_{\text{cap}}(G) + 1$ and $\chi(\mathcal{P}G) = \chi(G) + 1$.

PROOF. The statement about χ is straightforward: since $\chi(\mathcal{C}G) = \chi(G) + 1$ and since $\mathcal{P}G \supset \mathcal{C}G$, we get that $\chi(\mathcal{P}G) \geq \chi(G) + 1$. However, $\chi(\mathcal{P}G)$ cannot be any larger than this, for we can color each vane properly using

$\chi(G)$ colors and use a new color for the cone point to obtain a proper vertex coloring of $\mathcal{P}G$ using only $\chi(G) + 1$ colors.

Let $k = \chi_{\text{cap}}(G)$. We will now prove that $\chi_{\text{cap}}(\mathcal{P}G) = k + 1$ in two steps, first showing that $\chi_{\text{cap}}(\mathcal{P}G) \geq k + 1$ by producing an emulsion of $\mathcal{P}G$ using $k + 1$ colors. Denote by G_i the i th vane of $\mathcal{P}G$, and pick an emulsive k -coloring $c_i: E(G_i) \rightarrow [k + 1] \setminus \{i\}$ of the edges of G_i for $1 \leq i \leq k + 1$. From here we piece the vane colorings together into a coloring of $\mathcal{P}G$, defining $c: E(\mathcal{P}G) \rightarrow [k + 1]$ by

$$c(e) = \begin{cases} c_i(e) & \text{if } e \in E(G_i) \\ i & \text{if } e \text{ is a spoke to } G_i. \end{cases}$$

We claim that c is an emulsion. For suppose that $b: V(\mathcal{P}G) \rightarrow [k + 1]$ is compatible with c . The cone point is some color i , and it follows that none of the vertices in the vane G_i can be colored color i , for otherwise there would be a monochromatic spoke for G_i by the definition of c . Hence the vertices in the vane G_i must be colored using the k colors $[k + 1] \setminus \{i\}$. However, since the coloring c_i of $E(G_i)$ is emulsive (and uses the same color set as the vertex coloring of G_i), it follows that there must be some monochromatic edge in G_i . Therefore c is emulsive, and $\chi_{\text{cap}}(\mathcal{P}G) \geq k + 1$.

To prove that $\chi_{\text{cap}}(\mathcal{P}G) \leq k + 1$, let an edge coloring $c: E(\mathcal{P}G) \rightarrow [k + 2]$ be arbitrary. Define $f: [k + 2] \rightarrow [k + 1]$ by $f(k + 2) = k + 1$ and $f(x) = x$ for $x \neq k + 2$. Now the composition $f \circ c|_{E(G_i)}$ defines an edge $(k + 1)$ -coloring of the i th vane G_i of $\mathcal{P}G$. However, $\chi_{\text{cap}}(G_i) = k$, so there must exist a vertex coloring $b_i: V(G_i) \rightarrow [k + 1]$ that is compatible with $f \circ c|_{E(G_i)}$. We now combine the colorings of the vertex sets of the vanes into a coloring of $\mathcal{P}G$ by defining $b: V(\mathcal{P}G) \rightarrow [k + 2]$ by

$$b(v) = \begin{cases} b_i(v) & \text{if } v \in V(G_i) \\ k + 2 & \text{if } v \text{ is the cone point.} \end{cases}$$

None of the spokes is monochromatic since the cone point is a different color than every other vertex, while no edge in a vane is monochromatic since b_i is compatible with $f \circ c|_{E(G_i)}$, hence with $c|_{E(G_i)}$ as well (this is a less severe restriction since b_i does not use the color $k + 2$). Therefore b is compatible with c , so $\chi_{\text{cap}}(\mathcal{P}G)$ is at most $k + 1$. \square

With Theorem 6, we are able to easily characterize the effect of the cone operation on the chromatic capacity. For Corollary 7 and much of the rest of the paper we will require the fact that χ_{cap} is a monotonic increasing function.

That is, if G is a subgraph of H , which we write $G \subset H$, then $\chi_{\text{cap}}(G) \leq \chi_{\text{cap}}(H)$. The proof of this is simple: any emulsive edge coloring of G can be extended to an emulsive edge coloring of H by coloring the edges in $E(H) - E(G)$ arbitrarily.

Corollary 7 *The cone $\mathcal{C}G$ satisfies $\chi_{\text{cap}}(G) \leq \chi_{\text{cap}}(\mathcal{C}G) \leq \chi_{\text{cap}}(G) + 1$.*

PROOF. This follows immediately from the monotonicity of χ_{cap} together with Theorem 6 and the fact that $G \subset \mathcal{C}G \subset \mathcal{P}G$. \square

The above inequality is in fact tight. This can be seen since $\mathcal{C}K_4 = K_5$ and $\mathcal{C}K_3 = K_4$, while $\chi_{\text{cap}}(K_3) = 1$ and $\chi_{\text{cap}}(K_4) = \chi_{\text{cap}}(K_5) = 2$. Now we will use the pinwheel construction to generalize the result of Greene [8] that for every integer n there exists a graph G such that $n = \chi_{\text{cap}}(G) = \chi(G) - 1$.

Corollary 8 *For every positive integer k there exists a positive integer N such that for every $n \geq N$ there exists a graph G with $\chi(G) = n$ and $\chi_{\text{cap}}(G) = n - k$. Moreover, we can pick $N \leq \lceil 1 + k + \sqrt{1 + 2k} \rceil$.*

PROOF. We recall a result of Erdős and Gyárfás [7, Theorem 4], which states that $\chi_{\text{cap}}(K_n) < 1 + \sqrt{2n}$. Define $a_n = \chi(K_n) - \chi_{\text{cap}}(K_n)$ for every n . Since $\chi_{\text{cap}}(K_n) < 1 + \sqrt{2n}$ and $\chi(K_n) = n$, we find that a_n is an unbounded sequence of integers. Furthermore, $a_n - a_{n-1}$ is always either 0 or 1 by Corollary 7 since $K_n = \mathcal{C}K_{n-1}$. Since $a_1 = 1$, this implies that $\{a_n\}$ takes on every positive integer value. Let N be the least integer such that $a_N = k$. Taking the pinwheel of K_N repeatedly gives the first conclusion by Theorem 6. Also, because $a_n > n - 1 - \sqrt{2n}$ for every n we get that

$$a_{\lceil 1+k+\sqrt{1+2k} \rceil} > k + \sqrt{1+2k} - \sqrt{2+2k+2\sqrt{1+2k}} = k - 1,$$

so N is at most $\lceil 1 + k + \sqrt{1 + 2k} \rceil$. \square

The estimate $N \leq 2k$, which is stated in the abstract and introduction, follows easily from the inequality $\lceil 1 + k + \sqrt{1 + 2k} \rceil \leq 2k$ that holds for $k \geq 4$.

Next we will use the pinwheel construction to prove a theorem of an entirely different sort. In the following theorem, we improve on bounds originally stated in [8] by Greene for $\chi_{\text{cap}}(K_n^r)$, where K_n^r denotes the complete n -partite graph with r vertices in each part. Greene noted that

$$(1 - o(1))\sqrt{n} < \chi_{\text{cap}}(K_n^r) < \min\{n, \sqrt{2er(n-1)}\}$$

where the $o(1)$ term goes to 0 as $n \rightarrow \infty$ (independently of r). The above lower bound follows from the monotonicity of χ_{cap} and the inclusion $K_n \subset K_n^r$. Greene points out that for large r this reduces to

$$(1 - o(1))\sqrt{n} < \chi_{\text{cap}}(K_n^r) \leq n - 1,$$

leaving a large gap. The following theorem improves on the lower bound, demonstrating that $\chi_{\text{cap}}(K_n^r) = n - 1$ for sufficiently large r .

Theorem 9 *If $r \geq \lfloor e(m-1)! \rfloor$, where $n \geq m$, then $\chi_{\text{cap}}(\mathcal{C}K_n^r) \geq m$. In particular, if $r \geq \lfloor e(n-2)! \rfloor$, then $\chi_{\text{cap}}(K_n^r) = n - 1$.*

PROOF. Following the construction of Greene in [8], we let $G^0 = K_1$, and define $G^n = \mathcal{P}G^{n-1}$. By Theorem 6, it follows that $\chi(G^n) = n + 1$ and $\chi_{\text{cap}}(G^n) = n$. Since $\chi(G^n) = n + 1$, there exists a proper vertex coloring $c_1: V(G_1^{n-1}) \rightarrow [n]$ of the first vane $G_1^{n-1} \subset G^n$. For $2 \leq i \leq n$, define $c_i: V(G_i^{n-1}) \rightarrow [n]$ by $c_{i+1}(v) = c_i(v) + 1$, where the addition is modulo n . Defining $c: V(G^n) \rightarrow [n + 1]$ by $c(v) = c_i(v)$ for $v \in G_i^{n-1}$ and by coloring the cone point color $n + 1$, we see that c is a proper vertex coloring of G_n with $n + 1$ color classes. Moreover, the color classes $1, \dots, n$ each have an equal number of vertices, and the color class $n + 1$ has a single vertex. From the construction of G^n it is clear that $|V(G^n)| = n|V(G^{n-1})| + 1$, so it follows that the color classes $1, \dots, n$ each have $|V(G^{n-1})|$ vertices. Thus G^n can be embedded in $\mathcal{C}K_n^{|V(G^{n-1})|}$, and $\chi_{\text{cap}}(\mathcal{C}K_n^{|V(G^{n-1})|}) \geq n$ by the monotonicity of χ_{cap} . However, since $\chi_{\text{cap}}(\mathcal{C}K_n^{|V(G^{n-1})|}) \leq \chi(\mathcal{C}K_n^{|V(G^{n-1})|}) - 1 = n$, we have that $\chi_{\text{cap}}(\mathcal{C}K_n^{|V(G^{n-1})|}) = n$.

From the recurrence relation $|V(G^n)| = n|V(G^{n-1})| + 1$ it is easy to see that

$$|V(G^{n-1})| = (n-1)! \left(\sum_{k=0}^{n-1} \frac{1}{k!} \right) \leq \left\lfloor (n-1)! \left(\sum_{k=0}^{\infty} \frac{1}{k!} \right) \right\rfloor = \lfloor e(n-1)! \rfloor,$$

with equality when $n \geq 2$. Thus whenever $r \geq \lfloor e(n-1)! \rfloor$ it follows by the monotonicity of χ_{cap} that $\chi_{\text{cap}}(\mathcal{C}K_n^r) = n$. Moreover, if $m \leq n$ then $\mathcal{C}K_m^r \subset \mathcal{C}K_n^r$, so whenever $r \geq \lfloor e(m-1)! \rfloor$ we get $\chi_{\text{cap}}(\mathcal{C}K_n^r) \geq \chi_{\text{cap}}(\mathcal{C}K_m^r) = m$. \square

4 Graph products and vertex splitting

Our motivation for studying graph products primarily comes from the fact that K_n^r is precisely the lexicographic product of K_n with the empty graph on r vertices. Phrased slightly differently, our lower bound on $\chi_{\text{cap}}(K_n^r)$ demonstrates

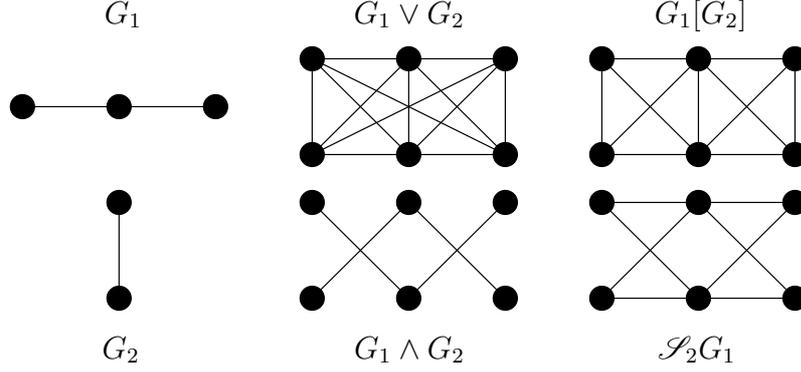


Fig. 2. The graph products of Definition 10 and the 2-split of the graph G_1 .

that if r is large enough then the chromatic capacity of the lexicographic product $K_n[\overline{K_r}]$ is just $n - 1$. Noting that $\chi(K_n^r) = n$, a natural question to ask is whether a similar theorem holds for general graphs. That is, for every graph G , does $\chi_{\text{cap}}(G[\overline{K_r}]) = \chi(G) - 1$ if r is sufficiently large? The answer turns out to be affirmative, as Theorem 11 demonstrates, and this fact turns out to be incredibly useful.

Definition 10 *Let G_1 and G_2 be graphs. We denote by $G_1 \wedge G_2$, $G_1 \vee G_2$, and $G_1[G_2]$ the conjunction (or categorical product), disjunction, and lexicographic product (or graph composition), respectively. In each of the three products, the vertex set is $V(G_1) \times V(G_2)$. The edge sets of the three products are defined as follows:*

$$\begin{aligned} E(G_1 \wedge G_2) &= \{(v_1, w_1)(v_2, w_2) : v_1 \sim v_2 \text{ and } w_1 \sim w_2\}, \\ E(G_1 \vee G_2) &= \{(v_1, w_1)(v_2, w_2) : v_1 \sim v_2 \text{ or } w_1 \sim w_2\}, \text{ and} \\ E(G_1[G_2]) &= \{(v_1, w_1)(v_2, w_2) : v_1 \sim v_2 \text{ or } (v_1 = v_2 \text{ and } w_1 \sim w_2)\}, \end{aligned}$$

where \sim denotes adjacency (see Figure 2).

For G a graph, the r -split of G , denoted $\mathcal{S}_r G$, is the graph $G \vee \overline{K_r}$, the disjunction of G with the empty graph on r vertices.

For instance, the r -split of K_n is just K_n^r . Equivalently, we could have defined $\mathcal{S}_r G \equiv G[\overline{K_r}]$. In order to make the notation less cumbersome, it will be helpful to view $\mathcal{S}_r G$ as r distinct copies G_1, \dots, G_r of G . For a vertex $v \in V(G)$, we will also denote by v the copy of v in G_i . However, we will be very careful to always mention precisely which copy of v we are considering, so this should not cause any confusion. Now vertices $v \in V(G_i)$ and $w \in V(G_j)$ are connected by an edge in $\mathcal{S}_r G$ if and only if v is adjacent to w in G .

It is obvious that \mathcal{S}_r obeys the relation $\chi(\mathcal{S}_r G) = \chi(G)$; given a proper vertex coloring of G we may define one on $\mathcal{S}_r G$ by coloring the vertex $v \in V(G_i)$ the same color as $v \in V(G)$ for every v and i . The following theorem

demonstrates that even though the chromatic number does not increase when we r -split G , the chromatic capacity will always achieve the maximum value of $\chi(G) - 1$ if r is large enough. Since $\mathcal{S}_r K_n = K_n^r$, the following theorem also essentially serves as a generalization of Theorem 9 (as noted at the beginning of this section), although Theorem 9 provides substantially better bounds when splitting the vertices of complete graphs.

Theorem 11 *Let G be a graph on n vertices, and let $k < \chi(G)$. If r is a positive integer with*

$$r \geq \frac{k^{\lfloor kn/2 \rfloor + 1} - 1}{k - 1},$$

then $\chi_{\text{cap}}(\mathcal{S}_r G) \geq k$. In particular, for r sufficiently large, $\chi_{\text{cap}}(\mathcal{S}_r G) = \chi(\mathcal{S}_r G) - 1 = \chi(G) - 1$.

PROOF. Define $d = \lfloor kn/2 \rfloor + 1$. By the monotonicity of χ_{cap} , it suffices to show that if

$$r = \frac{k^d - 1}{k - 1} = \sum_{j=0}^{d-1} k^j$$

then $\chi_{\text{cap}}(\mathcal{S}_r G) \geq k$. View $\mathcal{S}_r G$ as r copies of G , indexed by sequences of the form $1, a_2, \dots, a_p$, where $p \leq d$ and $a_i \in [k]$. Now vertices $v \in V(G_{1,a_2,\dots,a_p})$ and $w \in V(G_{1,b_2,\dots,b_q})$ are adjacent in $\mathcal{S}_r G$ exactly when v and w are adjacent in G . If $1, a_2, \dots, a_p$ and $1, b_2, \dots, b_q$ are sequences with $p < q$ such that $a_i = b_i$ for all $i \leq p$, color all the edges joining G_{1,a_2,\dots,a_p} to G_{1,b_2,\dots,b_q} the color b_{p+1} . For example, all the edges joining G_1 to $G_{1,2}$ would be colored color 2, and all the edges joining $G_{1,4}$ to $G_{1,4,5,7}$ would be colored color 5. Color all the other edges of $\mathcal{S}_r G$ arbitrarily (see Figure 3 for an example of this edge coloring). We claim that this edge coloring is an emulsion.

By way of contradiction, suppose that $b: V(\mathcal{S}_r G) \rightarrow [k]$ is compatible with the edge coloring. Since $k < \chi(G)$, there must exist an edge $v_1 w_1 \in E(G_1)$ with both vertices colored the same; say $b(v_1) = b(w_1)$, and define this color to be c_1 . Similarly, there must be an edge $v_2 w_2 \in E(G_{1,c_1})$ with $b(v_2) = b(w_2)$, and we define c_2 to be this color. Continuing inductively gives us a sequence of colors c_1, \dots, c_d and edges $v_1 w_1, \dots, v_d w_d$ such that the color c_i appears on both vertices of the edge $v_i w_i$ in $G_{1,c_1,\dots,c_{i-1}}$. We claim that the assumption that b is compatible implies that if copies of the vertex $v \in V(G)$ appear in two different edges $v_i w_i \in E(G_{1,c_1,\dots,c_{i-1}})$ and $v_j w_j \in E(G_{1,c_1,\dots,c_{j-1}})$ in this sequence, then $c_i \neq c_j$ (see Figure 4).

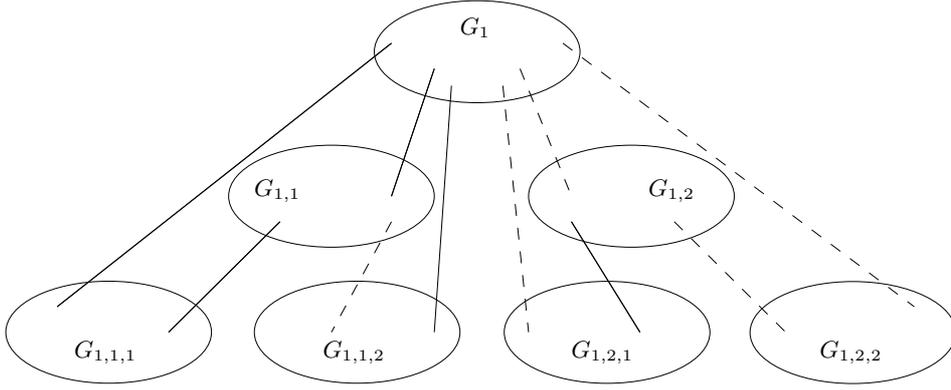


Fig. 3. The edge coloring of $\mathcal{S}_r G$ in the proof of Theorem 11 in the case where $n = 2$ and $k = 2$ so that $d = 3$ and $r = 7$. Solid lines represent color 1, while dashed lines represent color 2.

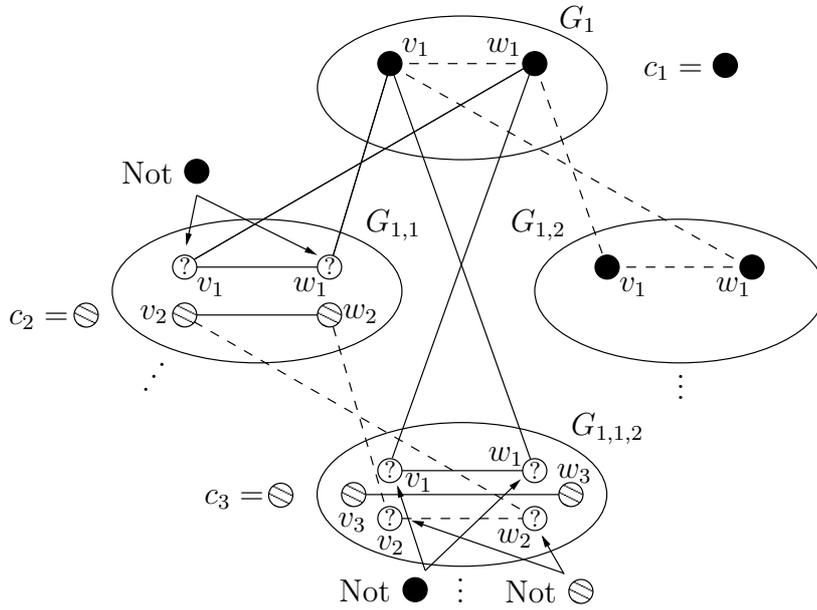


Fig. 4. A portion of the sequence of edges $v_i w_i$ and colors c_i for a compatible vertex coloring b in the proof of Theorem 11. Solid lines and vertices represent color 1, while dashed lines and vertices represent color 2. As both vertices v_1 and w_1 are color 1 in this example, we have that $c_1 = 1$. Similarly, $c_2 = c_3 = 2$. Notice that the copies of v_1 and w_1 in $G_{1,c_1} = G_{1,1}$ and $G_{1,c_1,c_2} = G_{1,1,2}$ are restricted from being color $c_1 = 1$ by the assumption that b is compatible. Similarly, the copies of v_2 and w_2 in $G_{1,c_1,c_2} = G_{1,1,2}$ are restricted from being color $c_2 = 2$.

For assume that $c_i = c_j$, and without loss of generality assume that $i < j$ and that both v_i and v_j are copies of v . Then v_i is joined to w_j by an edge, and this edge is colored c_j . However, both v_i and w_j are colored color c_j as well since $b(v_i) = c_i$ and $b(w_j) = c_j$, contradicting the compatibility of b .

Thus if $v \in V(G)$ appears in $k + 1$ different edges of the sequence then there must be $k + 1$ different colors with which these copies are colored. However, we

are restricted to using k colors, so this is not possible. The pigeonhole principle asserts that if d is at least $\lfloor kn/2 \rfloor + 1$ then there must be some vertex in the edge sequence that appears at least $k + 1$ times. But d was chosen this large, so b cannot be compatible, giving a contradiction. Therefore the edge coloring is an emulsion, and $\chi_{\text{cap}}(\mathcal{S}_r G) \geq k$. \square

We can use Theorem 11 to determine some lower bounds for the chromatic capacity of disjunctions, conjunctions, and lexicographic products immediately.

Corollary 12 *Let G_1 be a graph on n vertices and let $k < \chi(G_1)$. If r is a positive integer such that*

$$r \geq \frac{k^{\lfloor kn/2 \rfloor + 1} - 1}{k - 1},$$

then $\chi_{\text{cap}}(G_1[G_2]) \geq k$ and $\chi_{\text{cap}}(G_1 \vee G_2) \geq k$ whenever $|V(G_2)| \geq r$.

PROOF. If $|V(G_2)| \geq r$ then $G_2 \supset \overline{K_r}$. In particular, $G_1[G_2] \supset G_1[\overline{K_r}] = \mathcal{S}_r G_1$, so $\chi_{\text{cap}}(G_1[G_2]) \geq \chi_{\text{cap}}(\mathcal{S}_r G_1) \geq k$ by Theorem 11. The statement about disjunctions now follows from the monotonicity of χ_{cap} together with the inclusion $G_1[G_2] \subset G_1 \vee G_2$. \square

Corollary 13 *Let G_1 be a graph on n vertices and let $k < \chi(G_1)$. If r is a positive integer such that*

$$r \geq \frac{\chi(G_1)(k^{\lfloor kn/2 \rfloor + 1} - 1)}{k - 1},$$

then $\chi_{\text{cap}}(G_1 \wedge G_2) \geq k$ whenever $G_2 \supset K_r$. If $k = \chi(G_1) - 1$, then equality holds.

PROOF. Let $c: V(G_1) \rightarrow [\chi(G_1)]$ be a proper vertex coloring. Label the vertices of $K_{\chi(G_1)}$ as $1, \dots, \chi(G_1)$, and define $\varphi: G_1 \rightarrow G_1 \wedge K_{\chi(G_1)}$ by $\varphi(v) = (v, c(v))$. It is not hard to see that φ is in fact an embedding of G_1 in $G_1 \wedge K_{\chi(G_1)}$. Since we can embed one copy of G_1 in $G_1 \wedge K_{\chi(G_1)}$, it is easy to see that we can embed $\mathcal{S}_r G_1$ in $G_1 \wedge K_{r, \chi(G_1)}$. It therefore follows that

$$\mathcal{S}_{\lfloor \frac{r}{\chi(G_1)} \rfloor} G_1 \subset G_1 \wedge K_r.$$

If $G_2 \supset K_r$, then

$$\chi_{\text{cap}}(G_1 \wedge G_2) \geq \chi_{\text{cap}}(G_1 \wedge K_r) \geq \chi_{\text{cap}}\left(\mathcal{S}_{\lfloor \frac{r}{\chi(G_1)} \rfloor} G_1\right) \geq k$$

by Theorem 11.

If $k = \chi(G_1) - 1$, then since the projection $G_1 \wedge G_2 \rightarrow G_1$ preserves adjacency, we get that $\chi(G_1 \wedge G_2) \leq \chi(G_1)$. Hence $\chi_{\text{cap}}(G_1 \wedge G_2) \leq \chi(G_1) - 1$, and equality holds. \square

Theorem 11 also provides the framework for proving a seemingly unrelated result. In [8], Greene asks whether there exist graphs G with arbitrarily large girth and arbitrarily large chromatic number that achieve the maximum chromatic capacity of $\chi_{\text{cap}}(G) = \chi(G) - 1$. While this problem remains open, we prove a similar result here. Define $\text{og}(G)$ to be the length of the shortest odd cycle in G . By using the operation of vertex splitting, we will construct graphs with arbitrarily large values of the invariants χ and og such that the equality $\chi_{\text{cap}}(G) + 1 = \chi(G)$ holds. Before proceeding with a proof of this result, we will need a simple theorem that relates og to \mathcal{S}_r .

Theorem 14 *Let G be a graph, and let $r \geq 1$. Then $\text{og}(\mathcal{S}_r G) = \text{og}(G)$.*

PROOF. For a vertex $v \in V(G)$, we define $\mathcal{S}^v G$, the *split of G at the vertex v* , to be the graph obtained from the disjoint union $G \amalg \{v'\}$ by joining v' to all the neighbors of v with an edge. We will show that $\text{og}(\mathcal{S}^v G) = \text{og}(G)$ for every graph G ; we can then apply induction to this result to arrive at the desired conclusion since $\mathcal{S}_r G$ can be obtained from G by splitting each individual vertex $r - 1$ times.

Fix $v \in V(G)$. The inequality $\text{og}(\mathcal{S}^v G) \leq \text{og}(G)$ is evident from the relation $\mathcal{S}^v G \supset G$. Denote by v' the new vertex in $\mathcal{S}^v G$. Suppose that $v_1 v_2 \cdots v_k v_1$ is an odd cycle in $\mathcal{S}^v G$; we will show that there is an odd cycle in G with length at most k . If v and v' are not both contained in the cycle, then by replacing any instance of v' with v we can embed the cycle in G . Thus we may assume that both v and v' occur. Say $v = v_1$ and $v' = v_j$. Now one of the cycles $v v_2 \cdots v_{j-1} v$ or $v v_{j+1} \cdots v_k v$ is odd since the cycle $v v_2 \cdots v_{j-1} v' v_{j+1} \cdots v_k v$ is odd. However, this cycle also has length less than k , so $\text{og}(\mathcal{S}^v G) \geq \text{og}(G)$. \square

Corollary 15 *Let p and q be positive integers. There exists a graph G with $\chi(G) = \chi_{\text{cap}}(G) + 1 = p$ and $\text{og}(G) \geq q$.*

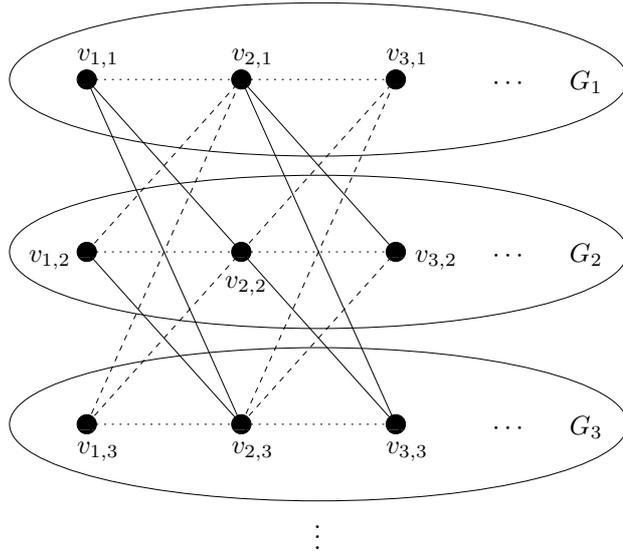


Fig. 5. A portion of the edge coloring c of $\mathcal{S}_{2 \cdot |E(G)|+1}G$ in the proof of Theorem 16. Solid lines represent color 1, dashed lines represent color 2, and dotted lines represent color 3. Here the subgraph of G induced by v_1 , v_2 , and v_3 is isomorphic to the path on 3 vertices.

PROOF. The case when $p \leq 2$ is trivial, so we may assume that $p \geq 3$. The *Kneser graph* $KG_{n,k}$ is the graph whose vertices are the n -element subsets of $[2n + k]$, with two vertices being adjacent if they are disjoint subsets. In [12], Lovász shows that $\chi(KG_{n,k}) = k + 2$, and he mentions that $\text{og}(KG_{n,k}) \geq 2n/k + 1$. In particular, $KG_{\lceil (q-1)(p-2)/2 \rceil, p-2}$ has chromatic number p and contains no odd cycles of length less than q . Now if we pick r large enough, the r -split of this graph has all the desired properties by Theorems 11 and 14.

For a nonconstructive proof, one could simply use Erdős' theorem on the existence of graphs with high girth and chromatic number (see [1, p. 35]). \square

We are uncertain as to how good the bounds in Theorem 11 are, although we suspect that they are very far from being tight. For graphs with chromatic number 4, we have been able to do much better, changing a bound that is exponential in the number of vertices to one that is roughly quadratic. As an interesting consequence of the following theorem, we will be able to show that determining whether a graph has chromatic capacity at most k is NP-hard.

Theorem 16 *Let G be a graph with $\chi(G) \geq 4$. Then*

$$\chi_{\text{cap}}(\mathcal{S}_{2 \cdot |E(G)|+1}G) \geq 3.$$

PROOF. Enumerate the vertices of G as v_1, \dots, v_n . Let $k = 2 \cdot |E(G)| + 1$, and view $\mathcal{S}_k G$ as k copies G_1, \dots, G_k of G together with the appropriate additional edges between them. Denote by $v_{i,j}$ the copy of v_i lying in G_j . Define $c: E(\mathcal{S}_k G) \rightarrow [3]$ as follows: let $v_{i,j}$ and $v_{k,l}$ be adjacent vertices in $\mathcal{S}_k G$, and assume that $i < k$. Now set

$$c(v_{i,j}v_{k,l}) = \begin{cases} 1 & \text{if } j < l \\ 2 & \text{if } j > l \\ 3 & \text{if } j = l. \end{cases}$$

We claim that this edge coloring, illustrated in Figure 5, is an emulsion.

By way of contradiction, suppose that the vertex coloring $b: V(\mathcal{S}_k G) \rightarrow [3]$ is compatible with c . Since $\chi(G) \geq 4$, for every j with $1 \leq j \leq k$ there exists an edge $e_j = v_{p_j,j}v_{q_j,j}$ with $p_j < q_j$ such that $b(v_{p_j,j}) = b(v_{q_j,j})$. Since $k = 2 \cdot |E(G)| + 1$, the pigeonhole principle asserts that there are 3 distinct numbers x, y , and z in $[k]$ for which $p_x = p_y = p_z$ and $q_x = q_y = q_z$. That is, there are distinct x, y , and z such that the edges e_x, e_y , and e_z are all isomorphic copies of the same edge of G under the canonical isomorphism. Now none of e_x, e_y , or e_z can have their vertices colored color 3 since $c(e_x) = c(e_y) = c(e_z) = 3$, so by the pigeonhole principle there are distinct $s, t \in \{x, y, z\}$ with $s < t$ such that the vertices of e_s and e_t are all colored using only one of the colors 1 or 2. By the definition of c , we have that $c(v_{p_s,s}v_{q_t,t}) = 1$ and $c(v_{p_t,t}v_{q_s,s}) = 2$. Hence if the vertices of e_s and e_t are all color 1 then the edge $v_{p_s,s}v_{q_t,t}$ is monochromatic, and if the vertices of e_s and e_t are all color 2 then the edge $v_{p_t,t}v_{q_s,s}$ is monochromatic. Either way, b cannot be compatible with c , which is a contradiction. Hence c is an emulsion. \square

Theorem 17 *Determining whether a graph has chromatic capacity at most 2 is NP-hard.*

PROOF. Since $|E(G)| < |V(G)|^2$ for every graph G , we can obtain $\mathcal{S}_{2 \cdot |E(G)| + 1} G$ from G in some number of steps bounded by a polynomial in $|V(G)|$. Together with the trivial bound $\chi_{\text{cap}}(G) \leq \chi(G) - 1$, Theorem 16 asserts that $\chi_{\text{cap}}(\mathcal{S}_{2 \cdot |E(G)| + 1}) \geq 3$ if and only if $\chi(G) \geq 4$. Therefore $\chi_{\text{cap}}(\mathcal{S}_{2 \cdot |E(G)| + 1}) \leq 2$ if and only if G is 3-colorable. Since graph 3-coloring is an NP-complete problem (see [15, p. 246]), this implies that determining when the chromatic capacity of a graph is at most 2 is NP-hard. \square

Corollary 18 *If $k \geq 2$, then determining whether a graph has chromatic capacity at most k is NP-hard.*

PROOF. Let G be a graph, define $G_0 = G$, and recursively define

$$G_n = \mathcal{C} \left(\prod_{i=1}^{2k} G_{n-1} \right).$$

If $\chi_{\text{cap}}(G) \leq k$, then $G_{k-2} \supset \mathcal{P}^{k-2}G$, where the exponent of \mathcal{P} denotes repeated application, so $\chi_{\text{cap}}(G_{k-2}) \geq \chi_{\text{cap}}(G) + k - 2$. On the other hand, an argument identical to the proof in Theorem 6 that $\chi_{\text{cap}}(\mathcal{P}G) \leq \chi_{\text{cap}}(G) + 1$ for every graph G shows that $\chi_{\text{cap}}(G_{j+1}) \leq \chi_{\text{cap}}(G_j) + 1$ for every j . Therefore $\chi_{\text{cap}}(G_{k-2}) \leq \chi_{\text{cap}}(G) + k - 2$, so $\chi_{\text{cap}}(G_{k-2}) = \chi_{\text{cap}}(G) + k - 2$ for every graph G with $\chi_{\text{cap}}(G) \leq k$. If $\chi_{\text{cap}}(G) \geq k + 1$, then since $G_{k-2} \supset G$ we have $\chi_{\text{cap}}(G_{k-2}) \geq k + 1$ as well. Hence $\chi_{\text{cap}}(G) \leq 2$ if and only if $\chi_{\text{cap}}(G_{k-2}) \leq k$. However, G_{k-2} can be obtained from G in a number of steps that is bounded by a polynomial in $|V(G)|$. Therefore, by Theorem 17, it follows that determining whether a graph has chromatic capacity at most k is NP-hard. \square

5 Local graph operations

In this section we study the interactions between χ_{cap} and some graph operations that are more local in nature than the pinwheels and graph products considered in Sections 3 and 4. Our first result concerns the deletion of vertices.

Proposition 19 *Suppose that G' is a graph obtained from G by deleting a vertex v_0 . Then $\chi_{\text{cap}}(G) - 1 \leq \chi_{\text{cap}}(G') \leq \chi_{\text{cap}}(G)$. If the degree $d(v_0)$ of v_0 is less than $\chi_{\text{cap}}(G)$, then $\chi_{\text{cap}}(G') = \chi_{\text{cap}}(G)$.*

PROOF. The inequality follows immediately from Corollary 7, the monotonicity of χ_{cap} , and the inclusions $G' \subset G \subset \mathcal{C}G'$. Set $k = \chi_{\text{cap}}(G)$, and suppose that $d(v_0) < k$. By the monotonicity of χ_{cap} , it suffices to show that $\chi_{\text{cap}}(G') \geq \chi_{\text{cap}}(G)$. Let $c: E(G) \rightarrow [k]$ be an emulsion, and let $c': E(G') \rightarrow [k]$ be induced by restriction. Suppose that $b': V(G') \rightarrow [k]$ is compatible with c' . Since $d(v_0) < k$, there is some color m that does not appear on any of the neighbors of v_0 . Define $b: V(G) \rightarrow [k]$ by

$$b(v) = \begin{cases} b'(v) & \text{if } v \neq v_0 \\ m & \text{if } v = v_0. \end{cases}$$

It follows immediately that b is compatible with c since b' is compatible with c' due to the way in which m was picked, which contradicts the assumption

that c is an emulsion. Therefore b' cannot be compatible with c' , and c' is an emulsion. \square

Corollary 20 *If $vw \in E(G)$ and $G' = G - vw$, then $\chi_{\text{cap}}(G) - 1 \leq \chi_{\text{cap}}(G') \leq \chi_{\text{cap}}(G)$. If either $d(v) < \chi_{\text{cap}}(G)$ or $d(w) < \chi_{\text{cap}}(G)$, then $\chi_{\text{cap}}(G') = \chi_{\text{cap}}(G)$.*

PROOF. Denote by G'' the graph obtained from G by deleting the vertex v . Then $G'' \subset G'$, so $\chi_{\text{cap}}(G) - 1 \leq \chi_{\text{cap}}(G'') \leq \chi_{\text{cap}}(G')$ by Proposition 19. The conclusion when $d(v) < \chi_{\text{cap}}(G)$ or $d(w) < \chi_{\text{cap}}(G)$ follows similarly. \square

A similar version of Proposition 19 holds with respect to the chromatic number; we state and prove it here as a lemma, since it will be an instrumental result in proving Theorem 22.

Lemma 21 *If G' is obtained from G by deleting a vertex v_0 with $d(v_0) < \chi(G) - 1$, then $\chi(G') = \chi(G)$.*

PROOF. By the monotonicity of χ , we have $\chi(G) \geq \chi(G')$; we show the reverse inequality. Set $j = \chi(G)$. To obtain a contradiction, suppose that $c': V(G') \rightarrow [j - 1]$ is a proper vertex coloring of G' using $j - 1$ colors. Since $d(v_0) < j - 1$, there is an $m \in [j - 1]$ such that none of the neighbors of v_0 is colored m . Therefore we can obtain a proper coloring $c: V(G) \rightarrow [j - 1]$ by setting $c(v_0) = m$ and $c(v) = c'(v)$ for all $v \neq v_0$, giving a contradiction since any proper coloring of the vertex set of G must use at least j colors. \square

A simple result regarding the chromatic number is that any graph of chromatic number k always contains a subgraph with minimum degree $k - 1$ and chromatic number k (see [6, p. 116]). An extremely similar result that we will prove shortly holds with respect to the chromatic capacity. Using Proposition 19 and Lemma 21, we can show that any graph G with chromatic capacity k must contain a subgraph G^* with minimum degree at least k and the same chromatic number and chromatic capacity as G . This result effectively demonstrates that whenever we are considering relations between χ_{cap} and χ for graphs G it suffices to consider the case where the graph G has minimum degree at least $\chi_{\text{cap}}(G)$.

Theorem 22 *For any graph G there exists a subgraph $G^* \subset G$ such that $\chi_{\text{cap}}(G^*) = \chi_{\text{cap}}(G)$, $\chi(G^*) = \chi(G)$, and minimum degree $\delta(G^*) \geq \chi_{\text{cap}}(G)$.*

PROOF. Set $G_0 = G$. Obtain G_{i+1} from G_i as follows: if $\delta(G_i) \geq \chi_{\text{cap}}(G)$, set $G_{i+1} = G_i$. Otherwise, pick a vertex of degree less than $\chi_{\text{cap}}(G)$ in G_i and

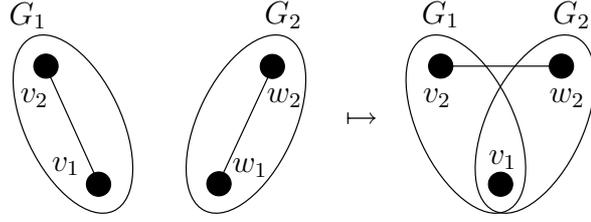


Fig. 6. The Hajós construction applied to two graphs G_1 and G_2 with respect to the edges v_1v_2 and w_1w_2 .

obtain G_{i+1} from G_i by deleting this vertex. Applying induction to Proposition 19 and Lemma 21 and using the inequality $\chi_{\text{cap}}(G) \leq \chi(G) - 1$, we see that $\chi_{\text{cap}}(G_i) = \chi_{\text{cap}}(G)$ and $\chi(G_i) = \chi(G)$ for every i . This sequence must eventually stabilize since G is a finite graph and since the number of vertices in G_i is nonincreasing in i , so there is some n for which $G_n = G_{n+1}$. Since $G_n = G_{n+1}$, we have that $\delta(G_n) \geq \chi_{\text{cap}}(G)$. Choosing $G^* = G_n$ completes the proof. \square

For the remainder of this section, we will consider two local graph operations that are extremely important in the theory of chromatic numbers. We define the class of *Hajós k -constructible graphs* recursively as follows:

- (1) K_k is Hajós k -constructible.
- (2) Suppose G_1 and G_2 are Hajós k -constructible graphs with $v_1v_2 \in E(G_1)$ and $w_1w_2 \in E(G_2)$. Then the graph G' obtained from the disjoint union $G_1 \amalg G_2$ by identifying the vertices v_1 and w_1 , deleting the edges v_1v_2 and w_1w_2 , and by adding an edge v_2w_2 is Hajós k -constructible (See Figure 6).
- (3) Let G be a Hajós k -constructible graph, and let v and w be nonadjacent vertices in G . Then the graph obtained from G by identifying v and w and removing any multiple edges that result is Hajós k -constructible.

The graph operation defined in the second part of the definition of a Hajós k -constructible graph is commonly referred to as the *Hajós construction*. We note that this construction relies in no way on the hypothesis that the original two graphs are Hajós k -constructible, so we can always consider the graph G' obtained by applying the Hajós construction to two arbitrary graphs G_1 and G_2 with edges v_1v_2 and w_1w_2 , respectively.

The reason why the class of Hajós k -constructible graphs and the two above operations are so important is the following theorem of Hajós.

Theorem 23 (Hajós [6, p. 102], [9]) *A graph G satisfies $\chi(G) \geq k$ if and only if it contains a Hajós k -constructible subgraph.*

One direction of the proof of Hajós' Theorem relies on the fact that if graphs G_1 and G_2 have chromatic number at least k then so does any graph obtained from them by applying the Hajós construction. Interestingly, an identical result holds when we replace "chromatic number" by "chromatic capacity," as Proposition 24 demonstrates.

Proposition 24 *Let G_1 and G_2 be graphs with edges v_1v_2 and w_1w_2 , respectively, and let G' be the graph obtained from G_1 and G_2 by applying the Hajós construction with respect to the edges v_1v_2 and w_1w_2 . If $\chi_{\text{cap}}(G_1) \geq k$ and $\chi_{\text{cap}}(G_2) \geq k$, then $\chi_{\text{cap}}(G') \geq k$.*

PROOF. Pick emulsive k -colorings $c_1: E(G_1) \rightarrow [k]$ and $c_2: E(G_2) \rightarrow [k]$. Define $c: E(G') \rightarrow [k]$ by $c(v_2w_2) = c_1(v_1v_2)$ and by letting $c = c_1 \cup c_2$ for all other edges. We claim that c is an emulsion. For let $b: V(G') \rightarrow [k]$ be arbitrary. Now b induces vertex colorings $b_i: V(G_i) \rightarrow [k]$ via restriction. Since each c_i is an emulsion, there is a monochromatic edge $x_1x_2 \in G_1$ with respect to b_1 and c_1 and a monochromatic edge $y_1y_2 \in E(G_2)$ with respect to b_2 and c_2 . If either $x_1x_2 \neq v_1v_2$ or $y_1y_2 \neq w_1w_2$, then b is clearly not compatible with c . However, if both v_1v_2 and w_1w_2 are monochromatic edges in G_1 and G_2 , respectively, then v_1, v_2, w_1 , and w_2 are all colored the same color in G' since v_1 is identified with w_1 in G' . Now the edge v_2w_2 is colored the same as the edge v_1v_2 , while the edge v_1v_2 was monochromatic, and w_2 is the same color as v_2 , so v_2w_2 is a monochromatic edge, b is not compatible with c , and c is an emulsion. \square

While the Hajós construction behaves nicely with respect to chromatic capacity, the same cannot be said for the vertex identification operation introduced in part (3) of the definition of Hajós k -constructible graphs. Unfortunately, the following result is the best that can be achieved.

Proposition 25 *Let G be a graph, and let G' be the multigraph defined by identifying two vertices of G (we do not delete any multiple edges or loops that arise from the quotient). Then $\chi_{\text{cap}}(G') \geq \chi_{\text{cap}}(G)$.*

PROOF. Any emulsive k -coloring of $E(G)$ induces an emulsive k -coloring of $E(G')$ in the canonical way. \square

Proposition 25 does not hold when we treat the quotient as a simple graph by deleting any multiple edges or loops that arise; one need look no further than the graph $P_3 + \overline{K_2}$ that has chromatic capacity 2 and the graph $P_3 + K_1$ of chromatic capacity 1 that can be obtained from $P_3 + \overline{K_2}$ by identifying the two vertices in the $\overline{K_2}$ (here P_3 denotes the path on three vertices). If it did

hold under this modification, Hajós' Theorem 23 would immediately imply Greene's Conjecture 2 since $\chi_{\text{cap}}(K_n) > (1 - o(1))\sqrt{n}$ for every n .

We note that in the event that the two vertices v and w to be identified in Proposition 25 are at a distance at least 3 from one another then the proposition still holds when we treat the quotient as a simple graph, for no multiple edges are created upon making an identification of vertices at distance 3 or more.

6 Open problems

In this section, we will discuss a number of open questions concerning the chromatic capacity.

6.1 Greene's Conjecture

While we have shown that Greene's Conjecture 2 holds for almost every graph, the conjecture itself remains open. This conjecture is quite possibly the most important open question in the study of chromatic capacities. Should it be true, we would have meaningful lower and upper bounds on the chromatic capacity in terms of the chromatic number alone, providing a much stronger correlation between the two chromatic quantities than is currently known.

6.2 Vertex splitting

Let G be a graph and let $r(G)$ be the smallest positive integer for which $\chi_{\text{cap}}(\mathcal{S}_{r(G)}G) = \chi(G) - 1$. Our proof of Theorem 11 demonstrates that $r(G)$ is smaller than some function of $|V(G)|$ and $\chi(G)$. It is interesting to consider whether or not the dependence on $|V(G)|$ is actually necessary. If it is not necessary, then this may lend some help in a proof of Conjecture 2, as there would be a function $h : \mathbb{N} \rightarrow \mathbb{N}$ such that $\chi_{\text{cap}}(\mathcal{S}_{h(\chi(G))}G) = \chi(G) - 1$ for every graph G . If one could then relate the chromatic capacity of a graph to the chromatic capacity of its r -split, Conjecture 2 could follow, depending on the particular bounds that are found.

Problem 26 *Does there exist a function $h : \mathbb{N} \rightarrow \mathbb{N}$ such that $r(G) \leq h(\chi(G))$ for every graph G ?*

If there does not exist such an h , we are naturally led to ask the following question:

Problem 27 *Does there exist a polynomial p such that $r(G) \leq p(|V(G)|)$ for every graph G ?*

However, we note that the answer to the following weaker question is not even known:

Problem 28 *Does there exist a polynomial p such that*

$$p(n) \geq \min\{|V(G)| : G \text{ a graph such that } \chi_{\text{cap}}(G) = \chi(G) - 1 = n\}$$

for every n ?

As Greene's construction in [8] of a sequence of graphs $\{G_n\}$ with $n = \chi_{\text{cap}}(G) = \chi(G) - 1$ satisfies $|V(G_n)| = \lfloor e(n-1)! \rfloor$, an answer to this question would be quite interesting in its own right.

6.3 Girth

Corollary 15 demonstrates that there are graphs G with $\chi(G)$ and $\text{og}(G)$ arbitrarily large for which $\chi_{\text{cap}}(G) = \chi(G) - 1$. In [8], Greene asked whether the same statement holds with respect to girth; this question still remains open.

Problem 29 *Let p and q be arbitrary positive integers. Does there exist a graph G with girth at least q such that $\chi_{\text{cap}}(G) = \chi(G) - 1 = p$?*

We note that any relatively simple proof of this statement would almost certainly be probabilistic, as any constructive proof of this statement would necessarily be at least as hard as a constructive proof of the Erdős result on the existence of graphs with arbitrarily large girth and chromatic number (see [1, p. 35]). While a constructive proof of this theorem is known, it is not nearly as simple as the probabilistic proof (see [13] for a constructive proof).

6.4 Maximality of the complete graphs

A well-known result regarding the chromatic number is that $\chi(G) \leq \Delta(G) + 1$ for every graph G , where $\Delta(G)$ denotes the maximum degree of G . One way of rephrasing this inequality is that there is no graph G such that $\chi(G) > \chi(K_{\Delta(G)+1})$. In some sense, the complete graphs have the smallest maximum

degree among all the graphs of the same chromatic number. It is unknown whether the same holds for χ_{cap} .

Problem 30 *Is there a graph G such that $\chi_{\text{cap}}(G) > \chi_{\text{cap}}(K_{\Delta(G)+1})$?*

6.5 Cartesian products

While we have obtained lower bounds for the chromatic capacity of disjunctions, lexicographic products and conjunctions, a reasonable lower bound for the chromatic capacity of the cartesian product of two graphs has yet to be found (vertices (v_1, w_1) and (v_2, w_2) are adjacent in the cartesian product if $v_1 = v_2$ and $w_1 \sim w_2$ or if $v_1 \sim v_2$ and $w_1 = w_2$). Using a proof technique similar to that of Theorem 11, it is possible to prove that if k is a positive integer and

$$r \geq \sum_{i=0}^k k^i = \frac{k^{k+1} - 1}{k - 1},$$

then $\chi_{\text{cap}}(G_1 \times G_2) \geq k$ whenever $G_2 \supset K_r$. However, due to the large order of r , whenever $k \geq 3$ we find that G_2 will have chromatic capacity much larger than k since it contains such a large complete subgraph. Since the cartesian product of two graphs contains isomorphic copies of each factor, the above result is therefore useless by the monotonicity of χ_{cap} .

6.6 Conjunctions and Hedetniemi's Conjecture

A long-standing conjecture of Hedetniemi [10] states that

$$\chi(G_1 \wedge G_2) = \min\{\chi(G_1), \chi(G_2)\}$$

for all graphs G_1 and G_2 . The inequality

$$\chi(G_1 \wedge G_2) \leq \min\{\chi(G_1), \chi(G_2)\}$$

is straightforward; it is the reverse inequality that has been open for nearly 40 years. We pose the following question about the chromatic capacity of a conjunction:

Problem 31 *Let G_1 and G_2 be graphs. Does the equality*

$$\chi_{\text{cap}}(G_1 \wedge G_2) = \min\{\chi_{\text{cap}}(G_1), \chi_{\text{cap}}(G_2)\}$$

necessarily hold?

Acknowledgements

I would like to thank Josh Greene, Matt Elder, Melanie Wood, and Joseph Gallian for numerous helpful discussions. This research was conducted in Duluth, Minnesota at Joseph Gallian's REU program under grants from the National Science Foundation (DMS-0137611) and the National Security Agency (H-98230-04-1-0050).

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