Advection-Diffusion Equation

$$\frac{d}{dt}\rho + V\cdot\frac{d}{dx}\rho = k\cdot\frac{d^2}{dx^2}\rho$$

Look at a general form of an Explicit difference equation

$$\frac{\rho(x, t+\Delta t) - \rho(x,t)}{\Delta t} + V\cdot\frac{w\cdot\rho(x+\Delta x, t) + (1-w)\cdot\rho(x,t) - w\cdot\rho(x,t) - (1-w)\cdot\rho(x-\Delta x, t)}{\Delta x} = k\cdot\left(\frac{\rho(x+\Delta x, t) + 2\cdot\rho(x,t) + \rho(x-\Delta x, t)}{\Delta x^2}\right)$$

Substitute in a perturbed density

$$\rho(x,t) := \rho_0 + \delta\rho(t)\cdot e^{i\cdot k\cdot x}$$

and divide the entire equation by e^{ikx}

$$\frac{\delta\rho(t+\Delta t) - \delta\rho(t)}{\Delta t} + \delta\rho(t)\cdot V\cdot\frac{w\cdot e^{i\cdot k\cdot\Delta x} + 1 - 2w - (1-w)\cdot e^{-i\cdot k\cdot\Delta x}}{\Delta x} = k\cdot\delta\rho(t)\cdot\left(\frac{e^{i\cdot k\cdot\Delta x}\cdot -2 + e^{-i\cdot k\cdot\Delta x}}{\Delta x^2}\right)$$

let $c = \frac{V\cdot\Delta t}{\Delta x}$ $\quad\gamma = \frac{k\cdot\Delta t}{\Delta x^2}$, and $\quad\Theta = k\cdot\Delta x$

$$\delta\rho(t+\Delta t) = [[1 + 2\cdot\gamma\cdot(\cos(\Theta) - 1) - c\cdot(2\cdot w - 1)\cdot(\cos(\Theta) - 1)] - c\cdot i\cdot\sin(\Theta)]\cdot\delta\rho(t)$$

The amplification factor is

$$G = [1 + (2\cdot\gamma - 2\cdot w\cdot c + c)\cdot(\cos(\Theta) - 1)] - c\cdot i\cdot\sin(\Theta)$$

Examining G, stability is not possible if the magnitude of either the real or imaginary parts exceeds 1
The imaginary term leads to the requirement that c be less than one

$$\frac{V\cdot\Delta t}{\Delta x} \le 1 \;\middle|\; \begin{array}{l}\text{solve}, \Delta t\\ \text{assume}, \Delta x > 0\\ \text{assume}, V > 0\end{array} \rightarrow \Delta t \le \frac{\Delta x}{V}$$

For the real term to be less than one we need:

$$2\cdot\gamma - 2\cdot w\cdot c + c \ge 0$$

which restricts possible weighting factors. For the real term to be greater than -1 we need:

$$1 - 2\cdot(2\cdot\gamma - 2\cdot w\cdot c + c) \ge -1$$

$$1 - 2\cdot\left(2\cdot\frac{k\cdot\Delta t}{\Delta x^2} - 2\cdot w\cdot\frac{V\cdot\Delta t}{\Delta x} + \frac{V\cdot\Delta t}{\Delta x}\right) \ge -1$$

or

$$\Delta t \le \frac{\Delta x^2}{2\cdot k - 2\cdot w\cdot V\cdot\Delta x + V\cdot\Delta x}$$

Note that the previous two conditions are necessary for stability but not sufficient. Fine tuning the stability conditions requires a little more finesse.

The product of G and its complex conjugate is

$$\left[1 + (2\cdot\gamma - 2\cdot w\cdot c + c)\cdot(\cos(\Theta) - 1)\right]^2 + c^2\cdot\left[1 - \cos^2(\Theta)\right]$$

Let $\mu = \cos(\Theta)$ and look the extreme values of $|G|^2$ with respect to μ

$$\frac{d}{d\mu}\left[\left[1 + (2\cdot\gamma - 2\cdot w\cdot c + c)\cdot(\mu - 1)\right]^2 + c^2\cdot\left(1 - \mu^2\right)\right] = 0 \;\left|\begin{array}{l}\text{solve}, \mu\\ \text{simplify}\end{array}\right. \rightarrow \frac{1}{4}\cdot(1 - 2\cdot\gamma + 2\cdot w\cdot c - c)\cdot\frac{(-2\cdot\gamma + 2\cdot w\cdot c - c)}{\left(\gamma^2 - 2\cdot\gamma\cdot w\cdot c + \gamma\cdot c + w^2\cdot c^2 - w\cdot c^2\right)}$$

Try the case of central differencing

$$\frac{1}{4}\cdot\frac{\left(-2\cdot\gamma + 2\cdot w\cdot c - c + 4\cdot\gamma^2 - 8\cdot\gamma\cdot w\cdot c + 4\cdot\gamma\cdot c + 4\cdot w^2\cdot c^2 - 4\cdot w\cdot c^2 + c^2\right)}{\left(\gamma^2 - 2\cdot\gamma\cdot w\cdot c + \gamma\cdot c + w^2\cdot c^2 - w\cdot c^2\right)} \;\left|\begin{array}{l}\text{substitute}, w = \frac{1}{2}\\ \text{simplify}\end{array}\right. \rightarrow -2\cdot\gamma\cdot\frac{(-1 + 2\cdot\gamma)}{\left(-4\cdot\gamma^2 + c^2\right)} = ▪$$

This gives a extreme for $|G|^2$ when

$$\mu = \frac{\left(4\cdot\gamma^2 - 2\gamma\right)}{\left(4\cdot\gamma^2 - c^2\right)} \quad \text{or}$$

$$\mu = \frac{\left(4\cdot\gamma^2 - 2\gamma\right)}{\left[4\cdot\gamma^2 - 2\cdot\gamma + \left(2\cdot\gamma - c^2\right)\right]}$$

or

$$\mu = \frac{1}{1 + \frac{2\cdot\gamma - c^2}{4\cdot\gamma^2 - 2\cdot\gamma}}$$

To make useful statements about stability we need a general idea of the behavior of $|G|^2$. First notice that $\left|G^2\right| \geq 0$ for $-1 \leq \mu \leq 1$, and when $\left|G^2\right| = 1$:

$$\frac{d}{d\mu}\left[\left[1 + 2\cdot\gamma\cdot(\mu - 1)\right]^2 + c^2\cdot\left(1 - \mu^2\right)\right] \;\left|\begin{array}{l}\text{simplify}\\ \text{collect}, \mu\\ \text{substitute}, \mu = 1\end{array}\right. \rightarrow -2\cdot c^2 + 4\cdot\gamma$$

$$\left[1 + 2\cdot\gamma\cdot(\mu - 1)\right]^2 + c^2\cdot\left(1 - \mu^2\right) \text{ substitute}, \mu = 1 \rightarrow 1$$

With bounding values and the slope at one end of the interval, the next thing to explore is the location and nature of the maximum or minimum of $|G|^2$. First note that $4\gamma^2$-2γ is always less than or equal to zero since $\gamma \leq \frac{1}{2}$. When $2\cdot\gamma - c^2 \geq 0$ or $c^2 \leq 2\cdot\gamma$ μ of the extreme value is greater than one. It is a maximum rather than minimum because the derivative of $|G|^2$ is positive and $|G|^2 = 1$ when $\mu=1$. Further behavior this situation can be studied in three regions.

Region 1

$$0 \ge \frac{2\cdot\gamma - c^2}{4\cdot\gamma^2 - 2\cdot\gamma} \ge -1$$ corresponding to $c^2 \le 2\cdot\gamma$ and $c \le 2\gamma$

In this case $|G|^2$ is montonically increasing between $\mu = -1$ and $\mu=1$, so has a maximum value of 1 (Remember that $|G|^2$ is greater than zero between $\mu = -1$ and $\mu=1$).

Region 2

$$-1 > \frac{2\cdot\gamma - c^2}{4\cdot\gamma^2 - 2\cdot\gamma} > -2$$

Here $|G|^2$ has a minumum at $\mu < -1$. This means that the maximum value of $|G|^2$ must again be 1 at $\mu=1$.

Region 3

$$-2 \ge \frac{2\cdot\gamma - c^2}{4\cdot\gamma^2 - 2\cdot\gamma} > -\infty$$

Here a local minimum exists in $|G|^2$ at a value of μ between -1 and 0. Since $|G|^2$ is always greater than zero, the largest possible values of $|G|^2$ are at $\mu = -1$ or $\mu = 1$. At $\mu = -1$

$\left|G^2\right| = [\,1 + 2\cdot\gamma\cdot(-2)\,]^2 \le 1$ which requires $\gamma \le \frac{1}{2}$ or $\Delta t \le \frac{\Delta x^2}{2\cdot k}$

What happens when $c^2 > 2\cdot\gamma$? Look at the details of the derivative of $|G|^2$

$$\frac{d}{d\mu}\left[[\,1 + 2\cdot\gamma\cdot(\mu - 1)\,]^2 + c^2\cdot\left(1 - \mu^2\right)\right] \;\Big|\;\begin{matrix}\text{simplify}\\ \text{collect}, \mu\end{matrix} \rightarrow \left(8\cdot\gamma^2 - 2\cdot c^2\right)\cdot\mu + 4\cdot\gamma - 8\cdot\gamma^2 =$$

When $\mu = 1$ and $c^2 > 2\cdot\gamma$ this slope is negative. However, G is one when μ is one, so G must be greater than one for some range of μ.

All of this leads us to two conditions that are necessary and sufficient for stability. The first is the standard condition for conduction:

$$\Delta t \le \frac{\Delta x^2}{2\cdot k}$$

and from $c^2 \le 2\cdot\gamma$ we get:

$$\Delta t \le \frac{2\cdot k}{V^2}$$

Note that for a pure convection problem (k=0) this gives the result that central differencing is unstable.

Also note that this is not the full story. The cell Peclet number (or cell Reynolds number is defined as:

$Pe = \frac{2\cdot c}{\gamma}$ or $Pe = \frac{V\cdot\Delta x}{k}$

we will later see that boundary conditions can add a stability requirement $Pe \le 2$