Casting Out Beams: Berkeley's Criticism of the Calculus

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First cast out the beam out of thine own eye; and thou shalt see clearly to cast out the mote out of thy brother's eye.

King James Bible, Matthew 7:5

Calculus burst upon the scientific community in the late 17th century with unparalleled success. Nevertheless it took another 200 years and the combined efforts of many first rate mathematicians to finally "get it right." In particular the fundamental concept of a limit did not take on its final form until the 1800's with the work of Karl Weierstrass.

The *Method of Fluxions* as it was known in Britain, was invented by Isaac Newton in the 1660's. For reasons known only to him, he did not publish his invention until much later. In the meantime Gottfried Wilhelm Leibniz published his own formulation of Calculus which he called *Calculus Differentialis* in 1684. Newton's approach was kinematic. Quantities were said to "flow" in time and the *fluxion* (derivative) of a flowing quantity was interpreted as a velocity. Leibniz's approach was based on the idea of infinitely small quantities: *infinitesimals*. Their public priority brawl is both legendary [11] and a stain on both of their reputations.

A slightly less well known public argument, but mathematically far more important, began in 1734 when Bishop George Berkeley published a treatise entitled *The Analyst*. Berkeley derided the lack of rigor in the fundamentals of both renderings of the topic. To be sure, neither Newton's nor Leibniz's formulation of the Calculus would stand up to the modern standard of rigor. But this is unfair. They could hardly be expected to meet standards which didn't exist at the time. Standards which, moreover, would come into being largely as a result of the "Analyst Controversy" itself.

Each man surely felt that his technique was sound but, surprisingly, it appears that they did not meet the standards of their own time either. As often happens in mathematics it was the simple fact that the Calculus *worked* that kept it alive. Exactly why it worked was murky to say the least, as Berkeley shows rather pointedly in *The Analyst*. Nevertheless, using Calculus, mathematicians were able to solve hitherto unsolvable problems. What is perhaps more significant to our current discussion is that they were also able to establish many known results much more simply than had been possible previously. So useful a tool had to be kept in the mathematical toolbox at all costs.

As we observed earlier, the final resolution of the "Analyst Controversy", took some 200 years and ended with Weierstrass' formulation of the limit concept. Lagrange tried to avoid limits entirely by founding Calculus on infinite series. Maclaurin harkened back to Greek geometry as the touchstone of rigor. D'Alembert tried and failed to formulate a precise limit definition. Cauchy almost got it right and Weierstrass finally did. In 1874 he published the formal definition of a limit which is still in use today. Small wonder then that our students struggle with the limit concept. It is deep. It is unintuitive. And it is hard.

Newton's Proof of the Power Rule

In 1734 George Berkeley, the Anglican Bishop of Cloyne and a philosopher of some renown published a treatise titled, *The Analyst*, in which he criticized "modern geometers" in general and Sir Isaac Newton in particular for his lack of rigor. His criticisms were tightly argued, deadly accurate, and demanded a response from the mathematical community. Unfortunately, an effective response was not immediately forthcoming for the simple reason that none was really possible at the time.

Berkeley was a theologian by training and a philosopher by inclination. Although mathematically knowledgeable, he was not a mathematician nor would he have claimed to be. Nevertheless, *The Analyst* is arguably one of the most important and influential publications in the history of mathematics. The following 200 years of work on foundations in general, and the Calculus in particular, can be traced directly back to Berkeley's criticism.

Looking back from the vantage point of the early twenty-first century, it is a little difficult to imagine what all of the fuss was about. So what exactly did Berkeley say? He offers several criticisms of the Calculus, but two of them in particular will be of interest to us. In the first he accuses Newto of faulty reasoning.

After some introductory and often sarcastic comments, Berkeley cites Newton's proof of the Power Rule from the introduction of *Quadrature of Curves* [8]: Let *x* be a uniformly flowing quantity and let *o* be a finite, positive increment (change) of x. Then $(x+o)^n = x^n + nox^{n-1} + \frac{n(n-1)}{2}o^2x^{n-2} + \cdots$, so that the increment of x^n is $nox^{n-1} + \frac{n(n-1)}{2}o^2x^{n-2} + \cdots$. Comparing the increment of x^n with the increment of *x*, we have

$$\frac{nox^{n-1} + \frac{n(n-1)}{2}o^2x^{n-2} + \cdots}{o} = nx^{n-1} + \frac{n(n-1)}{2}ox^{n-2} + \cdots$$

Now let the increment *o* vanish and in the "last proportion," we see that the fluxion (derivative) of x^n is nx^{n-1}.

Reading this in the early twenty-first century it can be difficult to overcome our training. We can easily see that a limit is being taken and thus there is nothing substantially wrong with the argument. This is probably how Newton himself saw his argument but he did not have the limit concept precisely formulated.

In any case, what Newton says is "Now let these Augments vanish," [8] by which he means that *o* actually becomes zero. This, says Berkeley, is improper reasoning because

> **Berkeley's Lemma.** "If with a View to demonstrate any Proposition, a certain Point is supposed, by virtue of which certain other Points are attained; and such supposed Point be it self afterwards destroyed or rejected by a contrary Supposition; in that case, all the other Points, attained thereby and consequent thereupon, must also be destroyed and rejected, so as from thence forward to be no more supposed or applied in the Demonstration."

In modern English, Berkeley's Lemma states that we can assume that any statement is true and that we can draw conclusions based on that assumption. However we may not at a later point negate the assumption but retain the conclusions. If the assumption is later taken to be false we must disregard any conclusions drawn from it unless they can be derived in some manner which does not depend on the original assumption. This, he asserts, is so clear as to need no proof.

Newton has clearly violated Berkeley's Lemma since he had originally assumed that *o* was a finite, positive increment, had drawn conclusions from that assumption, and keeps his conclusions even after he allows *o* to become zero. Therefore, says Berkeley, the reasoning is faulty. But the conclusion is correct. We know as well as Newton and Berkeley did that the derivative of x^n is nx^{n-1}. Berkeley states repeatedly in *The Analyst* that he has no quarrel with the results of the Calculus, only the reasoning behind them [1]:

> "It must be understood that I am not concerned about the truth of your Theorems, but only about the way of coming at them; whether it be legitimate or illegitimate, clear or obscure, scientific or tentative."

Initial Reactions

The very first reply to *The Analyst* came from James Jurin, a noted medical doctor of the time, and a student of Newton [2]. Writing under the pseudonym "Philalethes Cantabrigiensis," Jurin's primary concern seems to have been to vent his ire that anyone would presume to question his teacher.

Apparently Jurin did not learn Calculus any better than some modern students with less august teachers. At one point he asserts that $\frac{2x(2y+dy)}{2y} = \frac{2x(2y)}{2y+dy}$ since if $2x = 1000$ miles, then the difference between them is "not as much as the thousand-millionth part of an inch" [6]. Neither Newton nor Berkeley would have accepted such an argument. Indeed, in *The Analyst*, Berkeley cites Newton himself saying that "The most minute errors are not in mathematical matters to be scorned" [1].

John Walton, Professor of Mathematics at Dublin [2], made a valiant effort to address Berkeley's criticisms by attempting to elucidate Newton's doctrine of prime and ultimate ratios. Newton had avoided division by zero by considering only the ratios of quantities "as they vanish," or "come into being." These were variously called evanescent and nascent quantities, first and last proportions or ratios, and prime and ultimate ratios. That is, Newton would say that the ratio of $\sin 2x$ to *x* when *x* is zero is 2 because "in the last proportion" as *x* becomes "less than any assignable quantity" but is not yet zero the ratio of $\sin 2x$ to *x* actually *is* 2.

As Walton put it [12]:

> "The Magnitudes of the momentateous Increments or Decrements of Quantities are not regarded in the Method of Fluxions, but their first or last Proportions only; that is, the Proportions with which they begin or cease to exist:

These are not their Proportions immediately before or after they begin or cease to exist but the Proportions with which they begin to exist, or with which they vanish."

These "evanescent and nascent quantities" appear to be a first stab at the modern limit concept. Newton sometimes even refers to them as limits, and it is from his usage that the limit concept took hold and eventually became the invaluable tool it is today [2,3]. Newton surely understood what he meant but, unfortunately, he did not elaborate. It was left to mere mortals to follow the advice of Jean le Rhonde d'Alembert, "Persist and faith will come to you." [7].

The Compensating Errors argument

At this point, Berkeley has shown that Newton has derived correct results via incorrect reasoning. To resolve this incongruity he now insists that this has occurred because two mutually compensating errors were made Specifically he shows that correct results can be obtained through compensating errors. Curiously, he does this by analyzing one of Newton's arguments using Leibniz's notation.

Figure 1 is essentially the figure that appears in the introduction of Newton's *Quadrature of Curves* [8]. Berkeley examines this point in Section XXI of *The Analyst.* The task is to construct the tangent line to the parabola $y^2 = x$.

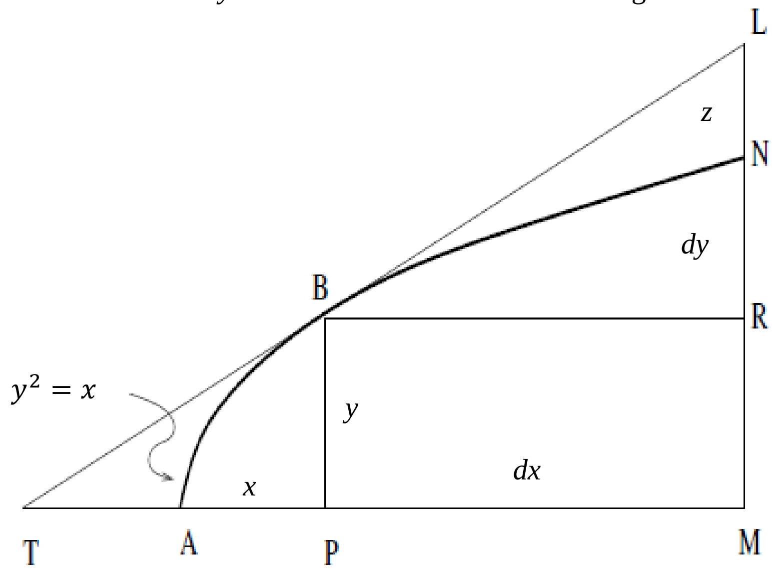


Figure 1

In the early 1700's, slope was not computed directly. The goal was to find the length of the subtangent *PT*. Observe that this is sufficient, since this gives two points of the tangent line, *B* and *T*, from which the slope can be computed trivially. According to the Leibnizian formulation since *dx* and *dy* are infinitely small, ~~then~~ the figure *BNR* is a triangle and in fact is similar to ΔTBP. Newton says is that in its "evanescent form" [8], the figure *BNR* "will become" proportional to ΔTBP which is not quite the same thing. But the conclusion is the same either way. Thus $\frac{RN}{RB} = \frac{PB}{PT}$, so that $PT = \frac{ydx}{dy}$. This, says Berkeley, is the first error. The curve $y^2 = x$ is not a straight line, and will not become one no matter how small *dx* is, so *BNR* is not a triangle. Therefore this formula is clearly wrong and, in fact, this makes this value of *PT* too large as *RN* is too small. Berkeley then correctly points out that ΔTBP is similar to ΔBLR and so *PT* should have been $\frac{ydx}{dy+z}$ and the error is *z*, the difference between the tangent line and the curve. Continuing with the calculus argument, differentiating $y^2 = x$, we obtain $2ydy = dx$ so that $\frac{dx}{dy} = 2y$ and the required length *PT* is given by $PT = \frac{ydx}{dy} = 2y^2 = 2x$. Note that this agrees with our modern formulation that the slope of the tangent line is given by $\frac{BP}{PT} = \frac{y}{2y^2} = \frac{1}{2}x^{-\frac{1}{2}}$. The fact that we differentiated $y^2 = x$ to get $2ydy = dx$ is what Berkeley says is the second (compensating) error. Specifically, since *N* lies on the curve, we have $(y + dy)^2 = x + dx$. Squaring this we obtain $y^2 + 2ydy + (dy)^2 = x + dx$. Since $y^2 = x$, we have $2ydy + (dy)^2 = dx$ and so $dy = \frac{dx}{2y} - \frac{(dy)^2}{2y}$. "Thus there was error of excess in making $dy = \frac{dx}{2y}$, which followed from the erroneous Rule of Differences [differentiation]. And the measure of this second error is $\frac{(dy)^2}{2y} = z$. Therefore the two errors being equal and contrary destroy each other; the first error of defect being corrected by the second error of excess." [1]

Berkeley then demonstrates that, in fact, $z = \frac{(dy)^2}{2y}$ using classical geometry. Berkeley cites Proposition 33 of Apollonius' *On Conics* [9] which says that the vertex of any parabola is midway between the intersection of the tangent line and the axis of symmetry and the orthogonal projection of the point of tangency onto the axis of symmetry. In other words, for this problem:

$TA = AP$ so $PT = 2x$. The reader may be wondering why Newton, Leibniz, Berkeley or anyone else would devote time to constructing the tangent to a parabola since this construction had been known since antiquity. In fact, mathematicians of the 17^{th} and 18^{th} centuries resolved many of these classical problems both to reaffirm the correctness of the calculus and to show its relative ease of use in solving these traditionally difficult problems.

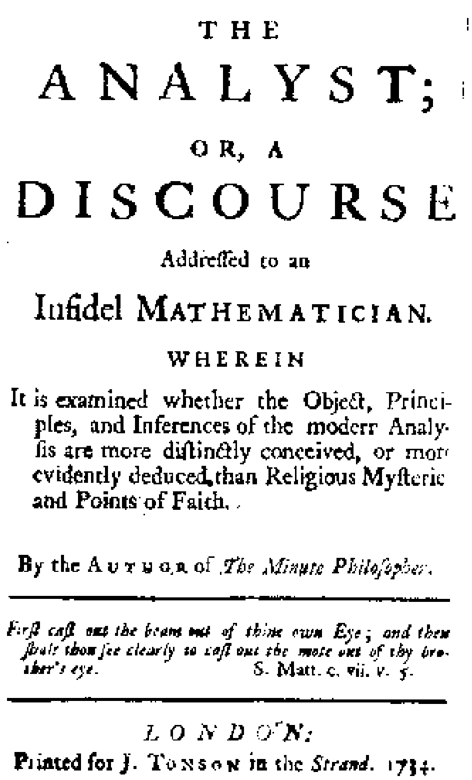
Berkeley already noted that ΔTBP is similar to ΔBLR, so $\frac{2x}{y} = \frac{dx}{dy+z}$. Solving for z, we have $z = \frac{ydx}{2x} - dy$. Since we already had $dx = 2ydy + (dy)^2$, we substitute to obtain $z = \frac{y(2ydy+(dy)^2)}{2x} - dy = \frac{2y^2dy}{2x} + \frac{y(dy)^2}{2x} - dy$. Using the fact that $y^2 = x$, we obtain $z = \frac{2xdy}{2x} + \frac{y(dy)^2}{2y^2} - dy = \frac{(dy)^2}{2y}$.

Berkeley's argument is sound and according to Grattan-Guinnes [5] can be adapted to any real analytic function.

The context of the controversy: Berkeley's Goal

Why did Berkeley write *The Analyst*? From the tone he uses, it is easy to imagine that he held some deep seated antipathy toward mathematics, or at least mathematicians. The latter may have been so, but the former can hardly be true. Berkeley was a well educated man and education at that time included a reverence for Euclid's Elements second only to reverence for the Bible. So what was this about?

Recall that this was the Age of Enlightenment. Western Europe had recently emerged from the darkness of the Middle Ages and the Scientific Revolution was in full swing. In the 17th and early 18th centuries the "free thinking" movement -- the belief that rational thought rather than divine revelation was the proper tool for understanding God's creation -- began questioning the methods of theology. The remarkable success of the Calculus to explain natural phenomena, in particular in the hands of a Newton or a Leibniz, was held up as the exemplar of how theologians should be investigating God and his creation. "In no area was the application of reason more needed, claimed the freethinkers, than in theology, where tradition, superstition, and vested interest had prevailed [4].


THE
ANALYST;
OR, A
DISCOURSE
Addressed to an
Infidel MATHEMATICIAN.
WHEREIN
It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.
By the AUTHOR of *The Minute Philosopher*.
First cast out the beam out of thine own Eye; and then shalt thou see clearly to cast out the mote out of thy brother's eye. S. Matt. c. vii. v. 5.
LONDON:
Printed for J. TONSON in the *Strand*. 1734.

As a theologian Berkeley was on the defensive and his defense strategy is laid out clearly on his title page http://www.gap-system.org/~history/Bookpages/Berkeley2.gif where he quotes the New Testament Book of Matthew, Chapter 7, Verse 5 [the quote from the beginning of this article]. By the way, the infidel mathematician referred to in the title is generally believed to be Edmund Halley, one of the more outspoken "free thinkers" although Berkeley never mentions anyone by name. Berkeley was not attacking mathematics, he was defending theology and his strategy was to hoist the free-thinkers by their own petard. If Calculus is really to be the new paradigm for how to investigate the world, let's take a close look at it. Can we really depend on it? Is it well-founded?

As a theologian, Berkeley was, of course, motivated to answer "no" to both questions, but in this, he was aided by the lack of any existing logical foundations for the Calculus. Leibniz's formulation introduced the notion of "infinitesimals" which was no more rigorous than Newton's appeal to first and last ratios. Newton's conception eventually morphed into the limit concept we use today [3]. Leibniz's infinitesimals were merely a convenient notational device until the invention of "Nonstandard Analysis" by Abraham Robinson in the middle of the 20th century [10]. However, neither approach was rigorous at the time and it was Berkeley's incisive, insightful, and very public criticisms that began the march toward mathematical rigor.

For Berkeley however this was secondary. He was not arguing for or against mathematical rigor, *per se*. Rather, he was arguing that the call for "mathematical rigor" in theology by self-described free-thinkers was not only misplaced, but dangerous inasmuch as it could be used to undermine religious faith. Moreover, Berkeley maintained that such rigor was already present in theology and wholly missing among "modern geometers." In his words,

> "All these Points, I say, are supposed and believed by certain rigorous Extractors of Evidence in Religion, Men who pretend to believe no further than they can see. That Men who have been conversant only about clear Points, would with difficulty admit obscure ones might not seem altogether unaccountable. But he who can digest a second or third Fluxion, a second or third Difference, need not, methinks, be squeamish about any Point in Divinity." [1]

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