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SPHERICAL IDENTITIES

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1. DEFINITIONS

In this note, we shall present the identies involving angles in two spherical coordinates: (azimuth,altitude) and (RA,Dec). All geometrical quantities are defined in the HETDEX wiki.

Of course, one can calculate the same quantities by using the trigonometry on sphere:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \tag{1}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\,, \tag{2}$$

where the upper case letters correspond to the angle looking at the corresponding lower case letters.

Throughout, we shall use the three-dimensional unit vectors to find the identies. In fact, in this setting, Eq. (1) comes from the inner product of two vertex vectors, and Eq. (2) comes from the scalar triple product. Here are the variables that we use. For a generic point in horizontal coordinate (azimuth, altitude) $= (a, \ell)$,

$$\hat{\mathbf{n}} = (\cos\ell\cos a, -\cos\ell\sin a, \sin\ell)\,. \tag{3}$$

For the points at the center of the HET focal plane, we use the subscript $_0$:

$$\hat{\mathbf{n}}_0 = (\cos\ell_0\cos a_0, -\cos\ell_0\sin a_0, \sin\ell_0)\,. \tag{4}$$

The tracker $+\hat{X}$ and $+\hat{Y}$ directions are defined on the tangent plane of $\hat{\mathbf{n}}_0$ and given by

$$\mathbf{X} = (-\sin a_0, -\cos a_0, 0) \tag{5}$$

$$\mathbf{Y} = (\sin\ell_0\cos a_0, -\sin\ell_0\cos a_0, -\cos\ell_0)\,. \tag{6}$$

The location of north star is at $(\alpha_N = 0, \ell_N = 30.681436°)$:

$$\hat{\mathbf{n}}_N = (\cos\ell_N, 0, \sin\ell_N)\,. \tag{7}$$

In order to define the direction toward the north star $(+\hat{\delta})$ and RA $(+\hat{\alpha})$, at the pointing center $\hat{\mathbf{n}}_0$, we use the wedge product:

$$\begin{aligned}\hat{\alpha} =& \frac{1}{\cos\delta_0}\,\hat{\mathbf{n}}_N \wedge \hat{\mathbf{n}}_0\\ =& \frac{1}{\cos\delta_0}\Big(\cos\ell_0\sin a_0\sin\ell_N,\\ &\quad -\cos\ell_N\sin\ell_0 + \cos a_0\cos\ell_0\sin\ell_N,\\ &\quad -\cos\ell_0\cos\ell_N\sin a_0\Big)\end{aligned} \tag{8}$$

$$\begin{aligned}\hat{\delta} =& \hat{\mathbf{n}}_0 \wedge \hat{\alpha} = \frac{1}{\cos\delta_0}\hat{\mathbf{n}}_0 \wedge [\hat{\mathbf{n}}_N \wedge \hat{\mathbf{n}}_0]\\ =& \frac{1}{\cos\delta_0}\left[\hat{\mathbf{n}}_N - (\hat{\mathbf{n}}_0\cdot\hat{\mathbf{n}}_N)\,\hat{\mathbf{n}}_0\right] = \frac{1}{\cos\delta_0}\left[\hat{\mathbf{n}}_N - \sin\delta_0\hat{\mathbf{n}}_0\right].\end{aligned} \tag{9}$$

Finally, to define the hour angle, we introduce the vector

$$\hat{\mathbf{n}}_p(\delta) = \left(\cos\ell_p, 0, \sin\ell_p\right) = (\sin(\delta-\ell_N), 0, \cos(\delta-\ell_N)) \tag{10}$$

which is along the meridian. When calculating HA for declination δ, we use $\ell_p = \ell_N + \pi/2 - \delta$.

2. DECLINATION

Declination is defined from the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}_N$:

$$\begin{aligned}&\cos(\pi/2-\delta) = \sin\delta = \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_N\\ &\sin\delta = \cos\ell\cos\ell_N\cos a + \sin\ell\sin\ell_N\end{aligned} \tag{11}$$

3. POSITION ANGLE

The position angle can be calculated as the angle between $+\mathbf{Y}$ and $+\hat{\delta}$:

$$\begin{aligned}\cos\mathrm{PA} =& \mathbf{Y}\cdot\hat{\delta} = \frac{1}{\cos\delta_0}\mathbf{Y}\cdot[\hat{\mathbf{n}}_N - \sin\delta_0\hat{\mathbf{n}}_0]\\ =& \frac{\mathbf{Y}\cdot\hat{\mathbf{n}}_N}{\cos\delta_0} = \frac{\sin\ell_0\cos\ell_N\cos a_0 - \cos\ell_0\sin\ell_N}{\cos\delta_0}\end{aligned} \tag{12}$$

4. HOUR ANGLE

Hour angle can be calculated by dot producting

$$\begin{aligned}\cos\mathrm{HA} =& \frac{1}{\cos^2\delta}\left(\hat{\mathbf{n}}_N\wedge\hat{\mathbf{n}}_p\right)\cdot(\hat{\mathbf{n}}_N\wedge\hat{\mathbf{n}})\\ =& \frac{1}{\cos^2\delta}\left[\left(\hat{\mathbf{n}}_p\cdot\hat{\mathbf{n}}\right) - \sin^2\delta\right]\\ =& \frac{\cos\ell_N\sin\ell - \cos\ell\sin\ell_N\cos a}{\cos\delta}\,.\end{aligned} \tag{13}$$

Alternatively, we can use $\hat{\mathbf{n}}_N \wedge \hat{\mathbf{n}}_p = -\cos\delta\,\hat{\mathbf{y}}$ to find that

$$\cos\mathrm{HA} = -\frac{(\hat{\mathbf{n}}_N\wedge\hat{\mathbf{n}})_y}{\cos\delta}\,, \tag{14}$$

In particular, when $\hat{\mathbf{n}} = \hat{\mathbf{n}}_0$, it becomes

$$\cos\mathrm{HA} = \alpha_y = -\frac{\cos\ell_N\sin\ell_0 + \cos a_0\cos\ell_0\sin_{\ell N}}{\cos\delta_0}\,. \tag{15}$$

5. GEODESICS

Geodesic line on the sphere starting from $\hat{\mathbf{n}}_0$ and moving along the $\hat{\mathbf{v}} = \cos\varphi\hat{\alpha}_0 + \sin\varphi\hat{\delta}_0$ by the length θ can be calculated by

$$\begin{aligned}\hat{\mathbf{n}}(\theta) =& \cos\theta\hat{\mathbf{n}}_0 + \sin\theta\hat{\mathbf{v}}\\ =& \cos\theta\hat{\mathbf{n}}_0 + \sin\theta\cos\varphi\hat{\alpha}_0 + \sin\theta\sin\varphi\hat{\delta}_0\,.\end{aligned} \tag{16}$$

Yes, of course, you can do everything by using the rotational matrices in $SO(3)$!