# Equations in my cosmology module

Donghui Jeong
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This note summarizes the equations used in the Julia module cosmology.jl. When no input is given as an argument, the code use the planck maximum likelihood cosmology as a default.

## I. CONSTANTS

We keep at least five significant digits for all constants. They are (source: NIST and PDG, the numbers in parenthesis denote the 1-$\sigma$ (68% C.L.) uncertainty)

$$\begin{aligned} c \equiv& 2.997\ 924\ 58 \times 10^{10}\,\mathrm{cm/s}\,, \\ \hbar =& 1.054\ 571\ 800(13) \times 10^{-27}\,\mathrm{erg\,s}\,, \\ k_B =& 1.380\ 648\ 52(79) \times 10^{-16}\,\mathrm{erg/K}\,, \\ \mathrm{eV} =& 1.602\ 176\ 6208(98) \times 10^{-12}\,\mathrm{erg}\,, \\ G =& 6.674\ 08(31) \times 10^{-8}\,\mathrm{cm^3/g/s^2}\,. \end{aligned} \tag{1}$$

Astronomical units are

$$\mathrm{AU} \equiv 149\ 597\ 870\ 700\ \mathrm{m} \tag{2}$$

$$\mathrm{pc} \equiv 1\mathrm{AU}/\tan(1'') = 3.085\ 677\ 581\ 49 \times 10^{18}\mathrm{cm} \tag{3}$$

$$\mathrm{yr} = 365.25\ \mathrm{days}, \tag{4}$$

$$c = 0.30660\ \mathrm{pc/yr} = 3.0660 \times 10^{-7}\ \mathrm{Mpc/yr}\,. \tag{5}$$

In natural unit ($c = \hbar = k_B = 1$), we can convert all units to eV (or GeV) as,

$$\begin{aligned} 1\,\mathrm{s} =& \left(1.0546 \times 10^{-27}\,\mathrm{erg}\right)^{-1} = 1.5193 \times 10^{15}\,\mathrm{eV}^{-1} \\ =& 1.5193 \times 10^{24}\,\mathrm{GeV}^{-1} \\ 1\,\mathrm{cm} =& \frac{1.5193 \times 10^{15}}{2.9979 \times 10^{10}}\,\mathrm{eV}^{-1} = 5.0679 \times 10^{4}\,\mathrm{eV}^{-1} \\ =& 5.0679 \times 10^{13}\,\mathrm{GeV}^{-1} \\ 1\,\mathrm{K} =& \frac{1.3807 \times 10^{-16}}{1.6022 \times 10^{-12}}\,\mathrm{eV} = 8.6175 \times 10^{-5}\,\mathrm{eV} \\ =& 8.6175 \times 10^{-14}\,\mathrm{GeV} \\ 1\,\mathrm{g} =& \left(2.9979 \times 10^{10}\right)^2\mathrm{erg} = 5.6094 \times 10^{32}\,\mathrm{eV} \\ =& 5.6094 \times 10^{23}\,\mathrm{GeV}\,. \end{aligned} \tag{6}$$

The Hubble constant is in the natural unit:

$$\begin{aligned} H_0 =& 100h\ [\mathrm{km/s/Mpc}] = \frac{100}{2.9979 \times 10^5}\ [h/\mathrm{Mpc}] \\ =& 0.00033356\ [h/\mathrm{Mpc}], \end{aligned} \tag{7}$$

or

$$H_0^{-1} = 2997.9\ [\mathrm{Mpc}/h]\,. \tag{8}$$

The critical density is then,

$$\begin{aligned} \rho_{\rm crit} =& \frac{3H_0^2}{8\pi G} \\ =& \frac{3 \times 10^4 h^2\,\mathrm{km^2/s^2/Mpc^2}}{8\pi \times 6.6726 \times 10^{-23}\,\mathrm{km^3/g/s^2}} \\ =& \frac{3 \times 10^4 h^2}{8\pi \times 6.6726 \times 10^{-23}}\,\mathrm{g/Mpc^2/km} \\ =& \frac{3 \times 10^4 h^2}{8\pi \times 6.6726 \times 10^{-23}}\frac{3.0857 \times 10^{19}}{1.9891 \times 10^{33}}\,\mathrm{M_\odot/Mpc^3} \\ =& 2.7751 \times 10^{11} h^2 \mathrm{M_\odot/Mpc^3}\,. \end{aligned} \tag{9}$$

Here, we use $M_\odot = 1.9891 \times 10^{33}$g. In eV, the critical density is also,

$$\begin{aligned} \rho_{\rm crit} =& \frac{3 \times 10^4 h^2}{8\pi \times 6.6726 \times 10^{-23}}\,\mathrm{g/Mpc^2/km} \\ =& \frac{3 \times 10^4 h^2}{8\pi \times 6.6726 \times 10^{-23}}\frac{1}{\left(3.0857 \times 10^{24}\right)^2 10^5}\,\mathrm{g/cm^3} \\ =& 1.8788 \times 10^{-29} h^2 \mathrm{g/cm^3} = 8.0968 \times 10^{-11}\,h^2 \mathrm{eV}^4\,. \end{aligned} \tag{10}$$

The CMB temperature is

$$T_{\rm cmb0} = 2.726\,\mathrm{K} = 0.23491\,\mathrm{meV} \tag{11}$$

and the neutrino temperature is

$$T_{\nu 0} = \left(\frac{4}{11}\right)^{1/3} T_{\rm cmb0} = 1.9457\,\mathrm{K} = 0.16767\,\mathrm{meV}\,, \tag{12}$$

when we assume that neutrinos acquire no entropy from electro-positron pair annihilation process. Because of the small leakage of entropy during the process led to the change in effective neutrino degrees of freedom $N_\nu^{\rm energy} = 3.046$ instead of 3. This fractional change is associated with the energy density. Therefore, when calculating the number density one must use $N_\nu^{\rm number} = 3.0345$, to satisfy $\delta N/N = 3/4\delta E/E$ for particle species following Fermi-Dirac distribution. Equivalently, the true temperature of neutrinos are $1 + 0.046/3/4 \simeq 1.00383$ times the *standard* value: $1.9457\,\mathrm{K} \to 1.9453\,\mathrm{K}$. We call this fudge factor as $t_\nu$ so that $T_\nu = t_\nu T_{\nu 0}(1 + z)$.

Energy density of photon is

$$\rho_{\gamma 0} = \frac{\pi^2}{15} T_\gamma^4 = 2.0036 \times 10^{-15}\,\mathrm{eV}^4, \tag{13}$$

which corresponds to

$$\Omega_\gamma h^2 \simeq 2.4746 \times 10^{-5} \,. \tag{14}$$

### A. Neutrinos

As for the massless neutrinos with degeneracy $g_\nu^{m_\nu=0} = 2N_\nu^{m_\nu=0}$, the energy density is

$$\begin{aligned}\rho_{\nu 0}^{m_\nu=0} &= \frac{7}{8}\frac{\pi^2}{15}\left(\frac{3.046}{3}\right) N_\nu^{m_\nu=0} T_\nu^4 \\ &= 4.6201 \times 10^{-16} N_\nu^{m_\nu=0}\,\mathrm{eV}^4,\end{aligned} \tag{15}$$

with

$$\Omega_{\nu 0}^{m_\nu=0} h^2 = 5.7061 \times 10^{-6} N_\nu^{m_\nu=0} \,. \tag{16}$$

If all neutrinos are either massive or massless (that is, no neutrino mass is around $T_{\nu 0}$), we can calculate the massive neutrino energy density as

$$\begin{aligned}\rho_{\nu 0}^{m_\nu} &= \sum n_\nu m_\nu = \frac{3\zeta(3)}{2\pi^2}\left(\frac{3.0345}{3}\right) T_{\nu 0}^3 \sum m_\nu \\ &= 8.71061 \times 10^{-13} \sum m_\nu \,\mathrm{eV}^3,\end{aligned} \tag{17}$$

and the density parameter becomes

$$\Omega_{\nu 0}^{m_\nu} h^2 = 0.010758 \frac{\sum m_\nu}{\mathrm{eV}} = \frac{\sum m_\nu}{93.0\,\mathrm{eV}} \,. \tag{18}$$

For the general case, we need to integrate Fermi-Dirac distribution function to find the expression for the energy density of massive neutrinos:

$$\begin{aligned}\rho_\nu^{m_\nu}(z) &= \sum_i 2\left(\frac{3.046}{3}\right)\int \frac{d^3p}{(2\pi)^3}\frac{\sqrt{m_{\nu,i}^2 + p^2}}{e^{p/T_\nu(z)}+1}, \\ &= \left(\frac{3.046}{3}\right)\sum_i \frac{T_\nu^4(z)}{\pi^2}\int_0^\infty dx \frac{x^2\sqrt{x^2+x_\nu^2}}{e^x+1}, \\ &= \sum_i 1.0042 \times 10^{-6} h^{-2} \rho_{\mathrm{crit}0}(1+z)^4 f(x_\nu) \,.\end{aligned} \tag{19}$$

where $x_\nu = m_\nu/T_\nu(z)$, and we define

$$f(x_\nu) = \int_0^\infty dx \frac{x^2\sqrt{x^2+x_\nu^2}}{e^x+1}. \tag{20}$$

Therefore, we find the expression for the total energy density of neutrinos as

$$\rho_\nu(z) = 1.0042 \frac{\rho_{\mathrm{crit}0}}{10^6 h^2}(1+z)^4 \sum_i f(x_\nu^i) \,. \tag{21}$$

Note that using following asymptotic behavior of $f(x_\nu)$:

$$\begin{aligned} f(x_\nu \to 0) &= \frac{7}{8}\frac{\pi^2}{15} \simeq 5.6822 \\ f(x_\nu \to \infty) &= \frac{3}{2}\zeta(3) x_\nu \simeq 1.8031 x_\nu \,,\end{aligned} \tag{22}$$

the expression reduces to the massless and massive cases, respectively. We use the asymptotic formula for $x_\nu < 0.0001$ and $x_\nu > 4000$, which corresponds to $\varepsilon \simeq 10^{-7}$.

## II. CONTENTS

The input value for the cosmological calculation is the present value of

$$\begin{aligned}&\Omega_{\mathrm{m}} = \Omega_{\mathrm{cdm}} + \Omega_{\mathrm{b}} + \Omega_\nu^{\mathrm{massive}}; \\ &\Omega_{\mathrm{cdm}} \\ &\Omega_{\mathrm{b}} \\ &\Omega_{\mathrm{de}} \\ &w_{\mathrm{de}} \\ &T_{\mathrm{cmb}} \\ &\vec{m}_\nu = \{m_\nu^1, m_\nu^2, m_\nu^3\} \\ &h \\ &\mathrm{unit} = \mathrm{Mpc}/h \text{ or Mpc},\end{aligned} \tag{23}$$

the code will then calculate $T_\nu = (4/11)^{1/3} T_{\mathrm{cmb}}$, and $\rho_\gamma$, $\rho_\nu$ based on the equations in the previous section. Note that setting $T_{\mathrm{cmb}} = 0$ would turn off the radiation contribution to the cosmic evolution.

We need two different modes for the Hubble: $Hz(z)$ and $Ha(a)$. For most cases, we shall use $Hz(z)$, but we shall use $Ha(a)$ when we need to calculate from $a = 0$ (when calculating the particle horizon, age of the Universe, for example).

- Ha(a)

$$\begin{aligned} H^2(a) =& H_0^2\Bigg[\left(\Omega_{\mathrm{cdm}} + \Omega_{\mathrm{b}}\right) a^{-3} + \Omega_{\mathrm{de}} a^{-3(w_{\mathrm{de}}+1)} + \Omega_{\mathrm{r}} a^{-4} \\ &+ \Omega_{\mathrm{k}} a^{-2} + \frac{\rho_\nu^{m_\nu}(a)}{\rho_{\mathrm{crit}0}}\Bigg]\end{aligned} \tag{24}$$

  Here, $\Omega_{\mathrm{k}} = 1 - \Omega_{\mathrm{m}} - \Omega_{\mathrm{de}} - \Omega_{\mathrm{r}}$ with $\Omega_{\mathrm{m}} = \Omega_{\mathrm{cdm}} + \Omega_{\mathrm{b}} + \Omega_\nu^{\mathrm{massive}}$ and $\Omega_{\mathrm{r}} = \Omega_\gamma + \Omega_\nu^{\mathrm{massless}}$. With the definitions in the previous section, we find that

$$\begin{aligned}\Omega_\nu^{\mathrm{massless}} h^2 &= 5.7061 \times 10^{-6} N_\nu^{\mathrm{massless}} \\ \Omega_\nu^{\mathrm{massive}} h^2 &= 1.0042 \times 10^{-6} \sum_i f(x_\nu^i),\end{aligned} \tag{25}$$

  where $x_\nu^i \equiv m_\nu^i/(0.16767\,\mathrm{meV})$.

- Hz(z) $Hz(z) = Ha(1/(1+z))$
- Omz(z)

$$\Omega_{\mathrm{m}}(z) = \frac{(\Omega_{\mathrm{cdm}} + \Omega_{\mathrm{b}})(1+z)^3 + \rho_\nu^{\mathrm{massive}}(z)/\rho_{\mathrm{crit}}}{H^2(z)/H_0^2} \tag{26}$$

- Odez(z)

$$\Omega_{\mathrm{de}}(z) = \frac{\Omega_{\mathrm{de}}(1+z)^{3(w_{\mathrm{de}}+1)}}{H^2(z)/H_0^2} \tag{27}$$

- Orz(z)

$$\Omega_{\mathrm{r}}(z) = \frac{\left[\Omega_\gamma + \Omega_\nu^{\mathrm{massless}}\right](1+z)^4}{H^2(z)/H_0^2} \tag{28}$$

- Okz(z)

$$\Omega_{\rm k}(z) = \frac{\Omega_{\rm k}(1+z)^2}{H^2(z)/H_0^2} \tag{29}$$

## III. GEOMETRY

- etaz(z)

$$\eta(z) = \int \frac{dt}{a(t)} = \int_0^{1/(1+z)} \frac{da}{a^2 H(a)} \tag{30}$$

- chiz(z)

$$\chi(z) = \int_0^z \frac{dz}{H(z)} \tag{31}$$

- dAz(z)

$$d_A(z) = \begin{cases} \frac{\chi(z)}{1+z} & |\Omega_k| < 10^{-6} \\ \frac{1}{1+z}\frac{\sinh\left[\chi(z)\sqrt{\Omega_k}H_0\right]}{\sqrt{\Omega_k}H_0} & \Omega_k > 0 \\ \frac{1}{1+z}\frac{\sin\left[\chi(z)\sqrt{-\Omega_k}H_0\right]}{\sqrt{-\Omega_k}H_0} & \Omega_k < 0 \end{cases}$$

- dAcz(z)

$$d_{Ac}(z) = (1+z)d_A(z) \tag{32}$$

- dLz(z)

$$d_L(z) = d_A(z)(1+z)^2 \tag{33}$$

- Volume(zi,zf,Area)

$$V(z_i, z_f, A) = A\left(\frac{\pi}{180}\right)^2 \int_{z_i}^{z_f} \frac{(1+z)^2 d_A^2(z)}{H(z)} dz \tag{34}$$

## IV. LINEAR GROWTH OF STRUCTURE

The equation govern the growth of linear perturbation for presureless matter component is given by

$$\ddot{D} + 2H\dot{D} - 4\pi G\rho_m D = 0. \tag{35}$$

Here, dot denotes the time derivative. One can reform this equation in terms of the logarithmic scale factor $\ln a$ as a variable as

$$\dot{D} = \frac{d\ln a}{dt}\frac{dD}{d\ln a} = H\frac{dD}{d\ln a}. \tag{36}$$

Now, instead of $D$, we shall use the growth factor of the gravitational potential $g = D/a$, which leads to the differential equation of

$$g'' + \frac{1}{2}\left[5 - 3w\Omega_V(a) - \Omega_r(a) + \Omega_k(a)\right]g' + \frac{1}{2}\left[3(1-w)\Omega_V(a) + 2\Omega_r(a) + 4\Omega_k(a)\right]g = 0. \tag{37}$$

Note that the prime denotes to the differentiation with respect to $\ln a$.

In the code, we solve the differential equation for $g(a)$ starting at deep matter domination epoch where $g(a_0) = 1$ and $g'(a_0) = 0$. Therefore, the relevant differential equation ignores the radiation contribution:

$$g'' + \left[\frac{5}{2} + \frac{1}{2}\left(\Omega_{\rm k}(a) - 3w_{\rm de}\Omega_{\rm de}(a)\right)\right]g' + \left[2\Omega_{\rm k}(a) + \frac{3}{2}(1-w_{\rm de})\Omega_{\rm de}(a)\right]g = 0, \tag{38}$$

which coincides with eq. (76) of Komatsu et al. (2009). The new variables are related to the standard growth functions as

$$\begin{aligned} D(a) &= g(a)a \\ f(a) &= \frac{d\ln D}{d\ln a} = 1 + \frac{d\ln g}{d\ln a} = 1 + \frac{1}{g}\frac{dg}{d\ln a} \\ \frac{df}{d\ln a} &= \frac{d^2\ln g}{d\ln a^2} = -\frac{1}{g^2}\left(\frac{dg}{d\ln a}\right)^2 + \frac{1}{g}\frac{d^2 g}{d\ln a^2}. \end{aligned} \tag{39}$$

- Dz(z)

$$D(z) = \frac{g(z)}{1+z} \tag{40}$$

- Dplusz(z)

$$D_+(z) = \frac{D(z)}{D(0)} \tag{41}$$

- fz(z)

$$f(z) = \frac{d\ln D}{d\ln a} = \frac{g'(z)}{g(z)} + 1 \tag{42}$$

- Dfz(z)

$$\frac{df}{d\ln a}(z) = \frac{d^2\ln D}{d\ln a^2} = \frac{g''(z)}{g(z)} - (f-1)^2 \tag{43}$$