Luminosity function estimator

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In this note, we shall derive the maximum-likelihood estimator of the luminosity function from the flux-limited galaxy surveys.

I. INTRODUCTION

First, we spell out the likelihood function for the average galaxy number density $\mathrm{d}n = \rho(L,z)\mathrm{d}L\mathrm{d}z$ within the redshift interval $\mathrm{d}z$ and the luminosity interval $\mathrm{d}L$. We first consider the simplest case where the luminosity function does not evolve in redshift: $\rho(L,z) = \Phi(L)$ for which case the famous $1/V_{\max}$ estimator [1, 2] is the MLE. We then consider the case where the function $\rho(L,z)$ is separable: $\rho(L,z) = f(z)\Phi(L)$, for which case the MLE becomes the C^- estimator [3].

We then consider the spectroscopic emission line samples for the HETDEX survey.

II. LIKELIHOOD FUNCTION

A. Poisson statistics

Let us denote the expected number of galaxies in the redshift bin $\mathrm{d}z$ and the luminosity bin $\mathrm{d}L$ as

$$\lambda(L,z)\mathrm{d}L\mathrm{d}z. \tag{1}$$

Taking an infinitesimally small bin size $\mathrm{d}L\mathrm{d}z$, one can always make $\lambda(L,z)\mathrm{d}L\mathrm{d}z$ small enough so that the galaxy samples become Poisson realizations of given expectation number $\lambda(L,z)\mathrm{d}L\mathrm{d}z$. The Poisson distribution with a mean number m is,

$$P_m(n) = \frac{m^n}{n!}e^{-m}\,, \tag{2}$$

for the probability of having n objects. Therefore, for the bin centered around (L,z), the probability of having only one galaxy is

$$P_1(L,z) = \lambda(L,z)\mathrm{d}L\mathrm{d}z e^{-\lambda(L,z)\mathrm{d}L\mathrm{d}z}\,, \tag{3}$$

and having no galaxy is

$$P_0(L,z) = e^{-\lambda(L,z)\mathrm{d}L\mathrm{d}z}\,. \tag{4}$$

Now, let us consider the discrete binning in the (L,z) space: (L_i, z_i) (say, we have total M such bins), and order the bin so that each of $i = 1,\cdots,N$ bin contains one galaxy. The likelihood function is then,

$$\begin{aligned}\mathcal{L} =& \prod_{i=1}^{N} P_1(L_i,z_i) \prod_{j=N+1}^{M} P_0(L_j,z_j) \\ =& \left(\prod_{i=1}^{N}\lambda_i \mathrm{d}L_i\mathrm{d}z_i\right)\exp\left[-\sum_{i=1}^{M}\lambda_j\mathrm{d}L_j\mathrm{d}z_j\right]\,.\end{aligned} \tag{5}$$

Without any further work, we can obtain an intuitive result here by minimizing the log-likelihood function:

$$\ln\mathcal{L} = \sum_{i=1}^{N}\left[\ln\lambda_i + \ln\left(\mathrm{d}L_i\mathrm{d}z_i\right)\right] - \sum_{j=1}^{M}\lambda_j\mathrm{d}L_j\mathrm{d}z_j\,, \tag{6}$$

from which we find that the likelihood function is maximized

$$\frac{\mathrm{d}\ln\mathcal{L}}{\mathrm{d}\lambda_k} = \frac{1}{\lambda_k} - \mathrm{d}L_k\mathrm{d}z_k = 0\,, \tag{7}$$

at $\lambda_i = (\mathrm{d}L_i\mathrm{d}z_i)^{-1}$, for the detected first N bins. For $i \in (N+1,\cdots,M)$, the MLE is unconstrained. That is expected because we assumed that there is only one galaxy per each bin. In fact, that's precisely how we weigh each galaxy when calculating the histogram.

B. With varying survey depth

Now let us consider the case where the survey depth changes over the footprint and denote them by the completeness function:

$$p(F,\hat{n}), \tag{8}$$

that quantifies the completeness level, the fraction of detected galaxies to total galaxies, at a given flux F. The mean number of galaxies at the direction $\hat{n}$ is then given by

$$p\left(\frac{L}{4\pi D_L(z)^2},\hat{n}\right)\rho(L,z)\,. \tag{9}$$

with the luminosity distance $D_L(z)$. Assuming the statistical isotropy of the distribution of galaxies, we can integrate the angular dependences to compute the expected mean number of galaxies in the luminosity, redshift bin as

$$\lambda(L,z) = \int\frac{\mathrm{d}^2\hat{n}}{4\pi}p\left(\frac{L}{4\pi D_L(z)^2},\hat{n}\right)\rho(L,z)\frac{\mathrm{d}V}{\mathrm{d}z}\,, \tag{10}$$

where we define $p(\hat{n}) = 0$ outside of the survey boundary. The comoving volume element $\mathrm{d}V/\mathrm{d}z$ is given as

$$\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{4\pi d_A^2(z)}{a_0 H(z)}\,, \tag{11}$$

with $a_0 = 1/(H_0\sqrt{|\Omega_k|})$ for $\Omega_k \neq 0$, and $a_0 = 1$ for a spatially flat universe. Note that using the comoving volume elements, we explicitly define the luminosity function $\rho(L,z)$ as the galaxy number density per comoving volumne. Alternatively, one can also define the luminosity function with the physical volume.

Refs. [4, 5] has defined a function $\Omega(L,z)$ denoting "the fractional area of the sky in which galaxies have fluxes large enough so that they can be detected" to arrive at

$$\lambda(L,z,\hat{n}) = \Omega(L,z)\rho(L,z)\frac{dV}{dz}\,, \tag{12}$$

with which we relate Ω to the completeness as

$$\Omega(L,z) = \int \frac{\mathrm{d}^2\hat{n}}{4\pi} p\left(\frac{L}{4\pi D_L(z)^2}, \hat{n}\right)\,. \tag{13}$$

With the new variables and functions, the log-likelihood function becomes, besides the constant part,

$$\ln\mathcal{L} = \sum_{i=1}^{N} \ln\left[\rho(L_i,z_i)\right] - \int \mathrm{d}z\frac{\mathrm{d}V}{\mathrm{d}z}\int \mathrm{d}L\,\Omega(L,z)\rho(L,z)\,, \tag{14}$$

for the N observed galaxies.

III. MAXIMUM LIKELIHOOD ESTIMATORS

A. The $1/V_{\max}$ estimator

First, let us assume that the luminosity function is invariant, and does not depend on the redshift:

$$\rho(L,z) = \Phi(L)\,, \tag{15}$$

and parameterize them as the sum of discrete Dirac-delta at the luminosity that we want to estimate the luminosity function:

$$\Phi(L) = \sum_{j=1}^{M} \Phi_j \delta^D(L - L_j)\,. \tag{16}$$

That makes the likelikhood function

$$\ln\mathcal{L} = \sum_{i=1}^{N} \ln\Phi_i - \sum_{j=1}^{M}\int \mathrm{d}z\frac{\mathrm{d}V}{\mathrm{d}z}\Omega(L_j,z)\Phi_j\,, \tag{17}$$

and the maximum likelihood estimator for Φ_i is

$$\begin{aligned}\Phi_i^{-1} &= \int \mathrm{d}z\frac{\mathrm{d}V}{\mathrm{d}z}\Omega(L_i,z) \equiv V_{\rm eff}(L_i)\\ &= \int \mathrm{d}z\frac{\mathrm{d}V}{\mathrm{d}z}\int\frac{\mathrm{d}^2\hat{n}}{4\pi}p\left(\frac{L_i}{4\pi D_L(z)^2},\hat{n}\right)\,.\end{aligned} \tag{18}$$

Here, $V_{\rm eff}(L_i)$ is the completeness-weighted volume for a given luminosity L_i.

1. *Sharp completeness function*

If the flux-limit is given as a sharp cut, the completeness function becomes a step function,

$$p(F,\hat{n}) = \Theta\left(F - F_{\rm lim}(\hat{n})\right)\,, \tag{19}$$

which is one for $F > F_{\rm lim}$ (above a flux limit), and zero otherwise. In this limit, the step function dictates the upper bound of the z-integration as

$$V_{\rm eff}(L_i) = \int\frac{\mathrm{d}^2\hat{n}}{4\pi}\int_0^{z_{\max}(L_i,F_{\rm lim}(\hat{n}))}\mathrm{d}z\frac{\mathrm{d}V}{\mathrm{d}z} \equiv V_{\max}(L_i) \tag{20}$$

This estimator is the famous $1/V_{\max}$ estimator where we weigh each galaxy of luminosity L_i by the maximum volume to which the galaxy can be still included in the sample.

B. The C^- estimator

When the amplitude of the luminosity function evolves in redshift while the slope stays the same, we may write it as

$$\rho(L,z) = f(z)\Phi(L)\,. \tag{21}$$

Using this definition, the log-likelihood function becomes

$$\ln\mathcal{L} = \sum_{i=1}^{N}\ln(f_i\Phi_i) - \int \mathrm{d}z\frac{\mathrm{d}V}{\mathrm{d}z}\int \mathrm{d}L\Omega(L,z)\Phi(L)f(z) \tag{22}$$

Again, we shall write the Luminosity function and its amplitude as

$$\Phi(L) = \sum_{\alpha=1}^{M}\Phi_\alpha\delta^D(L-L_\alpha) \tag{23}$$

$$f(z) = \sum_{\beta=1}^{M} f_\beta\delta^D(z-z_\beta)\,, \tag{24}$$

where we assume, again $1,\cdots,N$ observed bins and $N+1,\cdots,M$ empty bins. with which the likelihood function becomes

$$\ln\mathcal{L} = \sum_{i=1}^{N}\ln(f_{i_\alpha}\Phi_{i_\beta}) - \sum_{\alpha=1}^{M}\sum_{\beta=1}^{M}\frac{\mathrm{d}V}{\mathrm{d}z}(z_\beta)\Omega(L_\alpha,z_\beta)\Phi_\alpha f_\beta\,. \tag{25}$$

then the maximum likelihood estimator satisfies, for the detected sources,

$$\frac{\partial\ln\mathcal{L}}{\partial f_k} = \frac{1}{f_k} - \sum_{\alpha=1}^{M}\frac{\mathrm{d}V}{\mathrm{d}z}(z_k)\Omega(L_\alpha,z_k)\Phi_\alpha = 0\,, \tag{26}$$

$$\frac{\partial\ln\mathcal{L}}{\partial \Phi_k} = \frac{1}{\Phi_k} - \sum_{\beta=1}^{M}\frac{\mathrm{d}V}{\mathrm{d}z}(z_\beta)\Omega(L_k,z_\beta)f_\beta = 0\,, \tag{27}$$

or

$$f_k^{-1} = \sum_{\alpha=1}^{N} \frac{\mathrm{d}V}{\mathrm{d}z}(z_k)\Omega(L_\alpha, z_k)\Phi_\alpha \tag{28}$$

$$\Phi_k^{-1} = \sum_{\beta=1}^{N} \frac{\mathrm{d}V}{\mathrm{d}z}(z_\beta)\Omega(L_k, z_\beta) f_\beta\,, \tag{29}$$

for the detected sources $k \in (1, N)$. Note that we set unconstrained f_i and Φ_i to zero.

The normalization of the luminosity function is

$$N_{\mathrm{sample}} = \int \rho(L,z)\mathrm{d}L\mathrm{d}z\frac{\mathrm{d}V}{\mathrm{d}z} = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\Phi_\alpha f_\beta \frac{\mathrm{d}V}{\mathrm{d}z}(z_\beta)\,, \tag{30}$$

for the total number of galaxies, and

$$n = \int \rho(L,z)\mathrm{d}L = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\Phi_\alpha f_\beta\,, \tag{31}$$

for the number density of galaxie.

1. *Sharp completeness function*

For a step-function completeness curve with the fixed limiting flux F_{lim}, the C^- method [3] goes as follows. Here, we closely follow the calculation of [6]. In this case, the $\Omega(L, z)$ becomes the step function:

$$\Omega(L, z) = \Omega\Theta(z^{\max}(L) - z) \tag{32}$$

which is 1 for $z \le z_{\max}(L)$ and 0 otherwise. Here, Ω is the fraction of sky covered by the survey. Therefore, the MLE conditions become:

$$f_k^{-1} = \sum_{\alpha=1}^{N} \frac{\mathrm{d}V}{\mathrm{d}z}(z_k)\Phi_\alpha\Omega\Theta(z^{\max}(L_\alpha) - z_k) \tag{33}$$

$$\Phi_k^{-1} = \sum_{\beta=1}^{N} \frac{\mathrm{d}V}{\mathrm{d}z}(z_\beta) f_\beta\Omega\Theta(z^{\max}(L_k) - z_\beta)\,, \tag{34}$$

Let us order the galaxies in the descending luminosity so that $L_k \ge L_{k+1}$ for all k, and define $C_i \equiv C^-(L_i)$ as the total number of observed galaxies which satisfy following two conditions:

$$L_j > L_i\,,\ z_j \le z_i^{\max}\,, \tag{35}$$

that is, C_i encloses all galaxies that are brighter than L_i, and within the effective volume $V_{\max}$. Note that $C_1 = 0$ for all samples, as L_1 is the brightest of the sample. Using the definitions in this section, we calculate the C_k as

$$\begin{aligned} C_k =& C^-(L_k) \\ =& \Omega\int_0^{z_k^{\max}} \mathrm{d}z\frac{\mathrm{d}V}{\mathrm{d}z} f(z)\int_{L_k}^{\infty}\mathrm{d}L\Phi(L) \\ =& \Omega\sum_{i=1}^{N}\frac{\mathrm{d}V}{\mathrm{d}z}(z_i) f_i\Theta(z^{\max}(L_k) - z_i)\sum_{j=1}^{k}\Phi_j \\ =& \frac{1}{\Phi_k}\sum_{j=1}^{k-1}\Phi_j\,. \end{aligned} \tag{36}$$

Note that in the last equality we use the MLE condition Eq. (34). With the equation above, we find the recursion relation of

$$\begin{aligned} C_k + 1 =& \frac{1}{\Phi_k}\sum_{j=1}^{k}\Phi_j = \frac{\Phi_{k+1}}{\Phi_k}\left(\frac{1}{\Phi_{k+1}}\sum_{j=1}^{k}\Phi_j\right) \\ =& \frac{\Phi_{k+1}}{\Phi_k}C_{k+1}\,, \end{aligned} \tag{37}$$

which leads to

$$\Phi_{k+1} = \Phi_k\frac{C_k + 1}{C_{k+1}}\,, \tag{38}$$

or

$$\Phi_n = \Phi_1\prod_{i=1}^{n-1}\left(\frac{C_i + 1}{C_{i+1}}\right)\,. \tag{39}$$

The cumulative luminosity function is, then

$$\begin{aligned} \int_{L_n}^{\infty}\mathrm{d}L'\phi(L') =& \sum_{i=1}^{n}\Phi_i = \Phi_n\left(C_n + 1\right) \\ =& \Phi_1\prod_{i=1}^{n-1}\left(\frac{C_i + 1}{C_{i+1}}\right)(C_n + 1) \\ =& \Phi_1\prod_{i=2}^{n}\left(\frac{C_i + 1}{C_i}\right)\,. \end{aligned} \tag{40}$$

Alternatively, one can write

$$\int_{L_n}^{\infty}\mathrm{d}L'\phi(L') = \Phi_1\prod_{i=1}^{n}\left(\frac{C_i + 1}{C_i}\right)\,, \tag{41}$$

with

$$\frac{C_1 + 1}{C_1} = 1\,. \tag{42}$$

2. *Original derivation*

The way that Lynden-Bell introduced the method in Ref. [3] goes as follows. Let's assume a complete magnitude limited sample, for which the region defined as

$C^{-}(L)$ is also complete without any bias. We can therefore use the $C^{-}(L)$ region to estimate the luminosity function. Let's call $\psi(L_1)$ the true fraction of sources with $L \geq L_1$, and $\phi(L)$ the un-normalized luminosity function: $\phi = -\psi' = -\mathrm{d}\psi/\mathrm{d}L$. In reality, $X(L_1)$ is the observed number of galaxies above L_1. Then, Lynden-Bell begins with by relating

$$\frac{\delta\psi}{\psi} = \frac{\delta X}{C_1}, \tag{43}$$

here δX is the number of points in the infinitesimally small interval between L_1 and $L_1 + \delta L_1$ and C_1 is the number of points in the area defined above as $C^{-}(L_1)$.

Note that This equation holds if and only if $\rho(L, z) = \Phi(L)f(z)$ is separable so that the z-dependence cancels in Eq. (43). That is, the infinitesimal change in luminosity affects the same fractional change at all redshifts.

Integrating Eq. (43) yields

$$\psi(L_1) = A \exp\left(\int_{L_{\min}}^{L_1} \mathrm{d}L \frac{dX}{\mathrm{d}L}\frac{1}{C(L)}\right), \tag{44}$$

where one can obtain the normalization A from $\psi(\infty) = 1$, or matching the observed luminosity function with the number density. As X will jump at the observed galaxies with luminosity L_i, we could write

$$\frac{dX}{dL} = -\sum_i \delta^D(L - L_i)\,, \tag{45}$$

but that will cause error when $C(L)$ is not large. Lynden-Bell then pursued to find a way to include a galaxy at the edge of the $C^{-}(L)$ area in a contiguous way. Here's how he handles it. Let's consider the galaxy at luminosity L_i and some luminosity L close to L_i. Defining

$$x \equiv X(L) - X(L_i^{+})\,, \tag{46}$$

and $C^{-}(L_i)$ to be the value of C at L_i with the contribution of L_i omitted, we have

$$C(L) = C^{-}(L_i) + x\,, \tag{47}$$

then, the integration Eq. (44) across the point L_i becomes

$$\int_{L_i^{-}}^{L_i^{+}} \mathrm{d}L \frac{\mathrm{d}X}{\mathrm{d}L}\frac{1}{C(L)} = \int_0^1 \frac{dx}{C^{-} + x} = \ln\left(\frac{C^{-}+1}{C^{-}}\right). \tag{48}$$

Finally, substituting the result to Eq. (44), we arrive at the final expression:

$$\psi(L) = A \exp\left[\sum_i \ln\left(\frac{C^{-}+1}{C^{-}}\right)\right] = A\prod_i \frac{C^{-}(L_i)+1}{C^{-}(L_i)}. \tag{49}$$

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