Chapter 9 - Center of Mass and Linear Momentum

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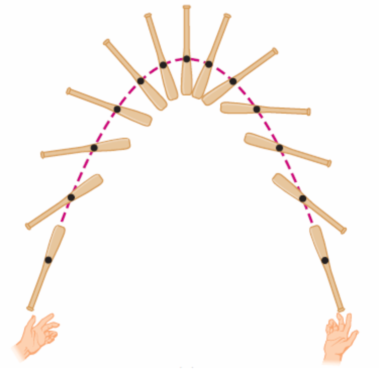
PHYS 211

Center of Mass

The center of mass of a system of particles is the point that moves as though

(a) *all of the mass were concentrated there;*

(b) *all external forces were applied there.*



Center of Mass

The center of mass of system of N particles is a weighted average of their positions:

$$x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{m_1 + m_2 + \cdots + m_N}.$$

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In fact, we can do this in any dimension:

$$y_{\text{com}} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_N y_N}{m_1 + m_2 + \cdots + m_N},$$

$$z_{\text{com}} = \frac{m_1 z_1 + m_2 z_2 + \cdots + m_N z_N}{m_1 + m_2 + \cdots + m_N}.$$

Center of Mass

In three dimensions, the center of mass is:

$$\vec{r}_{com} = x_{\text{com}}\hat{i} + y_{\text{com}}\hat{j} + z_{\text{com}}\hat{k}$$

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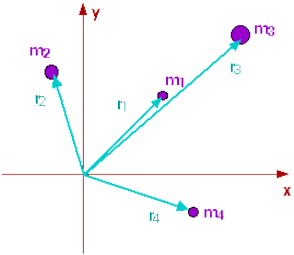

$$\vec{r}_{\text{com}} = \frac{1}{M}\sum_{i}^{N} m_i \vec{r}_i$$

Center of Mass

For solid bodies, the summation becomes an integral:

$$x_{\text{com}} = \frac{1}{M}\int x\,dm,$$
$$y_{\text{com}} = \frac{1}{M}\int y\,dm,$$
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The body is sectioned into point masses *dm*.

Center of Mass

The term **dm** *represents a small mass and depends on the problem at hand:*

$$\begin{aligned} \text{1D:} \quad dm &= \lambda\, dx, \\ \text{2D:} \quad dm &= \sigma\, dA, \\ \text{3D:} \quad dm &= \rho\, dV. \end{aligned}$$

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- λ is linear mass density kg/m
- σ is surface mass density kg/m^2
- ρ is volume mass density kg/m^3

Center of Mass

Lecture Question 9.1

Two girl scouts are sitting in a large canoe facing north on a still lake. The girl at the north end walks to her friend at the south end and sits down.

(a) The canoe will still be at rest, but it will be south of its original position.

(b) The canoe will still be at rest, but it will be north of its original position.

(c) The canoe will be moving toward the south.

(d) The canoe will be moving toward the north.

(e) The canoe will still be at rest at its original position.

Newton's 2nd Law - Revisited

For a system of particles (connected or not),

Newton's 2nd Law applies to the center of mass:

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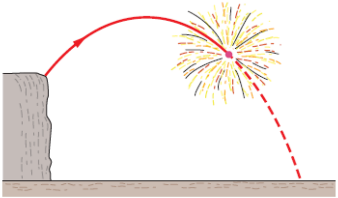

Newton's 2nd Law - Revisited

Proof:

$$\vec{r}_{\text{com}} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N)$$

$$M\vec{a}_{\text{com}} = \vec{F}_{net}$$

Newton's 2nd Law - Revisited

Proof:

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M\vec{a}_{\text{com}} &= m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_N\vec{a}_N \\
M\vec{a}_{\text{com}} &= \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N
\end{aligned}$$

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Linear Momentum and Impulse

The momentum of a particle is defined to be

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We can re-write Newton's 2nd Law using $\vec{p}$:

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

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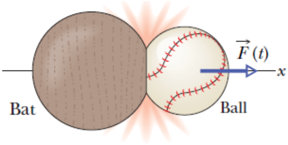
We then get

$$\vec{F}_{net} = \frac{d\vec{P}}{dt}$$

Linear Momentum and Impulse

When two objects collide there is a time varying force between them:



Linear Momentum and Impulse

The impulse is a summation of the total change in momentum.

$$\begin{aligned} d\vec{p}/dt &= \vec{F} \\ d\vec{p} &= \vec{F}dt \\ \int_{t_i}^{t_f} d\vec{p} &= \int_{t_i}^{t_f} \vec{F}(t)dt \end{aligned}$$

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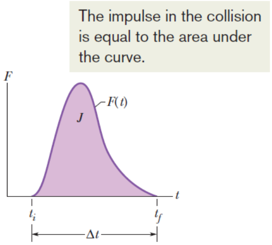
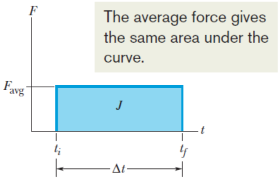


(a)



(b)

Linear Momentum and Impulse

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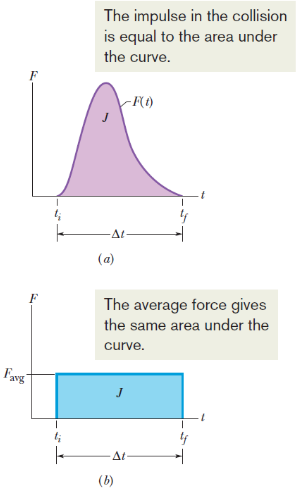
$$d\vec{p}/dt = \vec{F}$$

$$d\vec{p} = \vec{F}dt$$

$$\int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt$$

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt$$

$$\vec{J} = \vec{p}_f - \vec{p}_i$$



Linear Momentum and Impulse

Lecture Question 9.3

Two hockey players ($m_1 = 40$ kg and $m_2 = 60$ kg) are on either end of a 10 m rope. They slowly pull on the rope, bringing them together. When they finally meet, how far has the 60-kg player moved?

(a) 0 m

(b) 4.0 m

(c) 5.0 m

(d) 6.0 m

(e) 10 m

Conservation of Linear Momentum

If there are no external forces then momentum is conserved.

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Mathematically, we can write $\vec{P}_{before} = \vec{P}_{after}$ for

- Collisions
- Explosions

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Conservation of Linear Momentum

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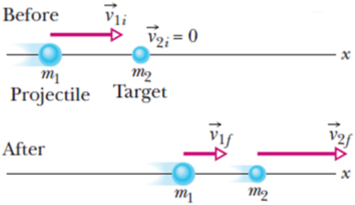
m_1 v_1 + m_2 v_2 \quad &= \quad m_1 v_1' + m_2 v_2'
\end{aligned}
$$

Examples:

- Bouncy-balls colliding
- Carts with springs for bumpers

Conservation of Linear Momentum

If an object of mass m_1 is shot at a stationary target of mass m_2 at a speed of v_{1i}, what are the speeds of the two objects after an elastic collision?

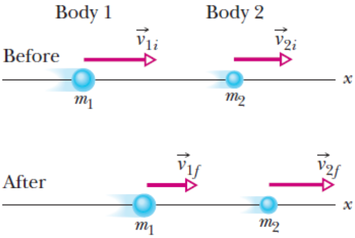


Conservation of Linear Momentum

If an object of mass m_1 is shot with speed v_{1i} at a moving target of mass m_2 at a speed of v_{2i}, what are the speeds of the two objects after an elastic collision?

Conservation of Linear Momentum

During an **inelastic collision**, *some kinetic energy is transferred to another form (e.g. heat or sound).*



$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

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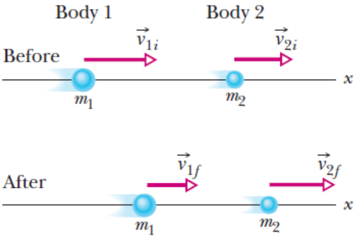
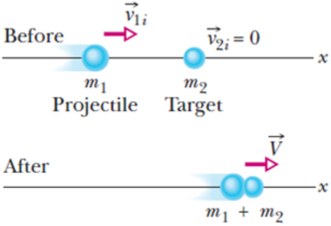


$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Example: two pool balls striking

Conservation of Linear Momentum

During a **completely inelastic collision**, *two bodies stick together and the kinetic energy loss is maximum.*



$$m_1 v_{1i} = m_1 V + m_2 V \rightarrow V = \frac{m_1}{m_1 + m_2} v_{1i}$$

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In a collision, the center of mass moves with constant velocity (no external forces!).

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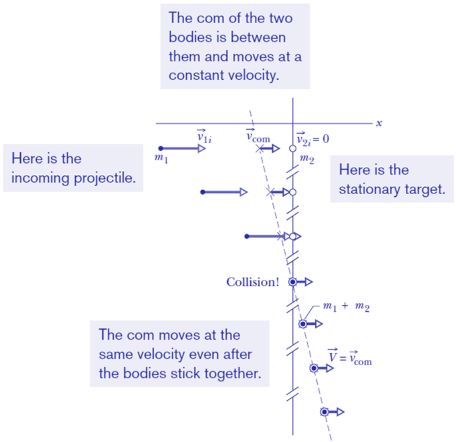

Conservation of Linear Momentum

Lecture Question 9.4

Two carts (m and $1.5m$) are placed on a horizontal air track. The lighter cart has a speed v just before it collides with the heavier cart at rest. What is the speed of the center of mass of the two carts after the collision?

(a) v

(b) $4v/5$

(c) $2v/5$

(d) $v/2$

(e) Cannot determine since we don't know if the collision is elastic or inelastic.