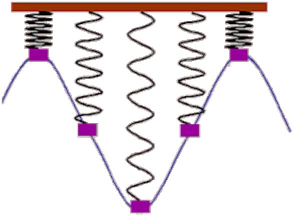
Chapter 15 - Oscillations



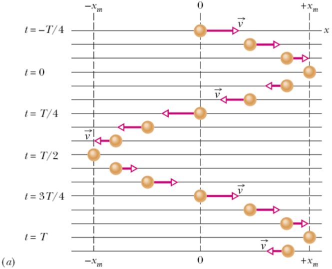
"The pendulum of the mind oscillates between sense and nonsense, not between right and wrong."

-Carl Gustav Jung

David J. Starling
Penn State Hazleton
PHYS 211

Simple Harmonic Oscillator (SHO)

Oscillatory motion is motion that is periodic in time (e.g., earthquake shakes, guitar strings).



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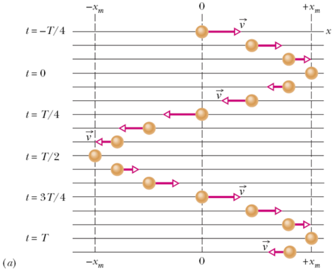
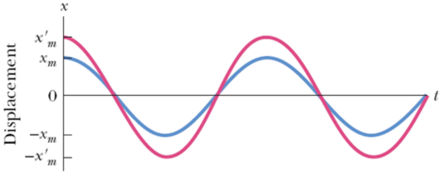


The period T measures the time for one oscillation.

Simple Harmonic Oscillator (SHO)

Oscillatory motion that is sinusoidal is known as **Simple Harmonic Motion**.

$$x(t) = x_m \cos(\omega t)$$



(a)

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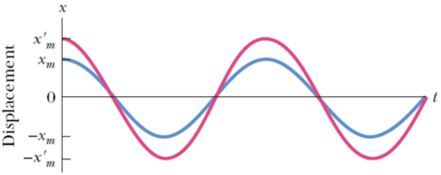

(a)

x_m is the maximum displacement

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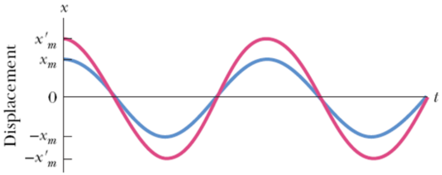


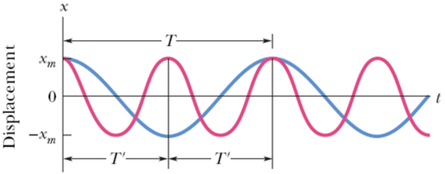
(*a*)

x_m is the maximum displacement
ω is the angular frequency: $\omega = 2\pi f = 2\pi / T$.

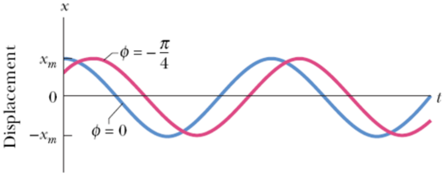
Simple Harmonic Oscillator (SHO)

Two oscillators can have different frequencies, or different phases:

$$x(t) = x_m \cos(\omega t + \phi)$$



(b)



(c)

Simple Harmonic Oscillator (SHO)

Sinusoidal oscillations are described by these definitions:

$$\begin{aligned} x(t) &= x_m \cos(\omega t + \phi) \\ T &= 1/f \\ \omega &= 2\pi f \end{aligned}$$

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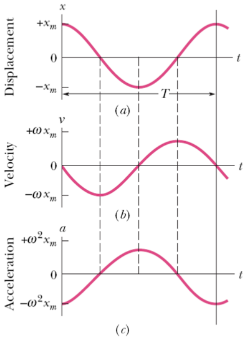
- x in meters
- T in seconds
- f in Hertz (1/s)
- ω in rad/s

Simple Harmonic Oscillator (SHO)

Since we know the position of an oscillating object, we also know its velocity and acceleration:

$$x(t) = x_m \cos(\omega t + \phi)$$
$$v(t) = \frac{dx}{dt} = -\omega x_m \sin(\omega t + \phi)$$
$$a(t) = \frac{dv}{dt} = -\omega^2 x_m \cos(\omega t + \phi)$$



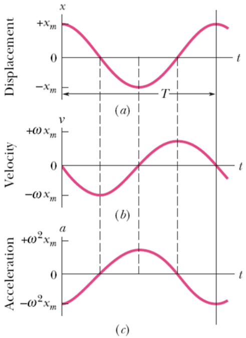
Simple Harmonic Oscillator (SHO)

The maximum velocity and acceleration depend on the frequency and the maximum displacement.

Max position: x_m

Max velocity: ωx_m

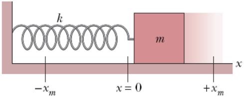
Max acceleration: $\omega^2 x_m$



Simple Harmonic Oscillator (SHO)

Simple harmonic motion is generated by a linear restoring force:

$$F = -kx = ma = m\frac{d^2x}{dt^2}$$

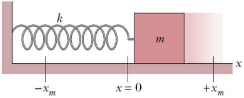


Simple Harmonic Oscillator (SHO)

Simple harmonic motion is generated by a linear restoring force:

$$F = -kx = ma = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$



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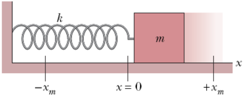


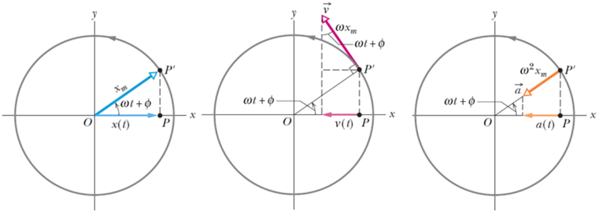
The solution to this differential equation:

$$x(t) = x_m \cos(\sqrt{k/m}t + \phi)$$

(so $\omega = \sqrt{k/m}$)

Simple Harmonic Oscillator (SHO)

The motion of a SHO is related to motion in a circle.



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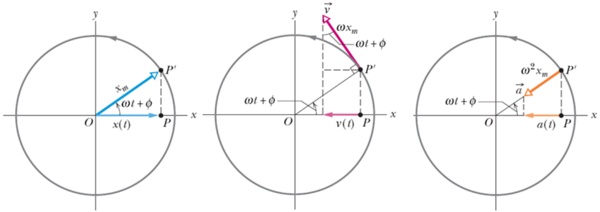

$$x(t) = x_m \cos(\omega t + \phi)$$
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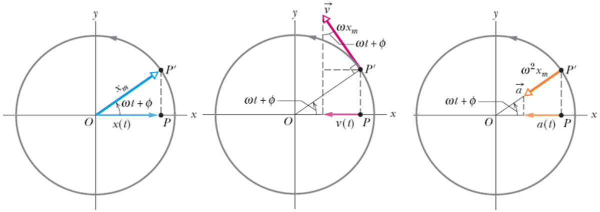
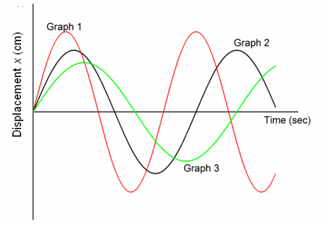


$$
\begin{aligned}
x(t) &= x_m \cos(\omega t + \phi) \\
v(t) &= -\omega x_m \sin(\omega t + \phi) \\
a(t) &= -\omega^2 x_m \cos(\omega t + \phi)
\end{aligned}
$$

Lecture Question 15.1

The graph below represents the oscillatory motion of three different springs with identical masses attached to each. Which of these springs has the smallest spring constant?



(a) Graph 1

(b) Graph 2

(c) Graph 3

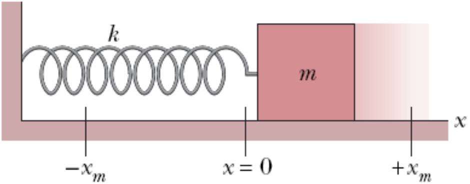
(d) Both 2 and 3 are smallest and equal

(e) All three have the same spring constant.

Energy in SHO

A spring stores potential energy. To find it, calculate the work the spring force does:

$$\Delta U = -W = -\int_{x_1}^{x_2} (-kx)dx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$



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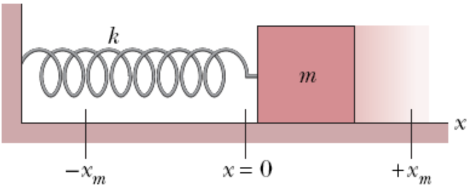
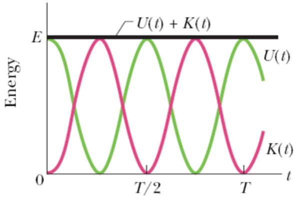
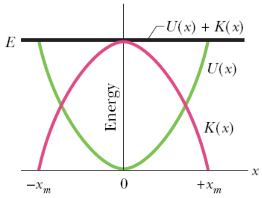


A spring compressed by x stores energy $U = \frac{1}{2}kx^2$.

Energy in SHO

As a mass oscillates, the energy transfers from kinetic to potential energy.





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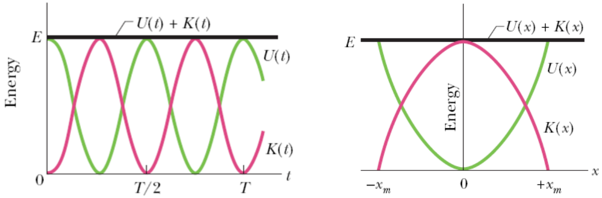
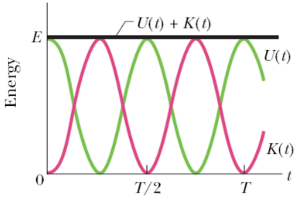
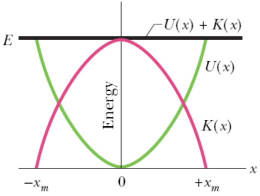


At the ends of the motion, velocity is zero, K is zero and U is maximum.

Energy in SHO





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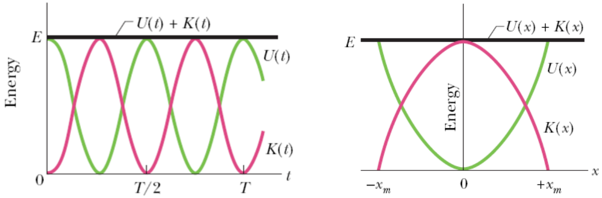
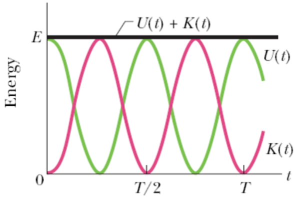
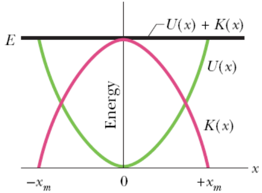


The energy oscillates between:

$$\begin{aligned} U &= \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2\cos^2(\omega t+\phi) \\ K &= \frac{1}{2}mv^2 = \frac{1}{2}\underbrace{m\omega^2}_{k}x_m^2\sin^2(\omega t+\phi) \end{aligned}$$

Energy in SHO





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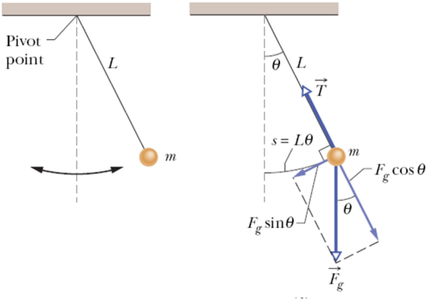
Pendulums

A simple pendulum is a ball on a string. It acts like a SHO for small angles.



The restoring force:

$$F = -mg\sin\theta \approx -mg\theta$$
$$I = mL^2$$

Pendulums

For small angles ($\theta < 20^\circ$), a pendulum is like a simple harmonic oscillator.

$$\tau = -(mg\theta)L \quad = \quad I\alpha = I\frac{d^2\theta}{dt^2}$$

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\end{aligned}$$

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\omega &= \sqrt{mgL/I} = \sqrt{g/L}
\end{aligned}$$

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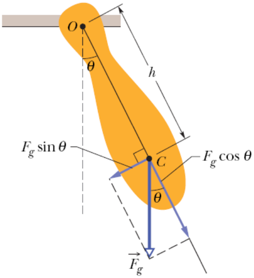
\omega &= \sqrt{mgL/I} = \sqrt{g/L} \\
T &= 1/f = 2\pi/\omega = 2\pi\sqrt{L/g}
\end{aligned}
$$

Pendulums

For a physical pendulum with moment of inertia I and small oscillations,



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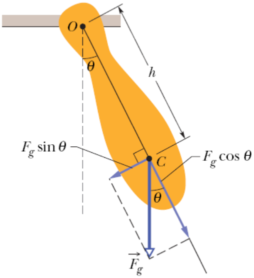

$$\frac{d^2\theta}{dt^2} + \frac{mgh}{I}\theta = 0$$

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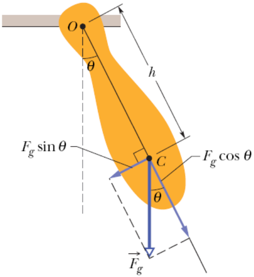

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$$\omega = \sqrt{mgh/I}$$

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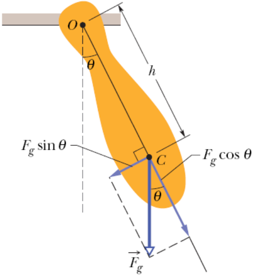


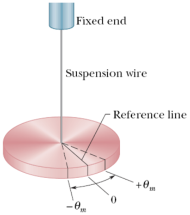
$$\frac{d^2\theta}{dt^2} + \frac{mgh}{I}\theta = 0$$

$$\omega = \sqrt{mgh/I}$$

$$T = 1/f = 2\pi/\omega = 2\pi\sqrt{I/mgh}$$

Pendulums

A torsion pendulum is a symmetric object where the restoring torque arises from a twisted wire.



$$\tau = -\kappa\theta \text{ (similar to spring)}$$

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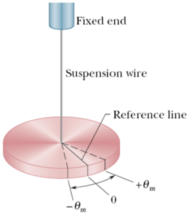

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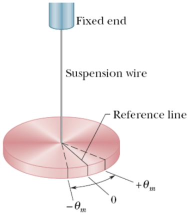

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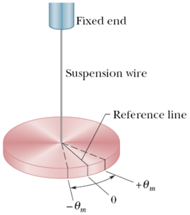


$$\begin{aligned}\tau &= -\kappa\theta \text{ (similar to spring)}\\ \frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta &= 0\\ \omega &= \sqrt{\kappa/I}\\ T &= 1/f = 2\pi/\omega = 2\pi\sqrt{I/\kappa}\end{aligned}$$

Lecture Question 15.3

A grandfather clock, which uses a pendulum to keep accurate time, is adjusted at sea level. The clock is then taken to an altitude of several kilometers. How will the clock behave in its new location?

(a) The clock will run slow.

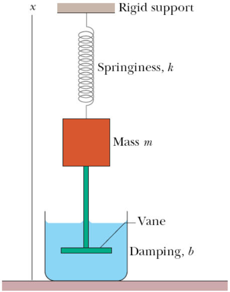
(b) The clock will run fast.

(c) The clock will run the same as it did at sea level.

(d) The clock cannot run at such high altitudes.

Damped Oscillations

Damped simple harmonic motion is the result of oscillatory behavior in the presence of a retarding force.



$$F_d = -bv$$

with b the damping constant.

Damped Oscillations

Applying Newton's Second Law to this situation,

$$\begin{aligned} F_{net} &= ma \\ -kx - bv &= ma \\ 0 &= \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x \end{aligned}$$

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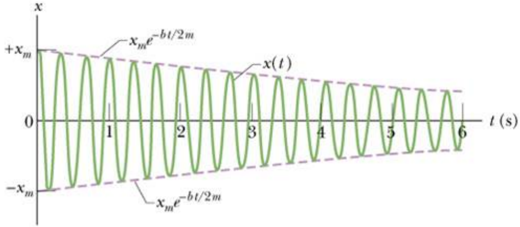
The solution:

$$\begin{aligned} x(t) &= x_m e^{-bt/2m}\cos(\omega' t + \phi) \\ \omega' &= \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \end{aligned}$$

Damped Oscillations

For damped harmonic motion, the oscillations will die out over time.



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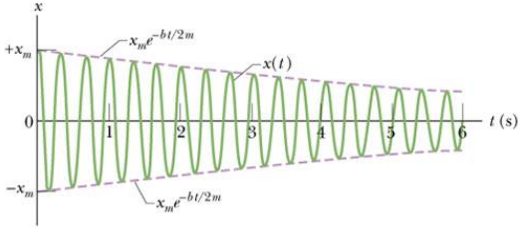
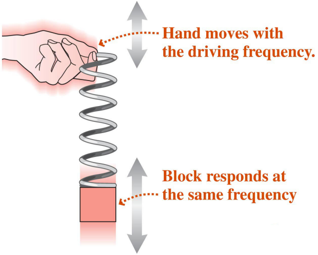


Side note: $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ is only valid if $k/m > b^2/4m^2$. What happens if $k/m \leq b^2/4m^2$?

Damped Oscillations

If an oscillator of angular frequency ω is **driven** *by an external force at a frequency ω_d, then the response will also be at ω_d.*



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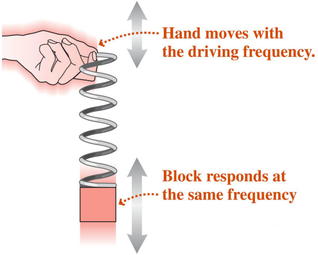
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = F_0 \cos(\omega t) \rightarrow x(t) = A\cos(\omega t + \phi)$$



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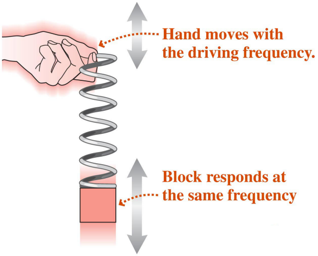
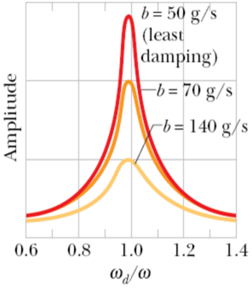


The amplitude A depends on the relationship between ω_d and ω.

Damped Oscillations

If a damped oscillator is driven at its natural frequency, the system is **on resonance** *and the oscillations are maximum.*



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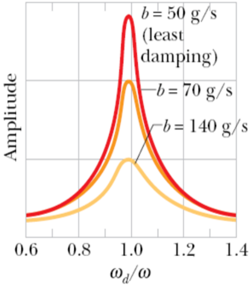

The width of the resonance peak depends on the damping constant.