A TRIGONOMETRIC SERIES

Suppose that \( f \) is a function defined on the interval \([-p, p]\) and can be expanded in an orthogonal series consisting of the trigonometric functions in the orthogonal set (1); that is,

\[
c_0 \phi_0(x) + c_1 \phi_1(x) + c_2 \phi_2(x) + \cdots,
\]

where the coefficients \( c_n \) are determined by using the inner product concept. The orthogonal set of trigonometric functions

\[
\left\{ 1, \cos \frac{\pi}{p} x, \cos \frac{2\pi}{p} x, \cos \frac{3\pi}{p} x, \ldots, \sin \frac{\pi}{p} x, \sin \frac{2\pi}{p} x, \sin \frac{3\pi}{p} x, \ldots \right\}
\]

will be of particular importance later on in the solution of certain kinds of boundary-value problems involving linear partial differential equations. The set (1) is orthogonal on the interval \([-p, p]\).

A TRIGONOMETRIC SERIES

Suppose that \( f \) is a function defined on the interval \([-p, p]\) and can be expanded in an orthogonal series consisting of the trigonometric functions in the orthogonal set (1); that is,

\[
f(x) = c_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right).
\]

The coefficients \( a_0, a_1, a_2, \ldots, b_1, b_2, \ldots \) can be determined in exactly the same manner as in the general discussion of orthogonal series expansions on page 401. Before proceeding, note that we have chosen to write the coefficient of 1 in the set (1) as \( c_0 \) rather than \( a_0 \). This is for convenience only; the formula of \( a_n \) will then reduce to \( a_0 \) for \( n = 0 \).

Now integrating both sides of (2) from \(-p\) to \( p\) gives

\[
\int_{-p}^{p} f(x) \, dx = a_0 \int_{-p}^{p} 1 \, dx + \sum_{n=1}^{\infty} \left( a_n \int_{-p}^{p} \cos \frac{n\pi}{p} x \, dx + b_n \int_{-p}^{p} \sin \frac{n\pi}{p} x \, dx \right).
\]

Since \( \cos(n\pi x/p) \) and \( \sin(n\pi x/p) \), \( n \neq 1 \) are orthogonal to 1 on the interval, the right side of (3) reduces to a single term:

\[
\int_{-p}^{p} f(x) \, dx = a_0 \frac{2}{p} \int_{-p}^{p} x \, dx = a_0 \frac{p}{2} = pa_0.
\]

Solving for \( a_0 \) yields

\[
a_0 = \frac{1}{p} \int_{-p}^{p} f(x) \, dx.
\]

Now we multiply (2) by \( \cos(m\pi x/p) \) and integrate:

\[
\int_{-p}^{p} f(x) \cos \frac{m\pi}{p} x \, dx = a_0 \frac{2}{p} \int_{-p}^{p} \cos \frac{m\pi}{p} x \, dx + \sum_{n=1}^{\infty} \left( a_n \int_{-p}^{p} \cos \frac{m\pi}{p} x \cos \frac{n\pi}{p} x \, dx + \int_{-p}^{p} \cos \frac{m\pi}{p} x \sin \frac{n\pi}{p} x \, dx \right).
\]
By orthogonality we have

\[
\int_{-p}^{p} \cos \frac{m \pi}{p} x \, dx = 0, \quad m > 0, \quad \int_{-p}^{p} \cos \frac{m \pi}{p} x \sin \frac{n \pi}{p} x \, dx = 0,
\]

and

\[
\int_{-p}^{p} \cos \frac{m \pi}{p} x \cos \frac{n \pi}{p} x \, dx = \begin{cases} 0, & m \neq n \\ p, & m = n. \end{cases}
\]

Thus (5) reduces to

\[
\int_{-p}^{p} f(x) \cos \frac{n \pi}{p} x \, dx = a_n p,
\]

and so

\[
a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n \pi}{p} x \, dx. \quad (6)
\]

Finally, if we multiply (2) by \(\sin(m \pi x / p)\), integrate, and make use of the results

\[
\int_{-p}^{p} \sin \frac{m \pi}{p} x \, dx = 0, \quad m > 0, \quad \int_{-p}^{p} \sin \frac{m \pi}{p} x \cos \frac{n \pi}{p} x \, dx = 0,
\]

and

\[
\int_{-p}^{p} \sin \frac{m \pi}{p} x \sin \frac{n \pi}{p} x \, dx = \begin{cases} 0, & m \neq n \\ p, & m = n, \end{cases}
\]

we find that

\[
b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n \pi}{p} x \, dx. \quad (7)
\]

The trigonometric series (2) with coefficients \(a_0\), \(a_n\), and \(b_n\) defined by (4), (6), and (7), respectively, is said to be the Fourier series of the function \(f\). The coefficients obtained from (4), (6), and (7) are referred to as Fourier coefficients of \(f\).

In finding the coefficients \(a_0\), \(a_n\), and \(b_n\), we assumed that \(f\) was integrable on the interval and that (2), as well as the series obtained by multiplying (2) by \(\cos(m \pi x / p)\), converged in such a manner as to permit term-by-term integration. Until (2) is shown to be convergent for a given function \(f\), the equality sign is not to be taken in a strict or literal sense. Some texts use the symbol \(\sim\) in place of \(=\). In view of the fact that most functions in applications are of a type that guarantees convergence of the series, we shall use the equality symbol. We summarize the results:

**DEFINITION 11.2.1 Fourier Series**

The Fourier series of a function \(f\) defined on the interval \((-p, p)\) is given by

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n \pi}{p} x + b_n \sin \frac{n \pi}{p} x\right), \quad (8)
\]

where

\[
a_0 = \frac{1}{p} \int_{-p}^{p} f(x) \, dx \quad (9)
\]

\[
a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n \pi}{p} x \, dx \quad (10)
\]

\[
b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n \pi}{p} x \, dx. \quad (11)
\]
EXAMPLE 1  Expansion in a Fourier Series

Expand
\[ f(x) = \begin{cases} 
0, & -\pi < x < 0 \\
\pi - x, & 0 \leq x < \pi 
\end{cases} \]  
(12)
in a Fourier series.

SOLUTION  The graph of \( f \) is given in Figure 11.2.1. With \( p = \pi \) we have from (9) and (10) that
\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \left[ \int_{0}^{\pi} \frac{\pi}{2} \, dx + \int_{0}^{\pi} (\pi - x) \, dx \right] = \frac{1}{\pi} \left[ \pi x \vphantom{\frac{\pi}{2}} \right]_0^{\pi} = \frac{\pi}{2} \]
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \left[ \int_{0}^{\pi} \cos nx \, dx + \int_{0}^{\pi} (\pi - x) \cos nx \, dx \right] \]
\[ = \frac{1}{\pi} \left[ \left( \pi - x \right) \frac{\sin nx}{n} \right]_0^{\pi} + \frac{1}{n} \int_{0}^{\pi} \sin nx \, dx \]
\[ = - \frac{\cos nx}{n \pi} \bigg|_0^{\pi} = \frac{1 - (-1)^n}{n^2 \pi}, \]
where we have used \( \cos n\pi = (-1)^n \). In like manner we find from (11) that
\[ b_n = \frac{1}{\pi} \int_{0}^{\pi} (\pi - x) \sin nx \, dx = \frac{1}{n}, \]
Therefore
\[ f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left( \frac{1 - (-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right). \]  
(13)

Note that \( a_n \) defined by (10) reduces to \( a_0 \) given by (9) when we set \( n = 0 \). But as Example 1 shows, this might not be the case after the integral for \( a_n \) is evaluated.

CONVERGENCE OF A FOURIER SERIES  The following theorem gives sufficient conditions for convergence of a Fourier series at a point.

THEOREM 11.2.1  Conditions for Convergence

Let \( f \) and \( f' \) be piecewise continuous on the interval \((-p, p); \) that is, let \( f \) and \( f' \) be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of \( f \) on the interval converges to \( f(x) \) at a point of continuity. At a point of discontinuity the Fourier series converges to the average
\[ \frac{f(x+) + f(x-)}{2}, \]
where \( f(x+) \) and \( f(x-) \) denote the limit of \( f \) at \( x \) from the right and from the left, respectively.*

For a proof of this theorem you are referred to the classic text by Churchill and Brown.†

*In other words, for \( x \) a point in the interval and \( h > 0, \)
\[ f(x+) = \lim_{h \to 0^+} f(x + h), \quad f(x-) = \lim_{h \to 0^-} f(x - h). \]
EXAMPLE 2 Convergence of a Point of Discontinuity

The function (12) in Example 1 satisfies the conditions of Theorem 11.2.1. Thus for every \( x \) in the interval \( (-\pi, \pi) \), except at \( x = 0 \), the series (13) will converge to \( f(x) \). At \( x = 0 \) the function is discontinuous, so the series (13) will converge to

\[
\frac{f(0^+) + f(0^-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}.
\]

PERIODIC EXTENSION Observe that each of the functions in the basic set (1) has a different fundamental period*—namely, \( 2p/n, n \geq 1 \)—but since a positive integer multiple of a period is also a period, we see that all of the functions have in common the period \( 2p \). (Verify.) Hence the right-hand side of (2) is \( 2p \)-periodic; indeed, \( 2p \) is the fundamental period of the sum. We conclude that a Fourier series not only represents the function on the interval \( (-p, p) \), but also gives the periodic extension of \( f \) outside this interval. We can now apply Theorem 11.2.1 to the periodic extension of \( f \), or we may assume from the outset that the given function is periodic with period \( 2p \); that is,

\[
f\left(\frac{x}{2p}\right) = f(x).
\]

When \( f \) is piecewise continuous and the right- and left-hand derivatives exist at \( x = -p \) and \( x = p \), respectively, then the series (8) converges to the average

\[
\frac{f(p-) + f(-p+)}{2}
\]

at these endpoints and to this value extended periodically to \( \pm 3p, \pm 5p, \pm 7p, \) and so on.

The Fourier series in (13) converges to the periodic extension of (12) on the entire \( x \)-axis. At 0, \( \pm 2\pi, \pm 4\pi, \ldots \) and at \( \pm \pi, \pm 3\pi, \pm 5\pi, \ldots \) the series converges to the values

\[
\frac{f(0^+) + f(0^-)}{2} = \frac{\pi}{2} \quad \text{and} \quad \frac{f(\pi^-) + f(-\pi+)}{2} = 0,
\]

respectively. The solid dots in Figure 11.2.2 represent the value \( \pi/2 \).

FIGURE 11.2.2 Periodic extension of function shown in Figure 11.2.1

SEQUENCE OF PARTIAL SUMS It is interesting to see how the sequence of partial sums \( \{S_n(x)\} \) of a Fourier series approximates a function. For example, the first three partial sums of (13) are

\[
S_1(x) = \frac{\pi}{4}, \quad S_2(x) = \frac{\pi}{4} + \frac{2}{\pi} \cos x + \sin x, \quad \text{and} \quad S_3(x) = \frac{\pi}{4} + \frac{2}{\pi} \cos x + \sin x + \frac{1}{2} \sin 2x.
\]

In Figure 11.2.3 we have used a CAS to graph the partial sums \( S_3(x), S_6(x), \) and \( S_{15}(x) \) of (13) on the interval \( (-\pi, \pi) \). Figure 11.2.3(d) shows the periodic extension using \( S_{15}(x) \) on \( (-4\pi, 4\pi) \).

*See Problem 21 in Exercises 11.1.
In Problems 1–16 find the Fourier series of \( f \) on the given interval.

1. \( f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \leq x < \pi \end{cases} \)

2. \( f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 2, & 0 \leq x < \pi \end{cases} \)

3. \( f(x) = \begin{cases} 1, & -1 < x < 0 \\ x, & 0 \leq x < 1 \end{cases} \)

4. \( f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \leq x < 1 \end{cases} \)

5. \( f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \leq x < \pi \end{cases} \)

6. \( f(x) = \begin{cases} \pi^2, & -\pi < x < 0 \\ \pi^2 - x^2, & 0 \leq x < \pi \end{cases} \)

7. \( f(x) = x + \pi, \quad -\pi < x < \pi \)

8. \( f(x) = 3 - 2x, \quad -\pi < x < \pi \)

9. \( f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \leq x < \pi \end{cases} \)

10. \( f(x) = \begin{cases} 0, & -\pi/2 < x < 0 \\ \cos x, & 0 \leq x < \pi/2 \end{cases} \)

11. \( f(x) = \begin{cases} 0, & -2 < x < -1 \\ -2, & -1 \leq x < 0 \\ 1, & 0 \leq x < 1 \\ 0, & 1 \leq x < 2 \end{cases} \)

12. \( f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases} \)

13. \( f(x) = \begin{cases} 1, & -5 < x < 0 \\ 1 + x, & 0 \leq x < 5 \end{cases} \)

14. \( f(x) = \begin{cases} 2 + x, & -2 < x < 0 \\ 2, & 0 \leq x < 2 \end{cases} \)

15. \( f(x) = e^x, \quad -\pi < x < \pi \)

16. \( f(x) = \begin{cases} 0, & -\pi < x < 0 \\ e^x - 1, & 0 \leq x < \pi \end{cases} \)

17. Use the result of Problem 5 to show that

\[
\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots
\]

and

\[
\frac{\pi^4}{12} = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots.
\]

18. Use Problem 17 to find a series that gives the numerical value of \( \pi^2/8 \).
19. Use the result of Problem 7 to show that
\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.
\]
20. Use the result of Problem 9 to show that
\[
\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \cdots.
\]
21. (a) Use the complex exponential form of the cosine and sine,
\[
\begin{align*}
\cos \frac{n \pi x}{p} &= \frac{e^{in \pi x/p} + e^{-in \pi x/p}}{2}, \\
\sin \frac{n \pi x}{p} &= \frac{e^{in \pi x/p} - e^{-in \pi x/p}}{2i},
\end{align*}
\]
to show that (8) can be written in the complex form
\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in \pi x/p},
\]
where
\[
c_0 = \frac{a_0}{2}, \quad c_n = \frac{(a_n - ib_n)}{2}, \quad \text{and} \quad c_{-n} = \frac{(a_n + ib_n)}{2},
\]
where \(n = 1, 2, 3, \ldots\).
(b) Show that \(c_0, c_n, \text{ and } c_{-n}\) of part (a) can be written as one integral
\[
c_n = \frac{1}{2p} \int_{-p}^{p} f(x) e^{-in \pi x/p} dx, \quad n = 0, \pm 1, \pm 2, \ldots.
\]
22. Use the results of Problem 21 to find the complex form of the Fourier series of \(f(x) = e^{-x}\) on the interval \([-\pi, \pi]\).

11.3 FOURIER COSINE AND SINE SERIES

REVIEW MATERIAL
- Sections 11.1 and 11.2

INTRODUCTION
The effort that is expended in evaluation of the definite integrals that define the coefficients the \(a_0, a_n, \text{ and } b_n\) in the expansion of a function \(f\) in a Fourier series is reduced significantly when \(f\) is either an even or an odd function. Recall that a function \(f\) is said to be
- even if \(f(-x) = f(x)\) and odd if \(f(-x) = -f(x)\).

On a symmetric interval such as \((-p, p)\) the graph of an even function possesses symmetry with respect to the \(y\)-axis, whereas the graph of an odd function possesses symmetry with respect to the origin.

EVEN AND ODD FUNCTIONS
It is likely that the origin of the terms even and odd derives from the fact that the graphs of polynomial functions that consist of all even powers of \(x\) are symmetric with respect to the \(y\)-axis, whereas graphs of polynomials that consist of all odd powers of \(x\) are symmetric with respect to origin. For example,
\[
f(x) = x^2 \quad \text{is even} \quad \text{since} \quad f(-x) = (-x)^2 = x^2 = f(x)
\]
\[
f(x) = x^3 \quad \text{is odd} \quad \text{since} \quad f(-x) = (-x)^3 = -x^3 = -f(x).
\]
See Figures 11.3.1 and 11.3.2. The trigonometric cosine and sine functions are even and odd functions, respectively, since \(\cos(-x) = \cos x\) and \(\sin(-x) = -\sin x\). The exponential functions \(f(x) = e^x\) and \(f(x) = e^{-x}\) are neither odd nor even.

PROPERTIES
The following theorem lists some properties of even and odd functions.