There are 7 problems. For each problem, please exhibit your work leading to the solution of the problem. Please write neatly and do not show scratch work.

1. Exhibit a finite state automaton $M, F$ with the following property: if $s$ is any finite sequence of letters from the alphabet \{a, b, c\}, then $M, F$ accepts $s$ if and only if each of the letters $a, b, c$ occurs at least once in $s$.

2. Consider the permutations $\pi = (1\ 2\ 3)(4\ 5\ 6\ 7\ 8\ 9)$ and $\sigma = (3\ 4)$.
   
   (a) What is the sign of $\pi$?
   
   (b) What is the order of $\pi$?
   
   (c) What is the shape of $\pi$?
   
   (d) Exhibit the permutation $\pi^{-1}$.
   
   (e) Exhibit the permutation $\pi^3$.
   
   (f) Exhibit the permutations $\pi^{43}$ and $\pi^{48}$.
   
   (g) Exhibit $\pi$ as a product of transpositions.
   
   (h) Exhibit the permutation $\sigma^{-1}\pi\sigma$.
   
   (i) Exhibit the permutations $\pi\sigma$ and $\sigma\pi$.
   
   (j) Exhibit the cyclic decomposition of $\pi\sigma$.

3. True or false. Do not give reasons for your answers.

   (a) For all integers $n \geq 2$ and all integers $a$, the multiplicative order of $a$ modulo $n$ is a divisor of $\phi(n)$.

   (b) For all integers $n \geq 2$, the group $G_n = \mathbb{Z}_n^*$ is Abelian.

   (c) For all integers $n \geq 2$, the permutation group $S(n)$ is Abelian.
(d) If $f : X \to Y$ is a bijection, then $f^{-1} : Y \to X$ exists and is a bijection.

(e) The union of any two relations is a relation.

(f) Given a set $X$ and an equivalence relation $E$ on $X$, there is a canonical injection $\phi : X \to X/E$.

(g) For all finite sets $X, Y, Z$ we have

$$|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |Y \cap Z| - |X \cap Z| + |X \cap Y \cap Z|.$$ 

(h) Any permutation in $S(n)$ can be written as a product of disjoint transpositions.

(i) The inverse of any function is a relation.

(j) Every relation is the inverse of some function.

4. Let $x$ be a real variable. Let $f$ and $g$ be the functions defined $f(x) = x^2$ and $g(x) = x + 5$.

   (a) What are the functions $fg$, $gf$, $f^2$, $g^2$, $f^2g$, $g^2f$, and $f^2g^2$?

   (b) What are the domains and ranges of $fg$ and $gf$?

5. Define the concept of disjoint permutations. Prove that for any two disjoint permutations $\pi$ and $\sigma$ we have $\pi \sigma = \sigma \pi$.

6. (a) Define what is meant by the shape of a permutation.

   (b) Define what it means for two permutations to be conjugate.

   (c) Explain the relationship between these two concepts.

   (d) Illustrate your explanation with an example.

7. Prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

holds for all positive integers $n$. 