§2.9 Ex. 5. In general, the coordinate vector of a given vector \( \mathbf{v} \) with respect to a given linearly independent set of vectors \( \mathbf{u}_1, \ldots, \mathbf{u}_p \) is defined to be the unique vector \( \mathbf{x} \) such that \( [\mathbf{u}_1 \cdots \mathbf{u}_p] \mathbf{x} = \mathbf{v} \). For this exercise, row reduction shows that

\[
\begin{bmatrix}
1 & -3 & 4 \\
5 & -7 & 10 \\
-3 & 5 & -7
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0.25 \\
0 & 1 & -1.25 \\
0 & 0 & 0
\end{bmatrix}
\]

and this implies that the coordinate vector of \( \begin{bmatrix} 4 \\ 10 \\ -7 \end{bmatrix} \) with respect to

the linearly independent set \( \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -7 \\ 5 \end{bmatrix} \) is \( \begin{bmatrix} 0.25 \\ -1.25 \end{bmatrix} \).

§2.9 Ex. 11. Row reduction shows that

\[
A = \begin{bmatrix}
1 & 2 & -5 & 0 & -1 \\
2 & 5 & -8 & 4 & 3 \\
-3 & -9 & 9 & -7 & -2 \\
3 & 10 & -7 & 11 & 7
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & -5 & 0 & -1 \\
0 & 1 & 2 & 4 & 5 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

so the pivot columns of \( A \) are columns 1, 2, 4. In other words, the column vectors

\[
\begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 11 \end{bmatrix}
\]

form a basis of \( \text{Col } A \). This implies that \( \dim \text{Col } A = 3 \) and \( \dim \text{Nul } A = 5 - 3 = 2 \). To obtain a basis of \( \text{Nul } A \), perform further row reduction
to get
\[ A \sim U = \begin{bmatrix} 1 & 0 & -9 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
as the reduced row echelon form of \( A \). We know that \( \text{Nul} \ A \) is the solution set of \( Ax = 0 \) which is the same as the solution set of \( Ux = 0 \). The standard parametric description of this solution set is
\[ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \]
where \( x_3 \) and \( x_5 \) are the free variables. Thus we see that the vectors
\[ \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \]
form a basis of \( \text{Nul} \ A \).

§2.9 Ex. 13. Let \( A \) be the matrix with the given vectors as its columns. By row reduction we have
\[ A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]
so columns 1, 3, 4 of \( A \) are the pivot columns of \( A \). We conclude that these three columns of \( A \) are a basis of the subspace spanned by all the columns of \( A \).

§2.9 Ex. 17. In general, for any \( m \times n \) matrix \( A \), the solution set of \( Ax = 0 \) is \( \text{Nul} \ A \), the null space of \( A \), and its dimension is \( n - \text{rank}(A) \). In the case of a \( 7 \times 6 \) matrix of rank 4, the dimension of \( \text{Nul} \ A \) is \( n - \text{rank}(A) = 6 - 4 = 2 \).
§2.9 Ex. 21.  (a) True. This is just the definition of $[\mathbf{x}]_B$, the coordinate vector of $\mathbf{x}$ with respect to the basis $B$.

(b) False. The 1-dimensional subspaces of $\mathbb{R}^n$ are the lines in $\mathbb{R}^n$ which pass through the origin.

(c) True. See Example 7 in §2.8 of the textbook.

(d) True. For any matrix $A$, the pivot columns of $A$ form a basis for Col $A$, and the non-pivot columns of $A$ correspond to the free variables in the standard parametric description of Nul $A$. See also Exercises 11 and 13 above, and Example 6 in §2.8, and Theorem 14 in §2.9.

(e) True. See Theorem 15 in §2.9.