Heuristics for convergence part of Khinchin

Suppose that the $k$–dimensional surface $S$ in $n$–dimensional space is parameterised by the

$$ (x_1, \ldots, x_k, f_{k+1}(x), \ldots, f_n(x)) $$

with $\alpha_j \leq x_j \leq \beta_j$. For a given $q$ consider those $x$ for which there are $a$ with

$$ |x_j - a_j/q| \leq \psi(q)/q \quad (1 \leq j \leq k) $$

and

$$ |f_j(x) - a_j/q| \leq \psi(q)/q \quad (k < j \leq n). $$

Presuming the $f_j$ are sufficiently smooth, the $a_j$, by substitution for the $x_j$, must satisfy

$$ \|q f_j(a/q)\| \ll \psi(q) \quad (k < j \leq n). $$

Moreover the $k$–dimensional measure of that part of $S$ which satisfies the inequalities (1) and (2) is $\ll (\psi(q)/q)^k$. Hence the total measure of the subset of $S$ which has an approximation with denominator $q$ with $Q < q \leq 2q$ is

$$ \ll \sum_{Q < q \leq 2Q} (\psi(q)/q)^k N(q) $$

where $N(q)$ is the number of $a$ satisfying (3). Assuming that $\psi(q)$ is decreasing, a bound of the kind

$$ \text{card}\{q, a : Q < q \leq 2Q, q \ll a_j \ll q, \|q f_j(a/q)\| \ll \Psi \quad (k < j \leq n)\} \ll \Psi^{-k} Q^{k+1} $$

combined with partial summation should show that the total measure of points having infinitely many approximations is arbitrarily small, provided that

$$ \sum_{q=1}^{\infty} \psi(q)^n $$

converges.