Due Tuesday 28th April

1. Suppose that \( k \geq 2 \). Let \( R_k \subset [0,1]^k \) be defined by \( R_k = \{ t : t_i \geq 0, t_1 + \cdots + t_k \leq 1 \} \), and let \( m \in \mathbb{N} \) and \( f(t) = (1-t_1-\cdots-t_k)^m \). Given \( t_1, \ldots, t_j-1, t_j+1, \ldots, t_k \in [0,1]^{k-1} \) with \( t_1 + \cdots + t_j-1 + t_{j+1} + \cdots + t_k \leq 1 \) let \( A_j \) denote the interval \([0,1-t_1-\cdots-t_{j-1}-t_{j+1}-\cdots-t_k]\) (and take it to be the empty set otherwise) and define

\[
I_j(f) = \int_0^1 \cdots \int_0^1 \left( \int_{A_j} f(t) dt_j \right)^2 dt_1 \cdots dt_{j-1} dt_{j+1} \cdots dt_k
\]
and

\[
J(f) = \int_{R_k} f(t)^2 dt.
\]

(i) Prove that \( \sum_{j=1}^k I_j(f) = \frac{k(2m+2)!}{(2m+1+k)!(m+1)^2} \) and \( J(f) = \frac{(2m)!}{(2m+k)!} \).

(ii) Prove that \( \frac{\sum_{j=1}^k I_j(f)}{J(f)} = 4 \left( 1 - \frac{1}{2m+2} \right) \left( 1 - \frac{2m+1}{2m+1+k} \right) \).

(iii) (Goldston, Pintz, Yıldırım) Prove that if the level \( \theta \) of distribution satisfies \( \theta > \frac{1}{2} \), then there are infinitely many bounded gaps in the sequence of primes.

2. Let \( R_k \) be as in question 1. For \( t \in R_k \) let \( \alpha_k(t) = t_1 + \cdots + t_k \) and \( \beta_k(t) = t_1^2 + \cdots + t_k^2 \).

(i) Suppose that \( a \) and \( a_j \) are non-negative integers. Prove that

\[
\int_{R_k} (1 - \alpha_k(t))^a \prod_{j=1}^k t_j^{a_j} dt = \frac{a! \prod_{j=1}^k a_j!}{(k+a+\sum_{j=1}^k a_j)!}.
\]

(ii) Suppose that \( a \) and \( b \) are non-negative integers. Prove that

\[
\int_{R_k} (1 - \alpha_k(t))^a \beta_k(t)^b dt = \frac{a! b!}{(k+a+2b)!} \sum_{b_1, \ldots, b_k = b} \prod_{j=1}^k \frac{(2b_j)!}{b_j!}.
\]
(The multinomial theorem applied to \( \beta_k^b \) is useful here.)

3. (Maynard) (i) Let \( k = 5 \). In the notation of question 1, when \( t \in R_5 \), let

\[
f(t) = (1 - \alpha_5(t)) \beta_5(t) + \frac{7}{10} (1 - \alpha_5(t))^2 + \frac{1}{14} \beta_5(t)^2 - \frac{3}{14} (1 - \alpha_5(t)).
\]

Prove that

\[
\frac{\sum_{j=1}^5 I_j(f)}{J(f)} = \frac{1417255}{708216}.
\]

(ii) Prove that if the level of distribution \( \theta \) is 1, then \( \lim \inf_{n \to \infty} p_{n+1} - p_n \leq 12 \).