Due Tuesday 21st April

1. As in homework 11 call a set \( h \) of distinct non-negative integers \( h_1, \ldots, h_k \) *sf–admissible* when there is no prime \( p \) such that every residue class modulo \( p^2 \) contains at least one of them. Let \( S(x; h) \) denote the number of \( n \leq x \) such that \( n + h_1, \ldots, n + h_k \) are simultaneously squarefree. Given a \( k \)-tuple of positive integers \( d = d_1, \ldots, d_k \) let \( d \) denote the number of \( n \) such that \( n + h_1, \ldots, n + h_k \) are simultaneously squarefree. Let \( \nu_p(h) \) denote the number of different residue classes modulo \( p^2 \) amongst the \( h_1, \ldots, h_k \).

   (i) Prove that \( \rho(d) = d \sum_{\text{lcm}[d_1, \ldots, d_k] = m} g(m) \) and \( \rho^*(d) = d \sum_{\text{lcm}[d_1, \ldots, d_k] = m} \frac{2 \omega(m)}{m^2} \) \( \ll y^{\varepsilon-1} \)

   and deduce that

   \[
   T_k(x, y) = x \sum_{m=1}^\infty \frac{g(m)}{m^2} + O(xy^{\varepsilon-1})
   \]

   where

   \[
   g(m) = \sum_{[d_1, \ldots, d_k] = m} \mu(d_1) \ldots \mu(d_k) \rho^*(d).
   \]

   (ii) Prove that \( g(m) \) is multiplicative and has its support on the squarefree numbers.

   (v) Deduce that

   \[
   \sum_{m=1}^\infty \frac{g(m)}{m^2} = \prod_p \left( 1 + g(p)p^{-2} \right).
   \]

   (vi) Prove that \( 1 + g(p)p^{-2} = 1 - \nu_p(h)p^{-2} \).

   (vii) Prove that

   \[
   S(x; h) = x \prod_p \left( 1 - \frac{\nu_p(h)}{p^2} \right) + O(x^{1-\varepsilon})
   \]

   and hence that if \( h \) is sf–admissible, then there are infinitely many \( n \) such that \( n + h_j \) are simultaneously square free for \( j = 1, \ldots, k \).

2. Find the minimal diameter of 20–tuples which are sf–admissible, i.e. \( \max h_j - h_i \) is minimal.