Due Tuesday 11th October

The results of the previous two homeworks will be useful here.

Suppose throughout that \( \alpha \) is a real number, \( a \in \mathbb{Z}, q \in \mathbb{N} \) with \( (a, q) = 1 \) and \( \alpha - a/q \leq q^{-2} \), \( X \geq 2 \) and \( 1 < u < \sqrt{X} \). We assume also that

\[
S = S(\alpha) = \sum_{n \leq X} \Lambda(n)e(\alpha n)
\]

satisfies \( S = S_1 + S_2 - S_3 + S_4 \) where

\[
S_1 = \sum_{m > u} \sum_{u < n \leq X/m} a_m \mu(n)e(\alpha mn), \quad S_2 = \sum_{m \leq u} \mu(m) \sum_{n \leq X/m} (\log n)e(\alpha mn), \\
S_3 = \sum_{m \leq u^2} c_m \sum_{n \leq X/m} e(\alpha mn), \quad S_4 = \sum_{n \leq u} \Lambda(n)e(\alpha n), \quad a_m = \sum_{k | m} \Lambda(k), \quad c_m = \sum_{k \leq u} \sum_{l \leq u} \Lambda(k)\mu(l).
\]

1. (i) Let \( \mathcal{M} = \{2^j u : 0 \leq j, 2^j \leq Xu^{-2}\} \). Write \( S_1 = \sum_{M \in \mathcal{M}} T(M) \) where

\[
T(M) = \sum_{M < m \leq 2M} \sum_{u < n \leq X/m} a_m \mu(n)e(\alpha mn).
\]

Prove that

\[
S_1 \ll \sum_{M \in \mathcal{M}} (M(\log X)^2)^{\frac{1}{2}} (X/M)^{\frac{1}{2}} (Xq^{-1} + M + X/M + q)^{\frac{1}{2}} (\log X)^{1/2}
\]

and hence that \( S_1 \ll (Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2})(\log X)^{5/2} \).

2. Prove that

\[
S_2 = \int_1^X \sum_{m \leq \min(u, X/v)} \mu(m) \sum_{v < n \leq X/m} e(\alpha mn) \frac{dv}{v}
\]

and hence that \( S_2 \ll (Xq^{-1} + X^{2/5} + q)(\log X)^2 \).

3. Prove that \( |c_m| \leq \log m \) and deduce that \( S_3 \ll (Xq^{-1} + X^{4/5} + q)(\log X)^2 \).

4. Prove that \( S \ll (Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2})(\log X)^{5/2} \).