These questions are interconnected.

1. Let \( \pi_2(N) \) denote the number of of prime numbers \( p \leq N \) for which \( p + 2 \) is also prime and let \( P = \prod_{p \leq \sqrt{N}} p \). Define \( a_n \) to be 0 unless \( (n(n + 2), P) = 1 \) in which case take \( a_n = 1 \), and define \( Z = \sum_{n \leq N} a_n \). Prove that \( \pi_2(N) \leq Z + \sqrt{N} \).

2. Let \( Q = \sqrt{N} \). Prove that if \( f(2) = 1 \) and \( f(p) = 2 \) when \( 2 < p \leq Q \), then

\[
\pi_2(N) \ll \frac{N + Q^2}{L} + Q
\]

where \( L = \sum_{q \leq Q} \mu(q)^2 \prod_{p|q} \frac{f(p)}{p-f(p)} \).

3. Prove that \( L = S(\sqrt{N}) + S((\sqrt{N})/2) \) where

\[
S(R) = \sum_{q \leq R} \mu(q)^2 \prod_{p|q} \frac{2}{p-2}.
\]

4. Prove that if \( p > 2 \), then \( \frac{2}{p-2} = \sum_{k=1}^{\infty} \frac{2^k}{p^k} \) and that if \( g \) is the multiplicative function with \( g(p^k) = 2^k \), then

\[
S(R) \geq \sum_{q \leq R} \frac{g(q)}{q}.
\]

5. Prove that \( g(q) \geq d(q) \) and that \( S(R) \geq \sum_{q \leq R} \frac{d(q)}{q} \).

6. Prove that if \( R \geq 2 \), then \( \sum_{q \leq R} \frac{d(q)}{q} \gg (\log R)^2 \) and hence that if \( N \geq 2 \), then

\[
\pi_2(N) \ll \frac{N}{(\log N)^2}.
\]

7. (Brun 1919) Let \( \mathcal{P}_2 \) denote the set of primes \( p \) for which \( p + 2 \) is also prime. Prove that \( \sum_{p \in \mathcal{P}_2} \frac{1}{p} \) converges.