Math 568 Number Theory II, Spring 2016, Problems 6

Due Tuesday 22nd February

1. (i) Show that if \( \alpha(s) = \sum a_n n^{-s} \) has abscissa of convergence \( \sigma_c < \infty \), then
   \[
   \lim_{\sigma \to \infty} \alpha(\sigma) = a_1.
   \]

(ii) Show that if \( a_1 = 0 \), then there is no halfplane in which \( 1/\alpha(s) \) can be written as a convergent Dirichlet series.

(iii) Show that there is no halfplane in which \( 1/\zeta'(s) \) can be written as a convergent Dirichlet series. (Of course, this corresponds to log not having an inverse in \( \mathbb{A}, * \).)

2. Determine \( \sum \varphi(n)n^{-s} \), \( \sum \sigma(n)n^{-s} \), and \( \sum |\mu(n)|n^{-s} \) in terms of the zeta function (here \( \sigma(n) = \sum_{m|n} m \)).

3. Let \( \sigma_a(n) = \sum_{d|n} d^a \). Show that
   \[
   \sum_{n=1}^{\infty} \sigma_a(n)\sigma_b(n)n^{-s} = \zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)/\zeta(2s-a-b)
   \]
   when \( \sigma > \max (1, 1 + \Re a, 1 + \Re b, 1 + \Re(a + b)) \).

4. Let \( t(n) = (-1)^{\Omega(n)-\omega(n)} \prod_{p|n} (p-1)^{-1} \), and put \( T(s) = \sum_n t(n)n^{-s} \).
   (a) Show that for \( \sigma > 0 \), \( T(s) \) has the absolutely convergent Euler product
   \[
   T(s) = \prod_p \left(1 + \frac{1}{(p^s - 1)(p^s + 1)}\right).
   \]

(b) Determine all (complex) zeros of the function \( 1 + 1/((p-1)(p^s + 1)) \).

(c) Show that the line \( \sigma = 0 \) is a natural boundary of the function \( T(s) \), that is, no point on the line can be a point of analyticity of the function (or its analytic continuation).