1. Summary of order axioms from class (slightly different from the textbook): There is a relation “<” which satisfies the following axioms. \( a, b, c \) denote real numbers.

O1. Exactly one of \( a < b, a = b, b < a \) holds. O2. If \( a < b \) and \( b < c \), then \( a < c \). O3. If \( a < b \), then \( a + c < b + c \) for all \( c \). O4. If \( a < b \) and \( 0 < c \), then \( ac < bc \).

The expression \( a > b \) means \( b < a \). We also use \( a \leq b \) to mean “either \( a < b \) or \( a = b \)”.

Prove, referring as necessary to the above axioms, that if \( x \) and \( y \) are real numbers with \( 0 < x < y \), then \( 0 < x^3 < y^3 \).

By O4), \( 0 = 0. x < x, x = x^2 \) and \( 0 = 0. x^2 < x, x^2 = x^3 \). Also \( x^3 = x^2, x < x^2, y \), \( 0 = 0. y < x, y, x^2, y = (x, y), x < (x, y), y = x, y^2, x, y^2 = y^2, x < y^2, y = y^3 \). Hence, by O2), \( x^3 < x^2, y < x, y^2 < y^3 \).

2. Find all real numbers \( x \) that satisfy \(-1 < 2|x - 1| - |3x + 2| < 1\)

There are three cases to consider. (i) \( x < -2/3 \). Then the inequality becomes \(-1 < 2(1 - x) + 3x + 2 < 1 \) which is equivalent to \(-5 < x < -3 \). (ii) \(-2/3 \leq x < 1 \). Then we have \(-1 < 2(1 - x) - 3x - 2, \) i.e. \(-1/5 < x < 1/5 \). (iii) \( 1 \leq x \). Now \(-1 < 2(x - 1) - 3x - 2 < 1 \) so that \(-1 < -x < 1 \), i.e. \(-1 < x < 1 \) which is excluded since \( 1 \leq x \). Hence \( x \in (-5, -3) \cup (-1/5, 1/5) \).

3. Determine the set \( A = \left\{ x : \frac{x + 5}{x^2 + 2} < \frac{2}{x} \right\} \).

We have \( x^2 + 2 \geq 2 > 0 \) always. (a) First consider \( x > 0 \). Then \( \frac{x + 5}{x^2 + 2} < \frac{2}{x} \) iff \( x(x + 5) < 2(x^2 + 2) \) iff \( x^2 + 5x < 2x^2 + 4 \) iff \( 0 < x^2 - 5x + 4 \) iff \( 0 < (x - 4)(x - 1) \). This holds iff either \( x - 4 > 0 \) and \( x - 1 > 0 \) OR \( x - 1 < 0 \) and \( x - 4 < 0 \). Thus \( x > 4 \) or \( 0 < x < 1 \). (b) Now suppose \( x < 0 \). Then \( \frac{x + 5}{x^2 + 2} < \frac{2}{x} \) holds iff \( 0 > (x - 4)(x - 1) \). This holds either \( x - 4 < 0 \) and \( x - 1 < 0 \) or \( x - 1 > 0 \) and \( x - 4 < 0 \). In either case \( x > 1 \) contradicting \( x < 0 \). Thus the complete answer is \( A = (0, 1) \cup (4, +\infty) \).

4. Let \( A, B \) be non-empty sets of real numbers which are bounded above, and let \( A + B \) denote the set of numbers of the form \( a + b \) with \( a \in A \) and \( b \in B \). (i) Prove that \( \text{sup}(A + B) \) exists. (ii) Prove that \( \text{sup}(A + B) \leq \text{sup}A + \text{sup}B \). (iii) Let \( \delta > 0 \). Prove that there are \( a \in A \) and \( b \in B \) such that \( a > \text{sup}A - \delta \) and \( b > \text{sup}B - \delta \). (iv) Deduce that \( \text{sup}(A + B) = \text{sup}A + \text{sup}B \).

(i) \( A \) and \( B \) are non-empty, so there exists an \( a \in A \) and a \( b \in B \) in \( A \). Hence \( a + b \in A + B \) so \( A + B \) is non-empty. Moreover every \( a \in A \) satisfies \( a \leq \text{sup}A \) and every \( b \in B \) satisfies \( b \leq \text{sup}B \), and every element \( c \) of \( A + B \) is of this form. Hence \( c \leq \text{sup}A + \text{sup}B \) (*). Thus \( A + B \) is non-empty and bounded above, so by the Continuum Property \( \text{sup}(A + B) \) exists. (ii) Moreover, by (*), \( \text{sup}A + \text{sup}B \) is an upper bound for \( A + B \). (iii) If we had \( a \leq \text{sup}A - \delta \) for every \( a \in A \), then \( \text{sup}A \) would not be the least upper bound for \( A \). Hence there is an element \( a \) of \( A \) with \( a > \text{sup}A - \delta \). Likewise there is a \( b \in B \) with \( b > \text{sup}B - \delta \). (iv) By (ii) \( \text{sup}(A + B) \leq \text{sup}A + \text{sup}B \). We argue by contradiction. Suppose we have strict inequality. Let \( \delta = \frac{1}{2} (\text{sup}A + \text{sup}B - \text{sup}(A + B)) \). By (iii) there are \( a \in A, b \in B \) such that \( a + b > \text{sup}A + \text{sup}B - 2\delta \). But by the definition of \( \delta \) the RHS is \( \text{sup}(A + B) \) and this contradicts the fact that \( a + b \leq \text{sup}(A + B) \).