Geometry of Kapitsa's potentials

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Abstract. In this short note we uncover the geometry behind the classical result on averaging high-frequency vibrations \( \ddot{x} + a(\frac{t}{\varepsilon}) V'(x) = 0 \). It is shown that the classical effective potential of Kapitsa is produced by a centrifugal force of a point mass constrained to a certain curve determined by \( V \).

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1. Introduction

The purpose of this short note is to give a geometrical derivation of Kapitsa's classical result [2, 3, 1] on effective potentials. We also find a new geometrical expression of the classical effective potential. This result addresses the motion of a particle in a one-dimensional potential with time dependence:

\[
\ddot{x} + a \left( \frac{t}{\varepsilon} \right) V'(x) = 0
\]  

with \( V(0) = V'(0) = 0 \) and \( a(\tau + 1) = a(\tau) \). It is assumed that \( \max |a| < \varepsilon^\alpha \), with \( \alpha > -2 \) with the average \( \langle a \rangle = 0 \) and that \( V' \) is at least \( C^5 \). The averaging over \( t \) gives, according to the results in [3, 1, 4], the averaged motion in the time-independent potential

\[
W = \frac{1}{2} (u^2) f^2(x), \quad \text{where} \quad f(x) = V'(x) \quad \text{and} \quad v(\tau) = \int a \mathrm{d}\tau, \quad \langle v \rangle = 0.
\]

The averaged equations are thus of the form

\[
\ddot{X} = -(u^2) f'(x) f(X),
\]  

after dropping the error terms; estimates on these can be found in [4]. The timescale of equation (2) is the same as that of (1). Equations (2) are obtained by a standard (although slightly tedious) reduction to normal form. In the next section we shall show how the formal part of this reduction can be carried out purely geometrically with almost no computations.

2. The geometrical result

**Theorem 1.** The averaged dynamics of (1) is given, to the leading order, by

\[
\ddot{X} = -(u^2) k^{1\perp}(X) \sin^2 \theta,
\]  

where the velocity \( u \), the curvature \( k^{1\perp} \) and the angle \( \theta = \theta(X) \) are defined in the following paragraphs.

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Definition of $v$. $v = \int a \, dt$, $(v) = 0$.

Definition of $k^\perp$. First we define the curve $W$ as the image of the graph $y = V(x)$ under the map $(x, y) \mapsto (\int_0^x \sqrt{1 - V'(\sigma)^2} \, d\sigma, y)$ of the $xy$-plane; let $F$ be the family of $y$-translates of $W$, and let $F^\perp$ be the normal family to $F$. Now $k^\perp(x)$ is the curvature of the curves from $F^\perp$ where $x$ is the arclength from the $y$-axis to the point, see figure 1.

Definition of $\theta(x)$. The angle between the $y$-axis and the tangent to $W$ at a distance $x$ along $W$ (figure 1).

Remark 2. Equivalently, the curve $W$ can be defined by the condition
\[ \sin \theta(x) = V'(x), \tag{4} \]
where $\theta(x)$ is the angle defined earlier, together with the requirement that the curve be tangent to the $x$-axis at the origin.

Remark 3. If $V = \cos x$, then $W$ is the unit circle centred at $(0, 1)$.

2.1. A heuristic explanation of (3)

The form (3) was guessed originally by the following heuristic argument. Let us force a rigid wire $W$ to vibrate by giving it acceleration $a = a(t/\varepsilon)$ in the $y$-direction (figure 1), and consider the motion of a bead confined to $W$. Observe that the bead's arclength coordinate $x = x(t)$ satisfies equation (1), precisely due to the construction of $W$. Indeed, the tangential component of the acceleration along the wire is $\ddot{x} = -a \sin \theta$, which, according to (4), results in $\ddot{x} = -a V'(x)$; this coincides with (1) as claimed. We have thus re-interpreted the original equation (1). With this new interpretation we proceed to 'derive' the form (3). Indeed, the force applied by the wire $W$ to the bead is normal to the family $F$, i.e. is tangent to the family $F^\perp$. We think of this force as being very large; moreover, the period of vibration is

$\dagger$ $a = O(\varepsilon^\alpha)$; take $\alpha < 0.$
short, so that the bead 'would like' to follow the direction of the force (not having enough
time to gain speed). That is, the bead 'would like' to oscillate back and forth along a short
arc $\alpha$ of a curve from the family $F_\perp$, had it not been for the bead's inertia (see figure 1).
An attempt to constrain the bead to $\alpha$ would require a centripetal force $mu^2/R = u^2k_\perp(x)$,
where $u$ is the normal velocity of the wire, $x$ is the arclength, and $k_\perp$ is the curvature of
$\alpha$. We conclude that the bead behaves as if a centrifugal force $k_\perp(x)u^2$ was being applied
to it. Averaging over the short period $\varepsilon$ gives the average force $k_\perp(x)u^2$. This yields (3)
once we observe that $u = v \sin \theta$.

Rather than turning the above argument into a rigorous proof, we show that (3) is
equivalent to (2); the validity of (2) is proven by a normal form argument elsewhere (see
[4]).

3. Equivalence of two averaged forms

We must show that

$$k_\perp \sin^2 \theta = f'f.$$  \hspace{1cm} (5)

Differentiating the geometric definition (4), we obtain $f'(s) = \cos \theta \cdot \theta'(s)$, and thus

$$f'f = \sin \theta \cos \theta k,$$  \hspace{1cm} (6)

where $k = k(s) = \theta'(s)$ is the curvature of $W$. The next lemma connects the curvatures of
the two normal families.

**Lemma 4.** Consider the family $F_\perp$ of curves normal to the family $F$ of translates of a curve
along a straight line. The curvature $k_\perp$ of the curves from $F_\perp$ is related to the curvature $k$
of the curves from $F$ via

$$k(s) = k_\perp(s) \tan \theta,$$  \hspace{1cm} (7)

where $\theta$ is the angle between the normal to the curve from $F$ and the direction of translation.

**Proof of lemma 4.** The proof of the lemma is seen from figure 1: consider two normals
$n_A$ and $n_B$ at two nearby points $A$ and $B$ on a curve from $F_\perp$, and let $AC$ be parallel to
the axis of translation. We have

$$k_\perp = \lim_{B \to A} \frac{\angle(n_A, n_B)}{|AB|}. \hspace{1cm} (8)$$

Now,

$$\angle(n_A, n_B) = \angle(n_C, n_B) = k \cdot |CB| + o(|CB|).$$  \hspace{1cm} (9)

Substituting this into (8) we obtain

$$k_\perp = \lim_{B \to A} k \frac{|CB|}{|AB|} = k \cot \theta.$$ \hspace{1cm} (10)

The proof of the lemma is complete. \hfill $\square$

**Proof of (3).** Substituting (7) into (6) we obtain the desired equivalence (3), thus completing
the proof of theorem 1. \hfill $\square$

$^\dagger$ $R$ is the radius of curvature, the mass $m = 1$ by assumption.
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References