The basic structure for the final exam will be as in the first two midterms, but the exam will of course be longer than the midterms. The exam is cumulative, but with slightly more emphasis on the later material since the last exam. For all material up to the second midterm, please review the guides for the first two midterms. The rest of the material is summarized in the rest of this guide. It is a good idea to review all of the quizzes as well.

§4.3: In this section, we learned the definition of a group. This is a unifying object in mathematics, consisting of a set together with a means of combining elements of the set (an operation) which satisfies a few important properties. You should know what these properties are, both by name and how to verify them in specific cases. This includes knowing what is not a group, as on the quiz and homework. Recall that the main examples of groups we had already seen before this section are \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}_n \) (under addition), \( \mathbb{Z}_n^\times \) (under multiplication), and \( S(n), A(n) \) (under composition). You should also know what abelian (or commutative) groups are, as well as the examples of \( \text{GL}(2, \mathbb{R}) \) (the set of \( 2 \times 2 \) invertible matrices under matrix multiplication) and \( D(3), D(4) \) (the symmetries of an equilateral triangle and a square, respectively, under composition).

Here are a few problems to review.

1. Review Problems 1–4 and 8 in the homework.
2. Show Theorem 4.1.1, which states that \( S(n) \) is a group.

§5.1: In this section, we learned several useful basic concepts and facts about groups. In particular, we learned what a subgroup is, as well as concepts like orders of elements in groups and the cyclic subgroups generated by fixed elements. You should be very comfortable using the subgroup criterion we learned (we learned two ways to check that something is a subgroup, three if you count the more tedious checking of all the group axioms), as well as computing cyclic subgroups. Recall that we use the notation \( H \leq G \) to denote that \( H \) is a subgroup of \( G \), whereas the book doesn’t have a notation for this. Here are a few problems to consider.

1. Review problems 1, 2, 3, 5 from the homework.
2. Show Corollary 5.1.2, which states that identities are unique, the inverse of an inverse is the original element, and the socks and shoes property for inverses of products.
3. Show Theorem 5.1.4 (recall that this is not effectively a new proof, but the words from the corresponding proof in the permutations section just have to be replaced to accommodate a general group.)
(4) Show Theorem 5.1.5, which gives two efficient ways to check if a subset of a group is a subgroup (under the same operation). Don’t forget that it fails without the condition that the subset is nonempty.

(5) Show that $A(n)$ is a subgroup of $S(n)$, and that $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ for any $n \in \mathbb{Z}$.

(6) Find the cyclic subgroup generated by the matrix \((\frac{1}{2} \frac{1}{2})\) inside of $GL(2, \mathbb{R})$. By “find”, I mean write down in a simple, explicit way what the set is.

§5.2: Here, we learned about cosets and Lagrange’s Theorem. So, you should feel comfortable with computing examples of cosets, and doing basic things with them along the lines of the following problems. Recall that we denote the index (number of cosets of $H$ in $G$) by $[G : H]$, whereas the book doesn’t have a notation for this.

(1) Solve problems 1, 2, 3, 5 from the homework.
(2) What are the cosets of $n\mathbb{Z}$ inside of $\mathbb{Z}$?
(3) Find the cosets of $A(n)$ in $S(n)$.
(4) Show that the cosets of a subgroup $H \leq G$ partition $G$.
(5) Show (cf. the notes in class as well as Theorem 5.2.1) that $aH = bH \iff ab^{-1} \in H$. Further show that $aRb$ if $aH = bH$ is an equivalence relation (thus giving an alternate proof that they partition the group).
(6) Show that all cosets have the same number of elements.
(7) Prove Lagrange’s Theorem.
(8) Prove that all groups of prime order are cyclic, and deduces Euler’s theorem from Lagrange’s theorem.

§5.3:
Here, we learned about homomorphisms (note that the book didn’t cover these; but they are just isomorphisms as in the book but without the bijectivity condition) and isomorphisms. Two groups are isomorphic if there exists an isomorphism between them, and in this case we think of them as “essentially the same”. A simpler, but less convenient, way of thinking is that two groups are isomorphic iff their Cayley tables are the same up to reordering. Recall that the book doesn’t have a notation for this, but I write $G \cong H$ when they are isomorphic. We also learned about direct products of groups, and also used Lagrange’s theorem and (sometimes tedious) checks to find all groups of size at most 7. Recall that I also covered the concept of normal subgroups and quotient groups, which is briefly summarized at the end of Section 5.3 in the book. For the exam, please review the following problems.

(1) Solve problems 1, 2, 3, 4, 5 on the homework.
(2) Show Theorem 5.3.1, which describes the images of identity elements and inverses under isomorphisms (although it is not necessary to assume bijectivity as the book does).
(3) Show that $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are the only groups of size 4.
(4) Show Theorem 5.3.4, which states that a group in which every element squares to 1 is abelian.

(5) Show (cf. the class notes) that if $N$ is a normal subgroup of $G$, then the quotient group structure is well-defined and does define a group.

(6) Show that the quotient $G/N$ is abelian if and only if $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.

(7) Compute the Cayley table for $\mathbb{Z}_{12}/\langle 4 \rangle$. Which standard group that we learned about in our classification of small order finite groups is this isomorphic to?