Classifying Best Approximates Using Magical Intervals

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Mathfest ‘04
Approximating $e$

\[
\frac{2718}{1000} = \frac{1359}{500} = 2.718
\]

\[
\frac{193}{71} = 2.7183\ldots
\]

\[e = 2.718281828\ldots\]
How do we know when a rational number is a best approximate of an irrational number?
Continued Fraction Expansion

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \ldots}}}}}$$

$$\alpha = [a_0, a_1, a_2, a_3, \ldots]$$
Continued Fraction Expansion of $e$

$$e = 2.71828...$$
Continued Fraction Expansion of $e$

\[ e = 2 + .71828... \]

\[ e = [2, ...] \]
Continued Fraction Expansion of $e$

\[ e = 2 + \cfrac{1}{ \cfrac{1}{ \cfrac{1}{ \cfrac{1}{ \cfrac{1}{ \cfrac{1}{ \cfrac{1}{ \cfrac{1}{ \cfrac{1}{ \ddots } } } } } } } } } \]

\[ e = [ 2, \ldots ] \]
Continued Fraction Expansion of $e$

$$e = 2 + \frac{1}{1.39221\ldots}$$

$$e = [2, \ldots]$$
Continued Fraction Expansion of $e$

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$$

$$e = [2, 1, \ldots]$$
Continued Fraction Expansion of $e$

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + ...}}}}}}$$

$$e = [2, 1, 2, 1, 1, 4, 1, ...]$$
Convergents

\[ \frac{p_3}{q_3} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3} }} = [a_0, a_1, a_2, a_3] \]

**The Law of Best Approximates (Lagrange):**
The convergents are the complete set of best approximates!
QUESTION

How do we know when a rational number is a convergent of an irrational number?
Legendre’s Theorem: If \( \frac{p}{q} \) with \( q > 0 \) is a rational approximation to \( \alpha \) satisfying

\[
\left| \alpha - \frac{p}{q} \right| < \frac{1}{2q^2}
\]

Then \( \frac{p}{q} \) is a convergent to \( \alpha \).
Previous Results

**Theorem** (Mobius, 1998): For \( \phi = \frac{1+\sqrt{5}}{2} = [1] \), (the Golden ratio), \( \frac{p}{q} \) is a convergent to \( \phi \) if and only if

\[
\left| \phi - \frac{p}{q} \right| < \frac{1}{q^2}
\]

**Theorem** (Komatsu, 2003): For \( \phi_a = \frac{a + \sqrt{a^2 + 4}}{2} = [\bar{a}] \), \( \frac{p}{q} \) is a convergent to \( \phi_a \) if and only if

\[
\left| \phi_a - \frac{p}{q} \right| < \frac{1}{aq^2}
\]
Theorem - $m_e$ and $m_o$

Let $\alpha = [a_0, a_1, a_2, \ldots]$ be a real, irrational number, and for some integer $N \geq 0$ define

$$m_e = \min_{2n \geq N} \{a_{2n}\}$$

$$m_o = \min_{2n+1 \geq N} \{a_{2n+1}\}$$
Theorem - Conditions

If $\alpha$ satisfies any of the following four cases...

(i) $m_e > 1$ and $m_o > 1$

(ii) If $m_e = 1$ and $m_0 > 1$, then

(a) $a_n \geq a_{n+1} + 2$ for all odd $n \geq N$

(b) $a_n \geq a_{n-1} + 2$ for all odd $n > N$

(iii) If $m_e > 1$ and $m_o = 1$, analogous conditions to case (ii), reversing odd and even.

(iv) If $m_e = m_o = 1$, then for all $n \geq N$

(a) If $a_n \neq 1$, then $a_{n-1} = a_{n+1} = 1$

(b) $a_n \neq 2$

(c) If $a_n = 3$, then there exists an integer $k \geq 1$ such that $a_{n \pm i} = 1$ for all integers $1 \leq i < 2k$ and either $a_{n \pm 2k} \neq 1$, or $a_{n + 2k} \neq 1$ and $a_{n - 2k} = a_{n - 2k - 1} = 1$, or $a_{n - 2k} \neq 1$ and $a_{n + 2k} = a_{n + 2k + 1} = 1$. 
Theorem

If \( \alpha \) satisfies any of the above four cases, then a rational number \( p/q \) expressed in lowest terms with \( q \geq \max \{2, q_N\} \), is a convergent (and hence a best approximate) of \( \alpha \) if and only if

\[
\frac{-1}{m_e q^2} < \alpha - \frac{p}{q} < \frac{1}{m_o q^2}
\]
Corollary: \( \sqrt{11} \)

\[ \sqrt{11} = [3, 3, 6] \]

\[ N = 1, \quad m_e = 6, \quad m_o = 3 \]

(i) \( m_e > 1 \) and \( m_o > 1 \)

**Corollary.** A rational number \( p/q \) in lowest terms with \( q \geq 3 \) is a best approximate of \( \sqrt{11} \) if and only if

\[ \frac{-1}{6q^2} < \sqrt{11} - \frac{p}{q} < \frac{1}{3q^2} \]
**Corollary:**

\[
\frac{1 + \sqrt{2}}{2} = \left[1, 4\right]
\]

\(N = 0, \ m_e = 1, \ m_0 = 4\)

(ii) If \(m_e = 1\) and \(m_0 > 1\), then

(a) \(a_n \geq a_{n+1} + 2\) for all odd \(n \geq N\)

(b) \(a_n \geq a_{n-1} + 2\) for all odd \(n > N\)

**Corollary.** A rational number \(p/q\) in lowest terms with \(q \geq 2\) is a best approximate of \(\frac{1 + \sqrt{2}}{2}\) if and only if

\[
\frac{-1}{q^2} < \frac{1 + \sqrt{2}}{2} - \frac{p}{q} < \frac{1}{4q^2}
\]
Corollary: \( e \)

\[
e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]
\]

\[
N = 3, \quad m_e = 1, \quad m_o = 1
\]

(iv) If \( m_e = m_o = 1 \), then for all \( n \geq N \)

(a) If \( a_n \neq 1 \), then \( a_{n-1} = a_{n+1} = 1 \)

(b) \( a_n \neq 2 \)

(c) If \( a_n = 3 \), then there exists an integer \( k \geq 1 \) such that

\[
a_{n \pm i} = 1 \quad \text{for all integers } 1 \leq i < 2k \text{ and either } a_{n \pm 2k} \neq 1,
\]

or \( a_{n + 2k} \neq 1 \) and \( a_{n - 2k} = a_{n - 2k - 1} = 1 \), or \( a_{n - 2k} \neq 1 \) and

\[
a_{n + 2k} = a_{n + 2k + 1} = 1.
\]
**Corollary: e**

\[ e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...] \]

\[ N = 3, \quad m_e = 1, \quad m_o = 1 \]

**COROLLARY.** A rational number \( p/q \) in lowest terms with \( q > 2 \) is a best approximate of \( e \) if and only if

\[
\left| e - \frac{p}{q} \right| < \frac{1}{q^2}
\]
Best Approximates of $e$

\[ \left| e - \frac{1359}{500} \right| = .00028... > \frac{1}{500^2} = .000004 \]

\[ \left| e - \frac{193}{71} \right| = .000028... < \frac{1}{71^2} = .000198... \]
References


