Making Best Approximates Appear Through Magical Intervals

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Regional MAA Meetings
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Continued Fraction Expansion

\[ \alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \ldots}}}}} \]

\[ \alpha = [a_0, a_1, a_2, a_3, \ldots] \]
Continued Fraction Expansion of $\pi$

$$\pi = 3.14159...$$
Continued Fraction Expansion of \( \pi \)

\[
\pi = 3 + 0.14159\ldots
\]

\[
\pi = [3, \ldots]
\]
Continued Fraction Expansion of $\pi$

$$\pi = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

$$\pi = [3, \ldots]$$
Continued Fraction Expansion of $\pi$

$$\pi = 3 + \frac{1}{7.06251...}$$

$$\pi = [3, \ldots]$$
Continued Fraction Expansion of $\pi$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{.06251...}}}}$$

$$\pi = [3, 7, ...]$$
Continued Fraction Expansion of $\pi$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cdots}}}}}$$

$$\pi = [3, 7, 15, 1, 292, 1, 1, \ldots]$$
Approximating \( \pi \)

\[
\frac{103993}{33102} = 3.14159265... \\
\pi = 3.14159265...
\]
Approximating π

If you only have $1000 ...

\[
\frac{355}{113} = 3.14159292... \\
\pi = 3.14159265...
\]
Convergents

\[
\frac{p_3}{q_3} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3}}}
\]

The Law of Best Approximates (Lagrange): The convergents are the complete set of best approximates!
3rd convergent to $\pi$

$$\pi \approx 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}}$$
3rd convergent to $\pi$

\[
\pi \approx 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = \frac{355}{113} = 3.14159292... 
\]
QUESTION

How do we know when a rational number is a best approximate (a convergent) to an irrational number?
Previous Results

Theorem (Mobius, 1998): For \( \alpha = \frac{1 + \sqrt{5}}{2} = [1] \), (the Golden number), \( \frac{p}{q} \) is a convergent to \( \alpha \) if and only if

\[
\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}
\]

Theorem (Komatsu, 2003): For \( \alpha = \frac{a + \sqrt{a^2 + 4}}{2} = [a] \), \( \frac{p}{q} \) is a convergent to \( \alpha \) if and only if

\[
\left| \alpha - \frac{p}{q} \right| < \frac{1}{aq^2}
\]
\[ \alpha = [a, b, a, b, a, b, \ldots] \]

\[ \alpha = a + \frac{1}{b + \frac{1}{a + \frac{1}{b + \frac{1}{a + \ldots}}}} \]

\[ \alpha = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2b} \]
**THEOREM**: Let \( \alpha \) be an irrational number of the form \( \alpha = [a, b] = \frac{a b + \sqrt{a^2 b^2 + 4 a b}}{2 b} \) where \( \alpha \neq [1,2] \), \( \alpha \neq [1,3] \), and \( \alpha \neq [2,1] \). Then \( \frac{p}{q} \) is a convergent to \( \alpha \) if and only if...

\[
\frac{-1}{a q^2} < \alpha - \frac{p}{q} < \frac{1}{b q^2}
\]
FACT: If $\frac{p_n}{q_n}$ is a convergent to $\alpha$, then...

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}q_n^2}$$

FACT: Even convergents $< \alpha <$ odd convergents
Idea of Proof (\(<=\)) \(a > 1, b > 1\)

\[
-\frac{1}{aq^2} < \alpha - \frac{p}{q} < \frac{1}{bq^2} \quad \Rightarrow \quad \left| \alpha - \frac{p}{q} \right| < \frac{1}{\min\{a, b\}q^2} \leq \frac{1}{2q^2}
\]

**Legendre’s Theorem:** If \(\frac{p}{q}\) with \(q > 0\) is an approximation to \(\alpha\) satisfying

\[
\left| \alpha - \frac{p}{q} \right| < \frac{1}{2q^2}
\]

Then \(\frac{p}{q}\) is a convergent to \(\alpha\).
Theorem (Robinson): If \( \frac{p}{q} \) with \( q > 0 \) is an approximation to \( \alpha \) satisfying

\[
\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}
\]

Then \( \frac{p}{q} \) is either a convergent to \( \alpha \), \( \frac{p_n}{q_n} \), or of the form

\[
\frac{p_n + p_{n-1}}{q_n + q_{n-1}} \quad \text{or} \quad \frac{p_n - p_{n-1}}{q_n - q_{n-1}}
\]

(where \( \frac{p_n}{q_n} \) and \( \frac{p_{n-1}}{q_{n-1}} \) are convergents).
Corollary

**THEOREM:** Let \( \alpha \) be an irrational number of the form

\[
\alpha = [a, b]
\]

where \( \alpha \neq [1,2] \), \( \alpha \neq [1,3] \), and \( \alpha \neq [2,1] \). Then given a positive integer \( q \), \( q \) is a denominator of a best approximate to \( \alpha \) if and only if there exists an integer in the interval...

\[
\left( \alpha q - \frac{1}{bq}, \alpha q + \frac{1}{aq} \right)
\]
References


