Handout: Arbitrarily Large Gaps Between Primes

Theorem: There are arbitrarily large gaps between consecutive primes. (In other words, it is possible to find arbitrarily large sets of consecutive non-prime numbers.)

Proof: In order to find a gap of length \( n \) between two consecutive primes, consider the following list of consecutive integers:

\[(n+1)! + 2, \quad (n+1)! + 3, \quad (n+1)! + 4, \quad \ldots, \quad (n+1)! + n, \quad (n+1)! + (n+1)\]

Using the V.I.F. that if \( d \mid a \) and \( d \mid b \), then \( d \mid (a+b) \), we know that the first number in that list is divisible by 2, the second number is divisible by 3, the third number is divisible by 4, \( \ldots \) and the last number is divisible by \( n+1 \). So all of these numbers are non-primes! And there are \( n \) numbers in this list. So the proof is complete.

Example: Let \( n = 4 \). Then we are looking at the list \( 5! + 2, 5! + 3, 5! + 4, \) and \( 5! + 5 \). These are \( 122, 123, 124, 125 \). The first is divisible by 2, the second divisible by 3, the third divisible by 4, and the last divisible by 5. So here we have four consecutive non-primes, meaning that we have a gap of at least four \( (4) \) between two consecutive primes.

Three comments are in order related to this result and the proof above.

1) Notice how this proof “borrows from” Euclid’s proof that there are infinitely many primes. The construction of the numbers above is very reminiscent of Euclid’s construction of the number \( N \) which is the product of all the primes in our list plus 1.

2) The proof above is known as “constructive” because it actually constructs for us what is needed to prove the theorem. This is very different from the proof technique using contradiction (which is the technique used to prove Euclid’s result that there are infinitely many primes).

3) The gaps that are provided in the proof above are by no means optimal. In fact, one can usually find similarly-sized gaps between much smaller primes by inspection of a list of primes! Even so, the nice thing about the proof above is that it *guarantees* that such gaps will exist no matter how large a gap you would like to have between consecutive primes.