MATH 436 PRACTICE MIDTERM 2

This practice midterm is approximately twice the length that the actual midterm will be. Solutions will be posted on ANGEL on Tuesday, November 3. No notes, calculators, or any other aids will be allowed on the midterm, and you are encouraged to at least first attempt these problems under exam conditions. (If you would like to first attempt one hour’s worth of problems, either the odd or even problems would be fairly representative of a midterm. The problems generally get harder as you go through them, so the first 5 or last 5 are not representative.)

Problem 1. Let $A$ be the matrix

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the operator $Tx = Ax$.

(1) Compute the eigenvalues of $T$. (Do not use determinants.)
(2) Determine if $T$ is diagonalizable. If it is, determine a basis of $V$ such that the matrix of $T$ with respect to that basis (on both the domain and codomain) is diagonal, and compute $\mathcal{M}(T)$ with respect to this basis.
(3) Compute $T^{100}(1,1,2)$.

Your answer may contain unsimplified powers of numbers.

Problem 2. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the transformation defined by the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 4 & 13 & 8 & 7 \end{pmatrix}.$$

(1) Determine a basis for the kernel of $T$.
(2) Determine an orthonormal basis for the kernel of $T$. (Use the standard inner product.)
(3) Explain how you could compute an orthonormal basis for $(\ker T)^\perp$. (You need not actually compute it.)

Problem 3. Consider the matrices

$$A_1 = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 5 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

and let $T_i : \mathbb{R}^3 \to \mathbb{R}^3$ be the corresponding operators. Which of the operators $T_i$ are diagonalizable? Prove your answer. For all of the operators which are diagonalizable, find a basis of $\mathbb{R}^3$ consisting of eigenvectors of $T_i$, and compute $\mathcal{M}(T_i)$ in terms of this basis.

Problem 4. Let $V$ be an inner product space over $\mathbb{C}$. Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$.

(1) Prove or give a counterexample: $T - 2I$ is injective.
(2) Prove or give a counterexample: $T - \frac{1}{2}I$ is injective.
(3) Generalize: for which $\lambda \in \mathbb{C}$ is $T - \lambda I$ necessarily injective? For which $\lambda \in \mathbb{C}$ is it possible that $T - \lambda I$ is not injective?
Problem 5. Let $V$ be a finite dimensional inner product space over $\mathbb{C}$, and let $T \in \mathcal{L}(V)$. Prove that $V$ has an orthonormal basis such that the matrix $\mathcal{M}(T)$ is upper triangular. (You may use the fact that there is some basis of $V$ such that $\mathcal{M}(T)$ is upper triangular.)

Problem 6. Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

1. Prove that $T$ and $S^{-1}TS$ have the same eigenvalues.
2. What is the relationship between the eigenvectors of $T$ and $S^{-1}TS$?

Problem 7. Let $V$ be a finite-dimensional inner product space, and let $W \subset U \subset V$ be subspaces. Write e.g. $P_U \in \mathcal{L}(V)$ for the orthogonal projection onto $U$.

1. Show that $P_U P_W = P_W$.
2. Show that $P_W P_U = P_W$.

Problem 8. Let $V$ and $W$ be finite dimensional inner product spaces, and let $T : V \rightarrow W$ be a linear transformation.

1. Define the adjoint $T^*$ of $T$.
2. Explain why your definition of the adjoint makes sense.
3. Show that if $T$ is surjective then $T^*$ is injective. (You may assume it has been proven that $T^*$ is linear, but otherwise your proof should be self-contained.)

Problem 9. Let $\mathcal{P}(\mathbb{C})$ be the vector space of polynomials over $\mathbb{C}$. Let $V$ be a finite-dimensional vector space over $\mathbb{C}$, and let $T \in \mathcal{L}(V)$. Consider the function $\Phi : \mathcal{P}(\mathbb{C}) \rightarrow \mathcal{L}(V)$

$\Phi(p(x)) = p(T)$.

1. Show that $\Phi$ is a linear transformation.
2. Show that $\Phi$ is not injective.
3. Let $p$ be a nonzero polynomial of smallest degree such that $p(T) = 0$ (such a polynomial exists by part (2)). Show that if $q$ is any polynomial such that $q(T) = 0$ then $p$ divides $q$.

(The polynomial $p$ is called the minimal polynomial of the operator $T$.)

Problem 10. Let $V = \mathcal{P}_5(\mathbb{R})$ be the vector space of polynomials of degree at most 5 with coefficients in $\mathbb{R}$. Define two functions $\langle \cdot, \cdot \rangle_i : V \times V \rightarrow \mathbb{R}$ by the rules

$\langle p(x), q(x) \rangle_1 = p(1)q(1)$  \hspace{1cm} $\langle p(x), q(x) \rangle_2 = p(1)q(1) + \int_0^1 p(x)q(x) \, dx$

1. Prove or disprove: $\langle \cdot, \cdot \rangle_1$ is an inner product on $V$.
2. Prove or disprove: $\langle \cdot, \cdot \rangle_2$ is an inner product on $V$.
3. Let $\| \cdot \|_2$ be the norm corresponding to $\langle \cdot, \cdot \rangle_2$. Find the linear polynomial $p(x)$ which minimizes

$\| x^5 - p(x) \|_2^2 = \langle x^5 - p(x), x^5 - p(x) \rangle_2$.

Note that the norm $\| \cdot \|_2$ heavily takes into account the value of the polynomials at $x = 1$. Thus you should expect that the linear polynomial which best approximates $x^5$ with respect to this norm “looks” like a mediocre approximation of $x^5$ at most values of $x$, but is a fairly close approximation to $x$ near $x = 1$. 