SOME COMPUTATIONS WITH THE Q-CURVATURE

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1. INTRODUCTION

The purpose of this note is to collect some computations involving the fourth order Q-curvature on an arbitrary n-manifold. The computations I present here are motivated by my desire to understand the natural Yamabe-type variational problem of finding a metric of constant Q-curvature in all dimensions. As I have not been able to find these computations anywhere in the literature, I have decided to write them up in the event that someone might find them useful. Hopefully there are no errors in the computations, but if you see any, or know of references where these formulae can be found, please let me know!

In its present iteration, this note has two specific goals. First, we want to compute the first variation of the total Q-curvature. Through a straightforward but tedious computation, we arrive at the following result.

Theorem 1.1. Let \((M^n, g)\) be a Riemannian manifold with \(n \geq 3\) and let \(h \in S^2 T^*M\) define a variation \(\delta g = h\) of the metric. Then

\[
\delta \int Q \, dvol = \int \langle E + \frac{n-4}{2n} Qg, h \rangle \, dvol,
\]

where

\[
E = \frac{2}{n-2} \left[ \delta dP + \langle W, P \rangle + (n - 4) \left( P^2 + \frac{n - 2}{4(n-1)} \nabla^2 J - \frac{n^2}{4(n-1)} JP \right) \right].
\]

For an explanation of our notation, see Section 2.

In general dimensions, the Bach tensor — usually defined by computing the first variation of \(\int |W|^2\) — is given by \(B = \delta dP + \langle W, P \rangle\). Thus, we easily see that this result includes the well-known formula for first variation of the total Q-curvature in the critical dimension \(n = 4\) as the Bach tensor. Since the Bach tensor is traceless in all dimensions, so too is the tensor \(E\) defined above, which is to be expected: Critical points of the total Q-curvature restricted to unit-volume metrics within a fixed conformal class have constant Q-curvature. It is also perhaps useful to compare (1.1) to the well-known formula for the first variation of the total scalar curvature:

\[
\delta \int R \, dvol = - \int \langle \text{Ric}_0 - \frac{n-2}{2n} Rg, h \rangle \, dvol,
\]

This immediately gives the Bianchi identity \(\delta \text{Ric}_0 = \frac{n-2}{2n} dR\), and in a similar way we see that (1.1) yields the Bianchi identity \(\delta E = -\frac{n-4}{2n} dQ\). An important use of (1.2) is that it gives rise to the Pohozaev identity for constant scalar curvature.
metrics (see, for example, [9]), and is instrumental in deriving estimates for such metrics. We expect that (1.1) should play a similar role for metrics of constant $Q$-curvature in dimensions at least five, but again, have not seen this in the literature.

The second goal of this note is to find a simple formula for the $Q$-curvature of a metric in a conformal class with respect to natural tractor quantities. Again, after a straightforward but tedious computation, we arrive at the following result:

**Theorem 1.2.** Let $(M^n, c)$ be a conformal manifold with $n \geq 3$ and let $g$ be the conformal metric. Given any positive $\sigma \in \mathcal{E}[2]$, the $Q$-curvature $Q_{\sigma^{-1}g}$ of the metric $\sigma^{-1}g \in c$ is given by

$$
Q_{\sigma^{-1}g} = -\frac{n(n+2)}{2}|L\sigma|^2 + \frac{n(n+4)}{2}\sigma|\nabla L\sqrt{\sigma}|^2 - \frac{n(n-1)(n+4)}{2}\sigma|L\sqrt{\sigma}|^2,
$$

where $L : \mathcal{E}[2] \rightarrow S^2_0\mathcal{T}$ and $L : \mathcal{E}[1] \rightarrow \mathcal{T}$ denote the respective first BGG splitting operators of the tractor bundles $S^2_0\mathcal{T}$ and $\mathcal{T}$.

For a further explanation of our notation, see Section 4.

This formula should be compared to the expression for the scalar curvature of the conformal metric $\sigma^{-2}g$ determined by a scale $\sigma \in \mathcal{E}[1]$,

$$
R_{\sigma^{-2}g} = -n(n-1)|L\sigma|^2.
$$

In particular, we see that the expression (1.3) is of the same form as the expression for the scalar curvature of a conformal metric, except that the former includes two additional terms. So far I have not been able to determine whether these terms have any interpretation in terms of the total $Q$-curvature and the related minimization problem, though would be quite interested to know if such an interpretation exists.

This note is organized as follows:

In Section 2, we recall the definitions of the Paneitz operator and the $Q$-curvature, and explain the notation of Theorem 1.1.

In Section 3, we compute the first variation of the total $Q$-curvature.

In Section 4, we recall some basic notions from the tractor calculus and introduce our notation for computing with (sections of) $S^2_0\mathcal{T}$, explaining in particular the notation of Theorem 1.2.

In Section 5, we compute the $Q$-curvature of a metric in a conformal class with respect to natural tractor quantities.

2. THE PANEITZ OPERATOR AND $Q$-CURVATURE

To begin, recall that on a Riemannian manifold $(M^n, g)$, the Paneitz operator $P_4$ is the fourth order operator

$$
P_4 = \Delta^2 + \delta (4P + (n-2)J g) + \frac{n-4}{2}Q,
$$

where $\delta = -d^*$ is the divergence (i.e. the negative of the adjoint of the exterior derivative), $P$ is the Schouten tensor, $J$ is its trace, and $Q$ is the $Q$-curvature, which are defined by

$$
P = \frac{1}{n-2} (\text{Ric} - J g),
$$

$$
J = \frac{R}{2(n-1)},
$$

$$
Q = -\Delta J - 2|P|^2 + \frac{n}{2}J^2.
$$
The importance of the Paneitz operator stems from the fact that it is a conformally covariant operator, a notion most easily defined using conformal density bundles. Regarding \((M^n, [g])\) as a conformal manifold — that is, \([g]\) is the conformal class of a metric \(g\), in that a metric \(h\) is in \([g]\) if and only if \(h = e^{2s}g\) for some \(s \in C^\infty(M)\) — one defines for each \(w \in \mathbb{R}\) the space \(E[w]\) of conformal densities of weight \(w\) as the set of all functions \(u: [g] \times M \to \mathbb{R}\) such that
\[
u(e^{2s}g, x) = e^{ws}u(g, x)
\]
for all \(s \in C^\infty(M)\). With this definition, it is easily checked that the Paneitz operator is an operator between conformal densities; precisely,
\[
P_4: E \left[-\frac{n-4}{2}\right] \to E \left[-\frac{n+4}{2}\right]
\]
In more familiar terms, this just states that, regarding \(P[g]\) as the Paneitz operator defined in terms of the metric \(g\),
\[
P_4[e^{2s}g] = e^{-\frac{n+4}{2}s}P_4[g]e^{\frac{n-4}{2}s}
\]
for all \(s \in C^\infty(M)\), where the exponential factors on the right hand side are to be regarded as multiplication operators.

The conformal covariance property (2.2) together with the definition of the \(Q\)-curvature of the metric \(g\) as
\[
Q_g = \frac{2}{n-4}P_4[g]w
\]
for any positive function \(w \in C^\infty(M)\), where \(\hat{g} = w^{\frac{4}{n-4}}g\). One can check that these definitions all make sense in the limit \(n \to 4\) (this is the so-called “analytic continuation in the dimension”), which is how one typically understands the \(Q\)-curvature (cf. [3]).

The total \(Q\)-curvature of a compact Riemannian manifold \((M^n, g)\) is defined by
\[
\int Q_g \, d\text{vol}_g.
\]
From (2.3), it is evident that if \(\hat{g} = w^{\frac{4}{n-4}}g\) is a conformally equivalent metric, then
\[
\int Q_{\hat{g}} \, d\text{vol}_{\hat{g}} = \frac{2}{n-4} \int P_4[g]w \, d\text{vol}_g.
\]
It is readily apparent from its definition (2.1) that the Paneitz operator is (formally) self-adjoint, and hence it is easy to deduce from (2.4) that the first variation of the total \(Q\)-curvature functional within a conformal class is
\[
\delta \int Q_{\hat{g}} \, d\text{vol}_{\hat{g}} = \frac{4}{n-4} \int (P_4[g]w)w' \, d\text{vol}_g = 2 \int Q_{\hat{g}}(\log w)' \, d\text{vol}_{\hat{g}}.
\]
Of course, our interest is in computing the first variation of the total \(Q\)-curvature over all metrics. To do that, we will need in addition the following notation: Given a Riemannian manifold \((M^n, g)\), let \(\text{Rm}\) be the Riemannian curvature tensor, and let \(\text{W}\) be the Weyl curvature, defined by
\[
\text{Rm} = \text{W} + P \wedge g,
\]
where $\wedge$ denotes the Kulkarni-Nomizu product,

$$(P \wedge g)(x, y, z, w) = P(x, z)g(y, w) + P(y, w)g(x, z) - P(x, w)g(y, z) - P(y, z)g(x, w).$$

Given sections $A \in S^2 \Lambda^2 T^* M$, $h \in S^2 T^* M$, we denote by $\langle A, h \rangle \in S^2 T^* M$ the natural contraction

$$\langle A, h \rangle(x, y) = \sum_{i,j=1}^{n} A(x, e_i, y, e_j)h(e_i, e_j)$$

for $\{e_i, e_j\}$ an orthonormal basis for $T_p M$. We will also denote by the subscript "naught" the projection $S^2 T^* M \rightarrow S^2_0 T^* M$; i.e. if $h \in S^2 T^* M$,

$$h_0 = h - \frac{1}{n} \text{tr}_g h.$$ 

Lastly, we denote by $\sharp$ the natural action of $\text{End}(TM)$ on $TM$, extended as a derivation to the tensor algebra $T^\infty = \bigoplus_{i,j=0}^{\infty} (T^* M)^{\otimes i} \otimes (TM)^{\otimes j}$ and, using the metric $g$ to identify $\text{End}(TM) \cong T^* M \otimes T^* M$, use the same symbol to denote the natural action of $T^* M \otimes T^* M$ on $T^\infty$. For example, if $T, h \in T^* M \otimes T^* M$, we have that

$$(T \sharp h)(x, y) = -\sum_{i=1}^{n} [T(e_i, x)h(e_i, y) + T(e_i, y)h(x, e_i)].$$

3. First variation of the total $Q$-curvature

Let us now prove Theorem 1.1. To begin, recall the following well-known pointwise variational formulae for the scalar and Ricci curvatures (see [2]).

**Lemma 3.1.** Let $(M^n, g)$ be a Riemannian manifold and let $h \in S^2 T^* M$ define a variation $\delta g = h$ of the metric. Then

$$\delta R = -\langle \text{Ric}, h \rangle + \delta^2 h - \Delta \text{tr} h$$

$$\delta \text{Ric} = -\frac{1}{2} \Delta h - \frac{1}{2} \text{Ric} \sharp h - \langle \text{Rm}, h \rangle + \frac{1}{2} L_{\delta h} g - \frac{1}{2} \nabla^2 \text{tr} h.$$

As immediate corollaries, we have the following pointwise variational formulae for the Schouten tensor and its trace.

**Corollary 3.2.** Let $(M^n, g)$ be a Riemannian manifold with $n \geq 3$ and let $h \in S^2 T^* M$ define a variation $\delta g = h$ of the metric. Then

$$\delta J = -\frac{1}{2(n-1)} \left( \langle (n-2)P + Jg, h \rangle - \delta^2 h + \Delta \text{tr} h \right)$$

$$\delta P = -\frac{1}{2} P \sharp h - \frac{1}{2(n-2)} \left[ \Delta h + 2 \langle \text{Rm}, h \rangle - L_{\delta h} g + \nabla^2 \text{tr} h \right]$$

$$- \frac{1}{n-1} \left( \langle (n-2)P + Jg, h \rangle - \delta^2 h + \Delta \text{tr} h \right),$$

where $W = \text{Rm} - P \wedge g$ is the Weyl curvature.

**Proof.** The only observation one needs to make is that $g \sharp h = -2h$. \qed
From this, we have the following formulae for the variations of the integrals of $J^2$ and $|P|^2$.

**Corollary 3.3.** Let $(M^n, g)$ be a Riemannian manifold with $n \geq 3$ and let $h \in S^2 T^* M$ define a variation $\delta g = h$ of the metric. Then

$$
\delta \int J^2 \, dvol = -\frac{1}{n-1} \int ((n-2)JP - \nabla^2 J + \Delta J g - \frac{n-3}{2} J^2 gh) \, dvol
$$

$$
\delta \int |P|^2 \, dvol = -\frac{1}{n-2} \int (G, h) \, dvol,
$$

where

$$
G = \delta dP + \langle W, P \rangle + (n-4)P^2 + \frac{n}{n-1} JP - \frac{n-2}{n-1} \nabla^2 J
- \frac{1}{n-1} J^2 g + \frac{n-2}{n-1} \Delta J g - \frac{n-4}{2} |P|^2 g.
$$

**Proof.** This follows immediately from Corollary 3.2 and the formulae

$$
\langle P \# h, P \rangle = \langle P \# P, h \rangle = -2 \langle P^2, h \rangle
$$

$$
\langle \langle \langle Rm, h \rangle, P \rangle = \langle \langle W, P \rangle - 2P^2 + JP + |P|^2 g, h \rangle
$$

$$
\Delta P = \delta dP + \nabla^2 J - \frac{1}{2} \text{Ric} \sharp P = \langle Rm, P \rangle,
$$

the last equation of which is a direct consequence of the Weitzenböck formula

$$
(d \delta + \delta d)T = \Delta T + \frac{1}{2} \text{Ric} \sharp T + \langle Rm, T \rangle
$$

on $S^2 T^* M$.

This is enough to prove Theorem 1.1.

**Proof of Theorem 1.1.** By the definition of $Q$, we have that

$$
\int Q \, dvol = \int \left(-2|P|^2 + \frac{n}{2} J^2 \right) \, dvol.
$$

The result then follows from Corollary 3.3 and some straightforward algebraic manipulations. $\square$

4. **Some tractor calculus**

For our purposes, the tractor calculus is a useful tool which simply and elegantly allows ones to perform computations in a manifestly conformally invariant way, as we now describe. For a more complete description of the tractor calculus, we refer to the original article [1], which is essentially in the same spirit of our presentation, and to the recent book [4], which contains a much more thorough account of the tractor calculus within the broader framework of parabolic geometries.

In Section 2 we introduced conformal densities associated to a conformal manifold $(M^n, c)$, which can be thought of as the most natural notions of “functions” within the conformal context. In analogy with the tangent bundle with its Levi-Civita connection in Riemannian geometry is the standard tractor bundle.
Definition 4.1. Let \((M^n, c)\) be a conformal manifold with \(n \geq 3\). The standard tractor bundle \(\mathcal{T}\) is the rank \(n + 2\) vector bundle such that a choice of metric \(g \in c\) determines an isomorphism \(\mathcal{T} \cong \mathbb{R} \otimes TM \otimes \mathbb{R}\), denoted here by

\[
\mathcal{T} \ni I = \begin{pmatrix} \sigma \\ \omega \\ \rho \end{pmatrix} \in \left( \mathbb{R} \otimes \mathbb{R} \right) 
\]

which transforms with a conformal change of metric \(g \mapsto e^{2s}g\) as

\[
\begin{pmatrix} \sigma \\ \omega \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} e^s \sigma \\ e^{-s} (\omega + \sigma \nabla s) \\ e^{-s} (\rho - g(\omega, \nabla s) - \frac{1}{2} \sigma |\nabla s|^2) \end{pmatrix},
\]

where \(\nabla\) and \(|\cdot|^2\) are computed using the metric \(g\).

The standard tractor metric \(h \in \mathcal{T}^* \otimes \mathcal{T}^*\) is defined by

\[
h(I, I) = 2\sigma \rho + |\omega|^2
\]

for any \(I \in \mathcal{T}\) and any choice of metric \(g \in c\), with \(I\) written in terms of \(\sigma, \omega, \rho\) according to (4.1).

The canonical tractor connection \(\nabla : \mathcal{T} \to \mathcal{T}^* M \otimes \mathcal{T}\) is defined by

\[
\nabla I = \begin{pmatrix} \nabla \sigma - g(\omega, \cdot) \\ \nabla \omega + \sigma P + pg \\ \nabla \rho - P(\omega, \cdot) \end{pmatrix}
\]

for any \(I \in \mathcal{T}\) and any choice of metric \(g \in c\).

Whenever a choice of metric \(g \in c\) is (implicitly) given, we shall denote standard tractors \(I \in \mathcal{T}\) by the “vector” (4.1). Moreover, we can use the tractor metric \(h\) to identify \(\mathcal{T} \cong \mathcal{T}^*\), and will in particular do this to regard \(I\) as the “row vector” \(I = (\rho, \omega, \sigma)\). The benefit of this notation is that evaluation \(h(I, J)\) is equivalent to the evaluation of the usual dot product of a row vector and a column vector.

Definition 4.2. Let \((M^n, c)\) be a conformal manifold with \(n \geq 3\). For any \(w \in \mathbb{R}\), the tractor-D operator \(D : \mathcal{E}[w] \to \mathcal{T}[w-1]\) is defined, given any choice of metric \(g \in c\), by

\[
D u = \frac{w(n + 2w - 2)u}{(n + 2w - 2)\nabla u - (\Delta u + wJ u)}.
\]

In particular, this gives rise to the so-called first BGG operator (see [7] for an easy-to-read account as well as original references).

Definition 4.3. Let \((M^n, c)\) be a conformal manifold with \(n \geq 3\). The first BGG operator associated to \(\mathcal{T}\) is the operator \(L : \mathcal{E}[1] \to \mathcal{T}\) given by \(L = \frac{1}{n} D\)
The main point of the first BGG operator is that it neatly relates the geometric properties of the standard tractor bundle to the natural quotient subbundle $E[1]$. Since the set of positive conformal densities of weight one can be identified with the conformal class via $\sigma \mapsto \sigma^{-2} g$, for $g$ the conformal metric (i.e. $g \in S^2 T^* M \otimes E[2]$ is defined by $g(g, \cdot) = g$ for all $g \in c$), it is natural to expect that this also encodes some information about metric in the conformal class. The is indeed the case: One can easily show that for any $\sigma \in E[1]$, the scalar curvature $R_\sigma$ of the metric $\sigma^{-2} g$ is given by

$$R_\sigma = -n(n-1)|L\sigma|^2$$

wherever the metric is defined, and moreover, that this metric is Einstein wherever defined if and only if

$$\nabla L\sigma = 0;$$

see, for example, [5].

The tractor calculus is the study of the tensor algebra of $T$. For our purposes, it will suffice to describe how this is carried out with the bundle $S_0^2 T$. By definition, any section $A$ of $S_0^2 T$ is such that

$$\langle A, I \otimes J \rangle = \langle A, J \otimes I \rangle, \quad \langle A, h \rangle = 0$$

for all $I, J \in T$, where $\langle \cdot , \cdot \rangle$ is the natural extension of the tractor metric to a section of $(S_0^2 T)^* \otimes (S_0^2 T)^*$. Moreover, since any section of $S_0^2 T$ can be written as a linear combination of symmetric products $I \otimes J := \frac{1}{2}(I \otimes J + J \otimes I)$ followed by a projection onto the tracefree part of $S^2 T$, it is natural to denote the section $A \in S_0^2 T$ in the “matrix notation”

$$A := \begin{pmatrix} y & \alpha & \sigma \\ \beta & T & \alpha \\ \rho & \beta & y \end{pmatrix} = \begin{pmatrix} \sigma \\ \alpha \\ \beta \end{pmatrix} \begin{pmatrix} T & y \\ \alpha \\ \beta \end{pmatrix},$$

(4.6)

here, the requirement that $A$ is tracefree is precisely the requirement that $\text{tr} T + 2y = 0$. In fact, this gives (provided a choice of metric $g \in c$) an isomorphism

$$S_0^2 T \cong \mathbb{R} \oplus TM \oplus (S^2 TM \oplus \mathbb{R}) \oplus TM \oplus \mathbb{R},$$

which, in conjunction with the following consequence of (4.2), can be used to define $S_0^2 T$ in an analogous way to our definition of the standard tractor bundle.

**Lemma 4.4.** Given a change of scale $g \mapsto e^{2s} g$, the vector formula (4.6) for a section of $S^2 T$ changes according to

$$\begin{pmatrix} \sigma \\ \alpha \\ y \\ \beta \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} e^{2s} \sigma \\ e^{2s} \alpha + \sigma \nabla s \\ e^{-2s} \left( \beta - (T - yg) \nabla s - \frac{1}{2} |\nabla s|^2 \alpha - \langle \nabla s, \alpha \rangle \nabla s - \frac{1}{2} |\nabla s|^2 \nabla s \right) \\ e^{-2s} \left( \rho - 2 \langle \nabla s, \beta \rangle + (T - yg) ds^2 + |\nabla s|^2 \nabla s, \alpha \rangle + \frac{1}{4} |\nabla s|^4 \right) \end{pmatrix}.$$

Using the notation (4.6), it is easy to derive the following formula for the tractor metric on $S_0^2 T$; note here that we are using the convention

$$|I \otimes J|^2 = \frac{1}{2} \left( |I|^2 |J|^2 + \langle I, J \rangle^2 \right),$$

so that if $\{ E_i \}$ is an orthonormal basis for $T$, then $\{ E_i \otimes E_j \}$ is an orthonormal basis for $S^2 T$. 

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Lemma 4.5. The metric induced on $S^2 T$ by the tractor metric is determined by the norm
\[
\left| \begin{array}{c|c}
\sigma & \alpha \\
T & y \\
\beta & \rho \\
\end{array} \right|^2 = |T|^2 + 2y^2 + 4\alpha \beta + 2\alpha \rho.
\]

Likewise, one can easily compute the explicit formula for the connection on $S_0^2 T$ induced by the canonical tractor connection.

Lemma 4.6. The connection induced on $S^2 T$ by the normal tractor connection is determined in scale by
\[
\nabla_z \left( \begin{array}{c|c}
\sigma & \alpha \\
T & y \\
\beta & \rho \\
\end{array} \right) = \left( \begin{array}{c|c}
\nabla_z \sigma - 2\alpha, z \\
\nabla_z \alpha - (T - yg)(z) + \sigma P(z) \\
\nabla_z \beta - (T - yg)(P(z)) + \rho z \\
\nabla_z \rho - 2P(\beta, z) \\
\end{array} \right).
\]

With these ingredients, together with the formula for the Kostant codifferential on $T^* M \otimes S_0^2 T$ (see [8]), it is not too difficult to compute the first BGG operator associated to $S_0^2 T$. In the interests of brevity and generality, we instead simply state the formula for the natural tractor-$D$ operator mapping conformal densities into weighted sections of $S_0^2 T$, which specializes to the first BGG operator $L: \mathcal{E}[2] \rightarrow S_0^2 T$.

Definition 4.7. Let $(M^n, c)$ be a conformal manifold with $n \geq 3$. The tractor-$D$ operator $\mathbb{D}: \mathcal{E}[w] \rightarrow S_0^2 \mathcal{E}[w - 2]$ is given by
\[
\mathbb{D} \sigma = \left( \begin{array}{c|c}
\frac{w(w - 1)(n + 2w - 2)(n + 2w - 4)\sigma}{(w - 1)(n + 2w - 2)(n + 2w - 4)\nabla \sigma} \\
\frac{(n + 2w - 4)(w(n + 2w - 2)\sigma P}{+ (n + 2w - 2)\nabla^2 \sigma - (\Delta \sigma + wJ\sigma)g} \\
\frac{- (w - 1)(n + 2w - 4)(\Delta \sigma + wJ\sigma)}{\nabla (\Delta \sigma + wJ\sigma) + (n + 2w - 2)P(\nabla \sigma)} \\
\frac{(\Delta^2 \sigma + 2\delta(n + 2w - 2)P + (w - 1)Jg)(\nabla \sigma) - (n + 2w - 4)(\nabla J, \nabla \sigma)}{+ w(\Delta J + (n + 2w - 2)P^2 +(w - 2)J^2)\sigma}
\end{array} \right),
\]
where $\Delta$ is the rough tractor Laplacian $\Delta = -\nabla^* \nabla = \text{tr} \nabla^2$.

In particular, the BGG splitting operator $L: \mathcal{E}[2] \rightarrow S_0^2 T$ is given by
\[
L = \frac{1}{2n(n + 2)} \mathbb{D} \sigma.
\]

Note in particular that taking $w = -\frac{n-4}{2}$ in the definition of the tractor-$D$ operator yields immediately the conformal covariance of the Paneitz operator (cf. [6]).

5. $Q$-Curvature and $S_0^2 T$

Let us now turn to the proof of Theorem 1.2. First, we compute the length $|L \sigma|^2$ for $\sigma \in \mathcal{E}[2]$. 

Proposition 5.1. Let \((M^n, c)\) be a conformal manifold with \(n \geq 3\). Given any \(\sigma \in \mathcal{E}^2\), it holds that

\[
|L\sigma|^2 = \frac{\sigma}{n(n+2)} \Delta^2 \sigma - \frac{1}{n+2} (\nabla \sigma, \nabla \Delta \sigma) + \frac{n+2}{n} \sigma (P, \nabla^2 \sigma) + \frac{1}{4} |\nabla^2 \sigma|^2
- P(\nabla \sigma, \nabla \sigma) - \frac{1}{4(n+2)} (\Delta \sigma)^2 - \frac{n-6}{n(n+2)} (\nabla \sigma, \nabla J) - \frac{n-2}{n(n+2)} J_\sigma \Delta \sigma
- \frac{2}{n+2} J |\nabla \sigma|^2 + \frac{n+2}{n} \sigma^2 |P|^2 + \frac{2}{n(n+2)} \sigma^2 \Delta J + \frac{1}{n+2} J^2 \sigma^2.
\]

Proof. It follows immediately from the definition of \(L : \mathcal{E}^2 \to S^2 T\) and Lemma 4.5 that

\[
|L\sigma|^2 = \frac{\sigma}{n(n+2)} [\Delta^2 \sigma + 2\delta ((n+2)P(\nabla \sigma) + J \nabla \sigma) - n(\nabla J, \nabla \sigma) + 2(\Delta J + (n+2) |P|^2) \sigma]
- \frac{1}{n+2} (\nabla \sigma, \nabla (\Delta \sigma + 2J \sigma) + (n+2)P(\nabla \sigma)) + \left| \sigma P + \frac{1}{2} \nabla^2 \sigma - \frac{\Delta \sigma + 2J \sigma}{2(n+2)} g \right|^2
+ \frac{1}{2(n+2)^2} (\Delta \sigma + 2J \sigma)^2.
\]

The result then follows by a straightforward algebraic manipulation. \(\square\)

On the other hand, (2.2) yields immediately the following formula for the \(Q\)-curvature of the metric \(\sigma^{-2} g\):

Proposition 5.2. Let \((M^n, g)\) be a Riemannian manifold and let \(\tilde{g} = \sigma^{-1} g\) be a conformally equivalent metric for some positive function \(\sigma \in C^\infty(M)\). Then the \(Q\)-curvature of \(\tilde{g}\) is given by

\[
Q_{\tilde{g}} = -\frac{1}{2} \sigma \Delta^2 \sigma + \frac{n}{2} (\nabla \sigma, \nabla \Delta \sigma) - 2 \sigma (P, \nabla^2 \sigma) + \frac{n^2}{4} |\nabla^2 \sigma|^2 + \frac{n}{4} P(\nabla \sigma, \nabla \sigma)
+ \frac{n}{8} \sigma (\nabla J, \nabla \sigma) + \frac{n-2}{2} \sigma J_\sigma \Delta \sigma - \frac{n(n-4)}{8} J_\sigma |\nabla \sigma|^2
+ (-\Delta J - \frac{n}{2} |P|^2) \sigma^2 - \frac{n(n+4)}{16} \sigma^{-1} |\nabla \sigma|^2 \Delta \sigma
- \frac{n(n+4)}{8} \sigma^{-1} (\nabla^2 \sigma, d\sigma \otimes d\sigma) + \frac{n(n+4)(n+8)}{128} \sigma^{-2} |\nabla \sigma|^4.
\]

Proof. From (2.2), it holds that

\[
Q_{\tilde{g}} = \frac{2}{n-4} \sigma^{\frac{n+4}{4}} P_\Lambda [\tilde{g}] \left( \sigma^{-\frac{n+4}{4}} \right).
\]

The result then follows by expanding out the right hand side using (2.1). \(\square\)

These two results are enough to prove Theorem 1.2.

Proof of Theorem 1.2. Fix a metric \(g \in c\), so that the metric \(\sigma^{-1} g\) is equivalent to the metric \(\tilde{g} = \sigma^{-1} g\) in the choice of scale determined by \(g\). In this way, combining
Proposition 5.1 and Proposition 5.2 yields

\[
Q_\hat{g} = -\frac{n(n+2)}{2} |L\sigma|^2 + \frac{n(n+4)}{2} \sigma\langle P, \nabla^2 \sigma \rangle + \frac{n(n+4)}{8} |\nabla^2 \sigma|^2 - \frac{n(n+4)}{4} P(\nabla \sigma, \nabla \sigma)
- \frac{n(n+4)}{8} J|\nabla \sigma|^2 + \frac{n(n+4)}{2} \sigma^2 |P|^2 - \frac{n(n+4)}{16} \sigma^{-1} |\nabla \sigma|^2 \Delta \sigma
- \frac{n(n+4)}{8} \sigma^{-1} \langle \nabla^2 \sigma, d\sigma \otimes d\sigma \rangle + \frac{n(n+4)(n+8)}{128} \sigma^{-2} |\nabla \sigma|^4.
\]

It is not hard to see that this can be simplified as

\[
Q_\hat{g} = -\frac{n(n+2)}{2} |L\sigma|^2 + \frac{n(n+4)}{2} \sigma P + \frac{1}{2} \nabla^2 \sigma - \frac{1}{4} \sigma^{-1} d\sigma \otimes d\sigma - \frac{\sigma^{-1}}{8} |\nabla \sigma|^2 g.
\]

However, it is straightforward to check that

\[
\sigma P_\hat{g} = \sigma P + \frac{1}{2} \nabla^2 \sigma - \frac{1}{4} \sigma^{-1} d\sigma \otimes d\sigma - \frac{\sigma^{-1}}{8} |\nabla \sigma|^2 g.
\]

On the other hand, for any \( u \in \mathcal{E}[1] \), it holds that

\[
|\langle P_{u \rightarrow \hat{g}} \rangle_0|^2 = |\nabla Lu|^2,
\]

whence follows

\[
|P_{u \rightarrow \hat{g}}|^2 = |\nabla Lu|^2 - (n-1)|Lu|^2.
\]

The result then follows by taking \( u = \sqrt{\sigma} \) in the above equation. \( \square \)

References


