Section 4.2 (Future Value of Annuities)

- At the end of each year Bethany deposits $2,000 into an investment account that earns 5% interest compounded annually. How much is in her account at the end of 5 years?

<table>
<thead>
<tr>
<th>Year</th>
<th>Starting Balance</th>
<th>Interest Earned</th>
<th>Deposit</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0</td>
<td>$0</td>
<td>$2,000</td>
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</tr>
<tr>
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<td>$2,000</td>
<td>$100</td>
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<td>$205</td>
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<tr>
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<td>$8,620.25</td>
<td>$431.01</td>
<td>$2,000</td>
<td>$11,051.26</td>
</tr>
</tbody>
</table>

- What if Bethany was depositing $1000 instead of $2000?

<table>
<thead>
<tr>
<th>Year</th>
<th>Starting Balance</th>
<th>Interest Earned</th>
<th>Deposit</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$500</td>
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<td>$3,152.50</td>
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<td>$3152.50</td>
<td>$157.63</td>
<td>$1,000</td>
<td>$4,310.13</td>
</tr>
<tr>
<td>5</td>
<td>$4,310.13</td>
<td>$215.51</td>
<td>$1,000</td>
<td>$5,525.64</td>
</tr>
</tbody>
</table>

- What do we notice about the Ending Balances in these 2 cases?
• What if we thought about investing $1? (and did no rounding)

<table>
<thead>
<tr>
<th>Year</th>
<th>Starting Balance</th>
<th>Interest Earned</th>
<th>Ending Deposit</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0</td>
<td>$0</td>
<td>$1</td>
<td>$1</td>
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<td>$4.310125</td>
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<tr>
<td>5</td>
<td>$4.310125</td>
<td>$0.21550625</td>
<td>$1</td>
<td>$5.52563125</td>
</tr>
</tbody>
</table>

• Look at the value $5.52563125 (the future value we would have after 5 years if we kept making $1 payments a year for 5 years at 5% annual interest)

  - What do we get if we multiply 5.52563125 times $2000?
  - What do we get if we multiply 5.52563125 times $1000?
• **Future Value Annuity Factor**
  
  – For a given interest rate, payment frequency, and number of payment periods, the **future value annuity factor** is the future value that would accumulate if each payment were $1.

  – We will use the symbol $s_{n|i}$, where $n$ is the number of payments and $i$ is the interest rate per payment period.
  – We will pronounce $s_{n|i}$ as ‘snigh’ (rhymes with sigh)

• In our example
  
  – So we found $s_{5|0.05} = 5.52563125$.
  – Were there any other $s_{n|i}$ values we know?

• **The Future Value of an Ordinary Annuity**

  \[ FV = PMT s_{n|i} \]

  where
  
  \( FV \) is the represents the future value of the annuity
  
  \( PMT \) represents the amount of each payment
  
  \( s_{n|i} \) is the future value annuity factor.

• **Example 1**: Find the future value of an ordinary annuity with annual payments of $8,115 for 5 years at 5% annually compounded interest.
• Two Ways to find Future Value Annuity Factors \( (s_{n|i}) \)

– Using a Table to Find Annuity Factors

(see for example http://accountinginfo.com/study/pv/table-fv-a-01.pdf)

* Example: How much will Isaiah have as a future value if he deposit $1,500 at the end of each year into an account that pays 3% interest compounded annually for 20 years.

– Using a Formula to Find Annuity Factors

* The Future Value Annuity Factor

\[
s_{n|i} = \frac{(1 + i)^n - 1}{i}
\]

Where \( i \) represents the interest rate per payment period and \( n \) represents the number of payment periods

• Example 2: Find \( s_{25|0.042} \)

– Option 1: Using the \( s_{n|i} \) formula all at once

\[
(1+.042)^{25}-1)/0.042=
\]

* Is cumbersome, and requires extra parenthesis.

* This is a simpler example than most we will do. Typically they will look like

\[
((1+.042/12)^{(25*12)}-1)/(0.042/12)=
\]

– Option 2: Three Step Method

(Works on TI – 30X Calculators)

To find \( s_{n|i} \) enter the following into the calculator:

Step 1:

\((1+i)^n=\)

Step 2:

\(-1=\)

Step 3:

\(/i=\)
When calculating Future Value Annuity Factors, you may not round anything. Period.

This is why we talk about an effective way to enter the $s_{\bar{n}}|i$ formula into your calculator so you don’t need to write down the intermediate steps.

**You Have Been Warned!**
• **Example 3:** Suppose you get married, and at the end of each year of your marriage, you and your spouse decide to deposit $500 into an account for a special 25th anniversary trip. If the account pays 4.2% interest compounded annually, how much will you have for your trip after 25 years?

1. Assume there is an annuity with weekly payments of $200, a term of 3 years, and an interest rate of 4.5%.
   (a) What is the i for this annuity?
   (b) What is the n for this annuity?
   (c) Find the future value annuity factor for this annuity.
   (d) What is the future value of this annuity?

2. Find the future value of a monthly annuity, assuming the term is 10 years and the interest rate is 7.2%, compounded monthly, and the payments are $25 a month.

3. At the end of each quarter David deposits $300 into a savings account that earns 3.92% interest compounded quarterly. Assume that the interest rate doesn’t change and that he keeps this up for 8 years. What is the future value of this account? How much total interest did he earn?

4. Suppose you get married, and at the start of each year of your marriage, you and your spouse decide to deposit $500 into an account for a special 25th anniversary trip. If the account pays 4.2% interest compounded annually, how much will you have for your trip after 25 years?

5. (Optional) Lilith gave up smoking, and decided to put the $15 a week she would have spent on cigarettes into a special savings account. Currently she earns 3.27% interest, assuming the interest rate doesn’t change, how much will she have in the account after 6 years?

6. (Optional) Find the future value and total interest earned on an Annuity Due with monthly payments of $120, for 5 years, and an interest rate of 6.02% compounded monthly.

7. (Optional) Find the future value and total interest earned on an annuity with monthly payments of $200, for 4 years, and an interest rate of 3\(\frac{3}{4}\)% compounded monthly.
• Future Value for an Annuity Due

Suppose Bethany is depositing $1000 at the start of each year (instead of at the end) into an account paying 5% interest compounded annually for 5 years. How much money would she have at the end of 5 years?

<table>
<thead>
<tr>
<th>Year</th>
<th>Deposit</th>
<th>After Deposit</th>
<th>Interest Earned</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1000</td>
<td>$1000</td>
<td>$50</td>
<td>$1050</td>
</tr>
<tr>
<td>2</td>
<td>$1000</td>
<td>$2050</td>
<td>$102.5</td>
<td>$2152.50</td>
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</table>

– Formula for Future Value for an Annuity Due

\[
FV = PMT \times s_{\bar{n}|i}(1 + i)
\]

where \( FV \) represents the future value of the annuity
\( PMT \) represents the amount of each payment
\( i \) is the interest rate per period
\( s_{\bar{n}|i} \) is the future value annuity factor

– Optional Example:

*Use the Formula for Future Value of an Annuity Due to answer:*

Suppose Bethany is depositing $1000 at the start of each year (instead of at the end) into an account paying 5% interest compounded annually for 5 years. How much money would she have at the end of 5 years?

*Does your answer agree with the table above?*