Last time we proved some rules for deductions.

- Generalization: $\Gamma \vdash \varphi \implies \Gamma \vdash \forall x \varphi$ (x not free in $\Gamma$)
- Rule T: $\Gamma \vdash \varphi_1, \ldots, \varphi_n$, $\{\varphi_1, \ldots, \varphi_n\}$ taut imply $\varphi \implies \Gamma \vdash \varphi$
- Deduction: $\Gamma; \varphi \vdash \psi \implies \Gamma \vdash \varphi \rightarrow \psi$
- Modus Ponens: $\Gamma \vdash \varphi \rightarrow \psi \implies \Gamma; \varphi \vdash \psi$

Today we will prove two more

- **Contraposition**: $\Gamma; \varphi \vdash \psi \iff \Gamma; \psi \vdash \neg \varphi$

  **Proof**: $\Gamma; \varphi \vdash \neg \psi \implies \Gamma \vdash \varphi \rightarrow \neg \psi$ Deduction
  $\implies \Gamma \vdash \psi \rightarrow \neg \varphi$ Rule T
  $\implies \Gamma; \psi \vdash \neg \varphi$ MP. $\square$

**Remarks**
- Rule T could be used since $\varphi \rightarrow \neg \psi$ is tautologically implied by $\psi \rightarrow \neg \varphi$
- Rule T is useful since it lets us say things like "$\varphi \rightarrow \psi$ and $\varphi \rightarrow \neg \psi$ are tautologically equivalent so I will replace the former with the latter"

- $\Gamma; \varphi \vdash \psi \iff \Gamma; \neg \psi \vdash \neg \varphi$
  is a combination of contrapositive and Rule T
Reductio ad Absurdum (Proof by contradiction)

If \( \Gamma; \varphi \) is inconsistent, that is \( \Gamma; \varphi \vdash \beta \) and \( \Gamma; \varphi \vdash \neg \beta \) for some \( \beta \) then
\( \Gamma \vdash \neg \varphi \).

Proof

1. \( \Gamma; \varphi \vdash \beta \) (By assumption)
2. \( \Gamma; \varphi \vdash \neg \beta \) (By assumption)
3. \( \Gamma \vdash \varphi \rightarrow \beta \) (Deduction 1)
4. \( \Gamma \vdash \varphi \rightarrow \neg \beta \) (Deduction 2)
5. \( \Gamma \vdash \beta \rightarrow \neg \varphi \) (Rule T)
6. \( \Gamma \vdash \beta \rightarrow \neg \varphi \) (Using contrapositive)
7. \( \Gamma \vdash \neg \varphi \) (Using that \( \neg \beta \rightarrow \neg \varphi, \beta \rightarrow \varphi \)) (using that \( \neg \beta \rightarrow \neg \varphi, \beta \rightarrow \varphi \))
Example \( \vdash \exists x \forall y \varphi \rightarrow \forall y \exists x \varphi \)

Proof (We work backwards)

By deduction, it suffices to show
\( \exists x \forall y \varphi \vdash \forall y \exists x \varphi \).

By generalization, it suffices to show
\( \exists x \forall y \varphi \vdash \exists x \varphi \).

This is by definition
\( \forall x \forall y \varphi \rightarrow \forall x \exists y \varphi \).

By contrapositive, it suffices to show
\( \exists x \forall y \varphi \vdash \forall x \forall y \varphi \).

By generalization, it suffices to show
\( \forall x \exists y \varphi \rightarrow \forall y \varphi \).

Now it is less straightforward.

By reductio ad absurdum, it suffices to show
\( \{ \forall x \exists y \varphi, \forall y \varphi \} \) is inconsistent.

This is true as follows:

1. \( \forall x \exists y \varphi \vdash \neg \varphi \) Axiom 2 (substitution) \& MP
2. \( \forall y \varphi \vdash \varphi \) Axiom 2 (substitution) \& MP
3. \( \{ \forall x \exists y \varphi, \forall y \varphi \} \) are inconsistent.
Strategy

The goal is to prove $\Gamma \vdash \varphi$

Case 1 Goal: $\Gamma \vdash \varphi \rightarrow \Theta$
Then show $\Gamma ; \varphi \rightarrow \Theta$ (Deduction)

Case 2 Goal: $\Gamma \vdash \forall x \varphi$
Then show $\Gamma \vdash \varphi$ (Generalization
Assuming $x$ not free in $\Gamma$)

Case 3(a) Goal: $\Gamma \vdash \neg (\varphi \rightarrow \Theta)$
Then show $\Gamma \vdash \neg \varphi$ and $\Gamma \vdash \Theta$

Case 3(b) Goal: $\Gamma \vdash \neg \neg \varphi$
Then show $\Gamma \vdash \varphi$

Case 3(c) Goal: $\Gamma \vdash \forall x \varphi$
Then show (if possible) $\Gamma \vdash \neg \psi_x$ (by substitution)
(Why? You are trying to show $\exists x \psi_x$
So it is enough to find a $t$ s.t. $\exists x \neg \psi_x$)

Formal proof:
1. $\Gamma \vdash \psi_x$
2. $\Gamma \vdash \forall x \varphi \rightarrow \psi_x$ Axiom 2 (substitution)
3. $\Gamma \vdash \neg \psi_x \rightarrow \neg \forall x \varphi$ Rule T (contrapositive)
4. $\Gamma \vdash \neg \forall x \varphi$ MP 1, 3

Note: This method (finding $\Gamma \vdash \neg \psi_x$) may not be possible.