Deductive Calculus

In the last section we worked with \( \Gamma \). There \( \Gamma \vdash \sigma \) implied something about all models of \( \Gamma \).

However, now we want to talk about proofs. Is there a direct way to prove \( \Gamma \vdash \sigma \)? Can we do it without talking about structures?

We will introduce \( \Gamma \vdash \sigma \), which means \( \Gamma \) proves or deduces \( \sigma \).

Our deductions have these parts

- Let \( \Gamma' \) be a set of formulas (hypotheses)
- Let \( \Lambda \) be the logical axioms (to be defined shortly)
- Modus ponens is the rule that from \( \alpha \) and \( \alpha \rightarrow \beta \) we may deduce \( \beta \). (This is our only rule of inference.)

(Informally)
We will say \( \Gamma \vdash \sigma \) if \( \sigma \) can be deduced from \( \Gamma' \) and \( \Lambda \) using only modus ponens.
Def. A deduction of \( \phi \) from \( \Gamma \) is a finite sequence \( \langle a_0, \ldots, a_n \rangle \) of formulas such that
\( a_n \) is \( \phi \) and for each \( k \leq n \), either
\( a_k \) is in \( \Gamma \cup \Lambda \), or
\( a_k \) is obtained by modus ponens from two earlier formulas in the sequence; that is, for some \( i \) and \( j \) less than \( k \), \( a_k \) is \( a_i \rightarrow a_j \).

Def. If a deduction exists, say (any of)
\( \phi \) is deducible from \( \Gamma \),
\( \phi \) is a theorem of \( \Gamma \), or
\( \Gamma \vdash \phi \).

This is very similar to inductive definitions from before. The big difference is that there is not a unique deduction of \( \phi \) from \( \Gamma \). However, we still can do induction.

Induction Principle
Suppose that \( S \) is a set of wffs that includes \( \Gamma \cup \Lambda \) and is closed under modus ponens. Then \( S \) contains every theorem of \( \Gamma \).

Takeaway. To prove a theorem of \( \Gamma \) has some property, first show \( \Gamma \) is true of all \( \psi \in \Gamma \), all \( \lambda \in \Lambda \) and then show if it is true of \( \lambda \) and \( \lambda \rightarrow \psi \), the \( \psi \) is true of \( \Psi \).
The logical axioms

Def: Say \( \varphi \) is a generalization of \( \psi \)
if \( \varphi = \forall x_1, \ldots, x_n \psi \) for some variables \( x_1, \ldots, x_n \).
(Also \( \psi \) is a generalization of itself.)

Def: The logical axioms are all generalizations of \( \forall \) \( \forall \) of the following forms:

1. Tautologies
2. \( \forall x \phi \rightarrow \forall x t \), where \( t \) is substitutable for \( x \) in \( \phi \).
3. \( \forall x (\phi \rightarrow \psi) \rightarrow (\forall x \phi \rightarrow \forall x \psi) \)
4. \( \phi \rightarrow \forall x \phi \), where \( x \) does not occur free in \( \phi \).

\( \forall \) Equality:

5. \( x = x \)
6. \( x = y \rightarrow x = x' \), where \( x \) is atomic and \( x' \) is obtained from \( x \) by replacing \( x \) in zero or more (but not necessarily all) places by \( y \).
**Substitution**

We will define two things

1. $x^x_t$ (meaning substitute $x$ with $t$)
2. "$t$ is substitutable for $x$ in $\alpha$"
   (meaning it is ok to substitute $x$ with $t$)

**Def.** Define $x^x_t$ recursively as follows

1. If $\alpha$ is atomic, $\alpha^x_t$ is the expression obtained by replacing $x$ with $t$.
   (Can define this via induction on terms. See exercise 1.)
2. $(\neg \alpha)^x_t = (\neg \alpha^x_t)$
3. $(\alpha \rightarrow \beta)^x_t = (\alpha^x_t \rightarrow \beta^x_t)$
4. $(\forall y \alpha)^x_t = \begin{cases} \forall y \alpha & \text{if } x = y \\ \forall y (\alpha^x_t) & \text{if } x \neq y \end{cases}$

**Examples**

1. $\phi^x_t = \emptyset$
2. $(Q x \rightarrow \forall x P x)^x_y$ is $(Q y \rightarrow \forall x P x)$
3. If $\alpha$ is $\neg \forall y x = y$, then $\forall x \alpha \rightarrow \alpha^x_t$ is
   $\forall x \neg \forall y x = y \rightarrow \forall y x = y$
4. If $\alpha$ is $\neg \forall y x = y$, then $\forall x \alpha \rightarrow \alpha^x_t$ is
   $\forall x \neg \forall y x = y \rightarrow \forall y y = y$  \text{ (Bad!)}

We must make conditions to avoid (4)
Def: Define "t is substitutable for x in α" by recursion as follows.

1. For atomic α, t is always substitutable for x in α.
2. t is substitutable for x in (¬α) iff t is substitutable for x in α.
3. t is substitutable for x in (α → β) iff t is substitutable for x in both α and β.
4. t is substitutable for x in \( \forall y \alpha \) iff either
   a) x does not occur free in \( \forall y \alpha \), or
   b) y does not occur free in t and t is substitutable for x in \( \alpha \).

Fact: t is always substitutable for x in α.

Proof: By induction,
1. If α is atomic, then x is substitutable for x.
2. If (¬α), then by induction hypothesis x is substitutable for \( \forall x \alpha \) and therefore is substitutable for \( \forall x \).
3. Same as 2
4. If \( \forall y \alpha \), then
   a) If \( y = x \), then x is not free in \( \forall x \alpha \) and therefore x is substitutable for x.
   b) If \( y \neq x \), then since y does not occur in \( \alpha \) and since x is substitutable for \( \alpha \) (Ind. hyp), we have x is substitutable for \( \forall y \alpha \).
Remark: $x^j_x$ just means "substitute $t$ for $x$" even when $t$ is not substitutable for $x$.

Examples

This is in axiom group 2

$$\forall x_3 \left( \forall x_1 (P_{x_1} \rightarrow \forall x_2 P_{x_2}) \rightarrow (P_{x_2} \rightarrow \forall x_2 P_{x_2}) \right)$$

This is not in axiom group 2

$$\forall x_1, \forall x_2 B_{x_1, x_2} \rightarrow \forall x_2 B_{x_2, x_2}$$
Tautologies

Def: A tautology (in first order logic) is a sentential tautology (in sentential logic) where the sentence symbols have been replaced with first order formulas.

Example: From \((A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)\)
we get the tautology
\((\forall y \rightarrow \neg P_y \rightarrow \neg P_x) \rightarrow (P_x \rightarrow \neg \forall y \rightarrow \neg P_y)\)

Here is a different approach:

Define a formula \(\alpha\) to be
1. prime iff it is atomic or \(\forall x \alpha\)
2. nonprime iff it is \((\neg \alpha)\) or \((\alpha \rightarrow \beta)\)

(This covers all the cases.)

Then treat each prime formula as a sentence symbol. The tautologies in this "sentential logic" are exactly the tautologies in first order logic.

\[
\begin{array}{cccccccc}
T & F & F & F & T & F & F & T \\
T & T & T & T & F & T & F & T \\
F & T & F & F & T & T & T & T \\
F & T & T & T & F & T & F & F \\
\end{array}
\]
Remarks

1. Some valid formulas are not tautologies, e.g.,
   \[ \forall x \, P_x \Rightarrow P_x \]
   is not a tautology, but it is in axiom group 2.
   Also \[ \forall x \, (P_x \Rightarrow P_x) \]
   is not a tautology, but it is in axiom group 1 since it is the generalization of the tautology \[ P_x \Rightarrow P_x \].

2. We didn't need to take all tautologies to be axioms. Actually, we could get by with a few tautologies being axioms.

3. Now we can connect the results of chapter 4 with chapter 2.