**Homomorphism Thm**

Let $h$ be a homomorphism of $\mathcal{A}$ into $\mathcal{B}$ and let $s : V \to |\mathcal{A}|$. ($V =$ variables)

(a) For any term $t$, we have

$$h(\overline{s}(t)) = \overline{h}s(t)$$

\[\uparrow \quad \text{computed in } \mathcal{A} \quad \uparrow \quad \text{computed in } \mathcal{B}\]

(b) For any quantifier-free formula $\alpha$ not containing the equality symbol,

$$F \alpha [s] \iff F \alpha [\overline{h}s]$$

(c) If $h$ is one-to-one (i.e., is an isomorphism into $\mathcal{B}$), then we may delete restriction, "not containing the equality symbol" in (b).

(d) If $h$ is onto, then we may delete the restriction "quantifier-free" in (b).

Proof will not be given today.

**Examples**

1. $\mathcal{A} = (N, +, 0)$, $\mathcal{B} = (\{e, 0\}, +, 0)$ $h$ as before.

   $h(v_1 \neq v_2 \in [0, 2] \text{ and } h(0) = h(2) = 0$, but $\not\models_\mathcal{B} v_1 = v_2 \in [0, 2]$.

2. $\mathcal{A} = (N, \leq)$, $\mathcal{B} = (\mathbb{Z}, \leq)$ $h = 1d$.

   $\not\models_\mathcal{A} \exists x \forall y x \leq y$, but $\models_\mathcal{B} \exists x \forall y x \leq y$.
Proof of homomorphism theorem

6. This is a homework exercise. Use induction on \( t \).

6. By induction on \( \alpha \),

- If \( \alpha \) is atomic, then \( \alpha = P \tau_1 \cdots \tau_n \)
  \[ \vdash_{\alpha} P \tau_1 \cdots \tau_n \subseteq \Leftrightarrow \langle S(\tau_1), \ldots, S(\tau_n) \rangle \in P^* \]
  \[ \Leftrightarrow \langle h(S(\tau_1)), \ldots, h(S(\tau_n)) \rangle \in P^* \text{ (h homo)} \]
  \[ \Leftrightarrow \langle h(S(\tau_1)), \ldots, h(S(\tau_n)) \rangle \in P^* \text{ (part 6m)} \]
  \[ \Leftrightarrow \vdash_{\alpha} P \tau_1 \cdots \tau_n \subseteq \text{homo} \]

- If \( \alpha = (\neg \beta) \), then
  \[ \vdash_{\alpha} \neg \beta \subseteq \Leftrightarrow \neg \vdash_{\alpha} \beta \subseteq \]
  \[ \Leftrightarrow \neg \vdash_{\alpha} \beta \subseteq \text{homo} \]
  \[ \iff \vdash_{\alpha} \neg \beta \subseteq \text{homo} \]

- If \( \alpha = (\beta \Rightarrow \gamma) \), then
  \[ \vdash_{\alpha} \beta \Rightarrow \gamma \subseteq \Leftrightarrow \vdash_{\alpha} \beta \Rightarrow \gamma \subseteq \text{homo} \]
  \[ \iff \vdash_{\alpha} \beta \Rightarrow \gamma \subseteq \text{homo} \]
(c) We just need to handle the atomic formula \( t_1 = t_2 \) (the rest is done in part (b))

\[
F_\emptyset t_1 = t_2 \[\psi\] \iff \overline{s}(t_1) = \overline{s}(t_2) \\
\iff h(\overline{s}(t_1)) = h(\overline{s}(t_2)) \iff h \text{ is onto} \\
\iff \overline{hos}(t_1) = \overline{hos}(t_2) \quad \text{(part (a))} \\
\iff \models \emptyset t_1 = t_2 \[\psi_{hos}\]
\]

(d) We just need to handle the quantifier induction step. The rest is done in part (b).

\( \alpha = \forall x \, \varphi \[\psi_{s}\] \), then for any \( \alpha \in \mathcal{A} \)

\[
\models \emptyset \forall x \, \varphi \[\psi_{hos}\] \Rightarrow \models \emptyset \forall x \, \varphi \[\psi(\overline{hos} \, \overline{x} \, h(x))\] \\
\iff \models \emptyset \varphi \[\psi(\overline{hos} \, \overline{x} \, h(a))\] \quad \text{(two functions are the same)} \\
\iff \models \emptyset \varphi \[\psi(\overline{s} \, \overline{x} \, h(a))\] \quad \text{(I H)}
\]

So always have \( \models \emptyset \forall x \, \varphi \[\psi_{hos}\] \Rightarrow \models \emptyset \forall x \, \varphi \[\psi\] \)

since \( \alpha \) was arbitrary.

For other direction, choose \( b \in \mathcal{B} \). Since \( h \) is onto, we have \( h(a) = b \) for some \( a \in \mathcal{A} \).

Then

\[
\models \emptyset \forall x \, \varphi \[\psi_{s}\] \iff \models \emptyset \varphi \[\psi(\overline{s} \, \overline{x} \, h(a))\] \\
\iff \models \emptyset \varphi \[\psi(\overline{hos} \, \overline{x} \, h(a))\] \quad \text{(I H)} \\
\iff \models \emptyset \varphi \[\psi(\overline{hos} \, \overline{x} \, 16) \quad (h(a) = b)\]
\]

Therefore \( \models \emptyset \forall x \, \varphi \[\psi_{s}\] \Rightarrow \models \emptyset \forall x \, \varphi \[\psi_{hos}\] \). \( \square \)
Def: Two structures $A$ and $B$ (for the same language) are elementarily equivalent ($A \equiv B$) if for all sentences $\phi$,

$$\forall \phi. \phi \iff \phi$$

Cor: Isomorphic structures are elementarily equivalent.

$$A \cong B \implies A \equiv B$$

The converse does not hold,

$$(\mathbb{R}; <) \equiv (\mathbb{Q}; <)$$

but $$(\mathbb{R}; <) \not\equiv (\mathbb{Q}; <)$$

(Proof is latter in the book.)

Def: An automorphism of $A$ is an isomorphism of $A$ onto $A$.

Cor: Let $h$ be an automorphism of $A$, and let $R$ be any $n$-ary definable relation in $A$. Then for any $a_1, \ldots, a_n$ in $|A|_0$,

$$\langle a_1, \ldots, a_n \rangle \in R \iff \langle h(a_1), \ldots, h(a_n) \rangle \in R.$$

Proof: By the homomorphism theorem, if $\phi$ defines $R$, then

$$\models_A \phi[[a_1, \ldots, a_n]] \iff \models_A \phi[[h(a_1), \ldots, h(a_n)]]$$
Showing relations are not definable

Examples

1. Consider the automorphism of \((\mathbb{R}; <)\) given by \(h(a) = a^3\). This does not preserve \(\mathbb{N}\), so the set \(\mathbb{N}\) is not definable in \((\mathbb{R}; <)\).

2. Consider the automorphism \(h: (\mathbb{E}; \vee) \to (\mathbb{F}; \wedge)\) given by swapping \(a\) and \(c\). Therefore, the point \(a\) is not definable.

3. Consider the automorphism of \((\mathbb{C}; +, \cdot, 0, 1)\) given by the conjugate map \(h(z) = \overline{z} \quad \forall z \in \mathbb{C}\).

Then \(i\) is not definable. (Although \(\exists \in \mathbb{R}, -i3 \in \mathbb{R}\).)